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Journal of Functional Analysis 208 (2004) 360–376

JOURNAL OF Functional Analysis

http://www.elsevier.com/locate/jfa

Spectral properties of the $\bar{\partial}$-canonical solution operator

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Received 25 November 2002; accepted 7 February 2003

Communicated by L. Carleson

Abstract

In this paper we consider a Hilbert space $\mathscr{H}^{(0,1)}$ of $(0,1)$-forms and a space $L^2(\mu)$ of square integrable functions with respect to a measure μ on a rotation invariant open set Ω in $\mathbb{C}^n$. We give necessary and sufficient conditions, in terms of the moments of the measure μ, for the canonical solution operator of the $\bar{\partial}$-equation to be bounded, compact and in the Schatten p-class from $\mathscr{H}^{(0,1)}$ into $L^2(\mu)$. Examples of $\mathscr{H}^{(0,1)}$ can be chosen to be the space of $(0,1)$-forms with coefficients in one of the classical Hilbert spaces of holomorphic functions such as the weighted Bergman space, the Hardy space, the Hardy–Sobolev space or the Möbius invariant space.

1. Introduction

Let Ω be a rotation invariant open set in $\mathbb{C}^n$ and let μ be a rotation invariant probability measure on Ω having moments of all orders; that is,

$$m_k := \int_\Omega |z|^{2k}\, d\mu(z) < \infty, \quad \text{for all } k \in \mathbb{N}_0.$$

Let $L^2(\mu)$ denote the space of all square integrable functions with respect to the measure μ. Such measures have been studied in [BT].

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doi:10.1016/S0022-1236(03)00139-3

We recall that the Fischer inner product $\langle \, , \rangle_{\mathscr{F}}$ is defined on the space of holomorphic polynomials by its restriction on the monomials by

$$\langle z^\alpha, z^\beta \rangle_{\mathscr{F}} := \begin{cases} \alpha! & \text{if } \alpha = \beta, \\ 0 & \text{if } \alpha \neq \beta. \end{cases}$$

This inner product turns the multiplication operator and the corresponding differentiation operator to be adjoint to each other. See [NS].

Consider a Hilbert space $\mathscr{H}$ of holomorphic functions on Ω such that there is a positive integer N such that $\mathscr{H}$ is the direct sum of the subspace of its polynomials of degree smaller than N and the closure of its subspace spanned by the homogeneous polynomials in $\mathscr{H}$ with degree greater than or equal to N. We also assume that the homogeneous polynomials in $\mathscr{H}$ with different degrees greater than or equal to N are orthogonal in $\mathscr{H}$. For $d \geqslant N$, we will denote by $\mathscr{H}_d$ the space of the homogeneous polynomials of degree d of $\mathscr{H}$. Assume, in addition, that there are a sequence $(h_d)_{d \geqslant N}$ of nonnegative real numbers and positive constants C_1 and C_2 such that

$$C_1 |\langle f, g \rangle_{\mathscr{H}}| \leqslant h_d |\langle f, g \rangle_{\mathscr{F}}| \leqslant C_2 |\langle f, g \rangle_{\mathscr{H}}|, \tag{1.1}$$

for all $d \geqslant N$ and f, g in $\mathscr{H}_d$.

Let $\mathscr{H}^{(0,1)}$ be the Hilbert space of all $(0,1)$-forms with holomorphic coefficients in $\mathscr{H}$. We equip $\mathscr{H}^{(0,1)}$ with the inner product

$$\langle f, g \rangle_{\mathscr{H}^{(0,1)}} := \sum_{j=1}^{n} \langle f_j, g_j \rangle_{\mathscr{H}},$$

for $f = \sum_{j=1}^n f_j \, d\bar{z}_j$ and $g = \sum_{j=1}^n g_j \, d\bar{z}_j$ in $\mathscr{H}^{(0,1)}$.

We shall study the spectral properties of the operator $S : \mathscr{D}om(S) \to L^2(\mu)$ defined on the dense subspace $\mathscr{D}om(S)$ of $\mathscr{H}^{(0,1)}$ consisting of those $(0,1)$-forms whose coefficients are polynomial elements of $\mathscr{H}$ and S solves the $\bar{\partial}$-equation $\bar{\partial}(S(g)) = g$, where $S(g)$ is the unique element of $L^2(\mu)$ which is continuous on Ω and orthogonal to holomorphic polynomials in $L^2(\mu)$. This operator will be called the $\bar{\partial}$-canonical solution operator.

The main purpose herein is to establish necessary and sufficient conditions for the $\bar{\partial}$-canonical solution operator to extend to be bounded, compact and in the Schatten p-class from $\mathscr{H}^{(0,1)}$ into $L^2(\mu)$.

We should point out that under particular choices of (h_d), the space $\mathscr{H}$ can be chosen to be the Bergman space, the Hardy space, the Hardy–Sobolev space or the Möbius invariant space.

The classical $\bar{\partial}$-canonical solution operator is the operator that associates to each $(0,1)$-form, with L^2-coefficients with respect to the Lebesgue measure, the L^2 solution to the $\bar{\partial}$-equation which is orthogonal to the Bergman space of all square integrable holomorphic functions. An explicit expression was given in terms of the

Bergman projection on arbitrary bounded domains in a recent work of Haslinger [Ha1]. The membership of this canonical solution operator to the Hilbert–Schmidt class was investigated by Haslinger in [Ha2,Ha3].

Our main results are the following.

Theorem A. *The $\bar{\partial}$-canonical solution operator $S : \mathscr{D}om(S) \to L^2(\mu)$ extends to a bounded on $\mathscr{H}^{(0,1)}$ if and only if*

$$\sup_{d \in \mathbb{N}_0, h_d > 0} \frac{m_{d+1}}{h_d (n+d)!} \left(d \left(1 - \frac{m_d^2}{m_{d-1} m_{d+1}} \right) + (n-1) \right) < +\infty. \tag{1.2}$$

Suppose that (1.2) is satisfied and denote again by S the extension of the $\bar{\partial}$-canonical solution operator to $\mathscr{H}^{(0,1)}$. Then we have

Theorem B. *The operator $S : \mathscr{H}^{(0,1)} \to L^2(\mu)$ is compact if and only if*

$$\lim_{d \to \infty, h_d > 0} \frac{m_{d+1}}{h_d (n+d)!} \left(d \left(1 - \frac{m_d^2}{m_{d-1} m_{d+1}} \right) + (n-1) \right) = 0. \tag{1.3}$$

Theorem C. *If $p > 0$ and*

$$\sum_{d \in \mathbb{N}_0, h_d > 0} \dim \mathscr{H}_d \left[\frac{m_{d+1}}{h_d (n+d)!} \left(d \left(1 - \frac{m_d^2}{m_{d-1} m_{d+1}} \right) + (n-1) \right) \right]^{\frac{p}{2}} < +\infty, \tag{1.4}$$

then the operator S is in the Schatten class $\mathscr{S}_p(\mathscr{H}^{(0,1)}, L^2(\mu))$.

Conversely, suppose that either $n = 1$ and $p > 0$ or $n \geqslant 2$ and $p \geqslant 2$ and the operator $S : \mathscr{H}^{(0,1)} \to L^2(\mu)$ is in the Schatten class $\mathscr{S}_p(\mathscr{H}^{(0,1)}, L^2(\mu))$. Then (1.4) *holds.*

We do not know whether (1.4) holds also in the remaining case $n \geqslant 2$ and $0 < p < 2$ under the assumption $S \in \mathscr{S}_p(\mathscr{H}^{(0,1)}, L^2(\mu))$.

2. Preparatory lemmas

In this section we shall express the $\bar{\partial}$-canonical solution operator and its adjoint in terms of reproducing kernels and some Hankel operators.

Lemma 2.1. *If $\alpha, \beta \in \mathbb{N}_0^n$, then*

$$\int_\Omega z^\alpha \bar{z}^\beta \, d\mu(z) = \begin{cases} \dfrac{(n-1)! m_{|\alpha|} \alpha!}{(n+|\alpha|-1)!} & \text{if } \alpha = \beta, \\ 0 & \text{if } \alpha \neq \beta. \end{cases} \tag{2.1}$$

Proof. To prove identity (2.1), let $\mathscr{U}$ be the unitary group consisting of all $n \times n$ unitary matrices and let dU denote the Haar probability measure on $\mathscr{U}$. Let σ be the rotation-invariant probability measure on the unit sphere $\mathbb{S}$ in $\mathbb{C}^n$. Due to the invariance of Ω and μ we see by Fubini's theorem and Proposition 1.4.7 in [Ru] that

$$\begin{aligned}\int_\Omega z^\alpha \bar{z}^\beta \, d\mu(z) &= \int_{\mathscr{U}} \int_\Omega (Uz)^\alpha (\overline{Uz})^\beta \, d\mu(z) \, dU \\ &= \int_\Omega \int_{\mathscr{U}} (Uz)^\alpha (\overline{Uz})^\beta \, dU \, d\mu(z) \\ &= \int_\Omega |z|^{|\alpha|+|\beta|} \, d\mu(z) \int_{\mathbb{S}} \zeta^\alpha \bar{\zeta}^\beta \, d\sigma(\zeta)\end{aligned}$$

from which (2.1) follows. □

If d is a nonnegative integer, let B_d be the reproducing kernel of the subspace of $L^2(\mu)$ consisting of the holomorphic polynomials of degree smaller than or equal to d with respect to the inner product of $L^2(\mu)$. It follows from Lemma 2.1 that for any polynomial f, the limit

$$(P_\mu f)(z) \coloneqq \lim_{d \to \infty} \int_\Omega B_d(z, \omega) f(\omega) \, d\mu(\omega) \tag{2.2}$$

is finite since the integral in (2.2) is the same for sufficiently large integers d. In addition, P_μ is a symmetric projection taking its values in the space of holomorphic polynomials; that is,

$$P_\mu \circ P_\mu = P_\mu \quad \text{and} \quad \langle P_\mu f, g \rangle_{L^2(\mu)} = \langle f, P_\mu g \rangle_{L^2(\mu)},$$

for all polynomials f, g and $P_\mu h = h$ for all holomorphic polynomials h. In general, this projection does not extend to be continuous on $L^2(\mu)$.

If φ is a holomorphic polynomial, then the Hankel operator with symbol $\bar{\varphi}$ is the operator given for all polynomial f by

$$H_{\bar{\varphi}} f \coloneqq (I - P_\mu)(\bar{\varphi} f),$$

where I is the identity operator.

Lemma 2.2. *For any homogeneous holomorphic polynomial g of degree $d+1$ we have the identity*

$$H_{\bar{z}_j}(g) = \bar{z}_j g - \frac{m_{d+1}}{(n+d) m_d} \frac{\partial g}{\partial z_j}. \tag{2.3}$$

Proof. To prove identity (2.3), observe by (2.1) that the polynomial $\bar{z}_j g(z)$ is orthogonal in $L^2(\mu)$ to any holomorphic homogeneous polynomial of degree different from d. Furthermore, if f is a homogeneous holomorphic polynomial of degree d, then by Lemma 2.1 and the fact that the multiplication operator by z_j and the differentiation operator $\frac{\partial}{\partial z_j}$ are adjoint of each other with respect to the Fischer inner product we have

$$\begin{aligned}\langle P_\mu(\bar{z}_j g), f\rangle_{L^2(\mu)} &= \langle \bar{w}_j g, f\rangle_{L^2(\mu)} \\ &= \langle g, w_j f\rangle_{L^2(\mu)} \\ &= \frac{(n-1)!m_{d+1}}{(n+d)!}\left\langle \frac{\partial g}{\partial z_j}, f\right\rangle_{\mathscr{F}} = \frac{m_{d+1}}{(n+d)m_d}\left\langle \frac{\partial g}{\partial z_j}, f\right\rangle_{L^2(\mu)}.\end{aligned}$$

This completes the proof of the lemma. $\square$

If $d \geqslant N$, we denote by $\mathscr{H}_d^{(0,1)}$ the subspace of $\mathscr{H}^{(0,1)}$ consisting of those $(0,1)$-forms with coefficients in $\mathscr{H}_d$ and let K_d be the reproducing kernel of $\mathscr{H}_d$. It is given by

$$K_d(z,w) := \sum_{g \in \mathscr{B}_d} g(z)\overline{g(w)},$$

where $\mathscr{B}_d$ is an orthonormal basis of $\mathscr{H}_d$.

Lemma 2.3. *Suppose that $n \geqslant 1$. Then the $\bar{\partial}$-canonical solution operator S is given for $g \in \mathscr{D}om(S)$ by*

$$S(g)(z) = \lim_{d\to\infty} \sum_{j=1}^n \int_\Omega B_d(z,w) g_j(w)(\bar{z}_j - \bar{w}_j)\, d\mu(w).$$

In particular, the restriction of S to $\mathscr{H}_d^{(0,1)}$ is given by

$$S(g)(z) = \langle g, \omega_d(z,\cdot)\rangle_{\mathscr{H}^{(0,1)}}, \quad g \in \mathscr{H}_d^{(0,1)},$$

where $\omega_d : \Omega \times \Omega \to \mathscr{H}^{(0,1)}$ is the mapping defined by

$$\omega_d(z,\xi) := \sum_{j=1}^n \overline{H_{\bar{z}_j}(K_d(z,\xi))}\, d\bar{\xi}_j \tag{2.4}$$

Proof. Let $g = \sum_{j=1}^n g_j\, d\bar{z}_j \in \mathscr{D}om(S)$. By (2.2) we see that

$$S_d(g)(z) = \sum_{j=1}^n \int_\Omega B_d(z,w) g_j(w)(\bar{z}_j - \bar{w}_j)\, d\mu(w)$$

is a polynomial independent of $d \geqslant d_g$ for some sufficient large integer $d_g \geqslant N$. In addition, a little computing shows that $S(g) := \lim_{d \to \infty} S_d(g)$ is orthogonal to holomorphic polynomials and satisfies the $\bar{\partial}$-equation $\bar{\partial}(S(g)) = g$. Now if the components g_j of g are in $\mathscr{H}_d$, then applying the reproducing formula for each g_j, we see that for $k \geqslant d_g$,

$$\begin{aligned} S(g)(z) &= \sum_{j=1}^{n} \int_{\Omega} B_k(z,w) \langle g_j, K_d(.,w) \rangle_{\mathscr{H}} (\bar{z}_j - \bar{w}_j) \, d\mu(w) \\ &= \sum_{j=1}^{n} \int_{\Omega} \langle g_j, \overline{B_k(z,w) K_d(w,\cdot)(\bar{z}_j - \bar{w}_j)} \rangle_{\mathscr{H}} \, d\mu(w) \\ &= \sum_{j=1}^{n} \left\langle g_j, \int_{\Omega} \overline{B_k(z,w) K_d(w,\cdot)(\bar{z}_j - \bar{w}_j)} \, d\mu(w) \right\rangle_{\mathscr{H}} \\ &= \sum_{j=1}^{n} \langle g_j, \overline{H_{\bar{z}_j}(K_d(z,.)} \rangle_{\mathscr{H}} = \langle g, \omega_d(z,\cdot) \rangle_{\mathscr{H}^{(0,1)}}. \end{aligned}$$

This completes the proof of the lemma. □

When $d \geqslant N$, we let $\mathscr{N}_d$ be the subspace of $\mathscr{H}$ consisting of polynomials of degree smaller than d and denote by $\mathscr{N}_d^{(0,1)}$ the corresponding space of $(0,1)$-forms having coefficients in $\mathscr{N}_d$. We also let $\mathscr{R}_d$ denote the closure in $\mathscr{H}$ of the direct sum of the subspaces $\mathscr{H}_k, k \geqslant d$, and denote by $\mathscr{R}_d^{(0,1)}$ the space of $(0,1)$-forms having coefficients in $\mathscr{R}_d$. Then by hypothesis $\mathscr{H}$ and $\mathscr{H}^{(0,1)}$ can be written as the following topological direct sums:

$$\mathscr{H} = \mathscr{N}_d \oplus \mathscr{R}_d \quad \text{and} \quad \mathscr{H}^{(0,1)} = \mathscr{N}_d^{(0,1)} \oplus \mathscr{R}_d^{(0,1)}. \tag{2.5}$$

We denote by the same symbol π_d the projections from $\mathscr{H}$ onto $\mathscr{N}_d$ and from $\mathscr{H}^{(0,1)}$ onto $\mathscr{N}_d^{(0,1)}$ corresponding to both of these two direct sums. Thus, if $g = \sum_{j=1}^{n} g_j \, d\bar{z}_j \in \mathscr{H}^{(0,1)}$, then

$$\pi_d g := \sum_{j=1}^{n} \pi_d g_j \, d\bar{z}_j. \tag{2.6}$$

Lemma 2.4. *The domain $\mathscr{D}om(S^*)$ of S^* contains all polynomials.*

Proof. It is sufficient to show that $\mathscr{D}om(S^*)$ contains the monomials $z^\alpha \bar{z}^\beta$. Choose $k \geqslant |\alpha| + |\beta| + 2$. Thus by (2.6) and Lemmas 2.1 and 2.2, we see that for each $g = \sum_{j=1}^{n} g_j \, d\bar{z}_j \in \mathscr{D}om(S)$ we have that

$$\langle (S \circ (I - \pi_k))(g), z^\alpha \bar{z}^\beta \rangle_{L^2(\mu)} = \sum_{j=1}^{n} \langle H_{\bar{z}_j}((I - \pi_k) g_j), z^\alpha \bar{z}^\beta \rangle_{L^2(\mu)} = 0.$$

Since the operator $S \circ \pi_k$ is of finite rank, this implies that

$$|\langle Sg, z^\alpha \bar{z}^\beta \rangle_{L^2(\mu)} = |\langle (S \circ \pi_k)(g), z^\alpha \bar{z}^\beta \rangle_{L^2(\mu)} \leqslant C\|g\|_{\mathscr{H}^{(0,1)}},$$

where C does not depend on g. This completes the proof of lemma. □

Next, let S_N be the restriction of S to $\mathscr{D}om(S_N) := \mathscr{D}om(S) \cap \mathscr{R}_N^{(0,1)}$. Then S_N is a densely defined operator in the Hilbert space $\mathscr{R}_N^{(0,1)}$. In view of Lemma 2.4, we see that the domain $\mathscr{D}om(S_N^*)$ of the adjoint S_N^* of S_N contains all polynomials. In addition, we have the following.

Lemma 2.5. *Suppose that $d \geqslant N$. Then for all $(0,1)$-forms f in $\mathscr{H}_d^{(0,1)}$ we have*

$$(S_N^* S_N)(f)(\xi) = \sum_{j=1}^n \langle f, \eta_j^d(\cdot, \xi) \rangle_{\mathscr{H}^{(0,1)}} \, d\bar{\xi}_j,$$

where $\eta_j^d(z,\xi) := \sum_{k=1}^n \eta_{j,k}^d(z,\xi)\, d\bar{z}_k$, and

$$\eta_{j,k}^d(z,\xi) = \int_\Omega \overline{H_{\bar{w}_k}(K_d(w,z))} H_{\bar{w}_j}(K_d(w,\xi))\, d\mu(w).$$

Proof. Let $f \in \mathscr{H}_d^{(0,1)}$ and $g \in \mathscr{H}_k^{(0,1)}$. Applying Lemmas 2.1 and 2.2 a little computing shows that $\langle S_N(f), S_N(g) \rangle_{L^2(\mu)} = 0$, if $d \neq k$. Otherwise, by (2.2) we have

$$\begin{aligned} \langle S_N(f), S_N(g) \rangle_{L^2(\mu)} &= \int_\Omega \langle f, \omega_d(z,\cdot) \rangle_{\mathscr{H}^{(0,1)}} \langle \omega_d(z,\cdot), g \rangle_{\mathscr{H}^{(0,1)}} \, d\mu(z) \\ &= \left\langle \int_\Omega \langle f, \omega_d(z,\cdot) \rangle_{\mathscr{H}^{(0,1)}} \omega_d(z,\cdot)\, d\mu(z), g \right\rangle_{\mathscr{H}^{(0,1)}} \\ &= \left\langle \sum_{j=1}^n \langle f, \eta_j^d(\cdot,\xi) \rangle_{\mathscr{H}^{(0,1)}} \, d\bar{\xi}_j, g \right\rangle_{\mathscr{H}^{(0,1)}}. \end{aligned}$$

This proves the lemma. □

Consider an orthonormal basis $\mathscr{B}_d$ of $\mathscr{H}_d, d \geqslant N$. Then the collection of the $(0,1)$-forms $\{f d\bar{z}_k, f \in \mathscr{B}_d, k = 1, \dots, n\}$ is an orthonormal basis of $\mathscr{H}_d^{(0,1)}$. In addition, we have

Lemma 2.6. *For all $d \geqslant N$, the operator $S_N^* S_N$ maps $\mathscr{H}_d^{(0,1)}$ into itself. More precisely, if $f \in \mathscr{B}_d$ and $k = 1, \ldots, n$ then*

$$S_N^* S_N(f d\bar{z}_k)(\xi) = \frac{(n-1)! m_{d+1}}{(n+d)!} \left(\sum_{g \in \mathscr{B}_d} g(\xi) \langle f, g \rangle_{\mathscr{F}} \, d\bar{\xi}_k + \sum_{j=1}^{n} \sum_{g \in \mathscr{B}_d} g(\xi) \left(1 - \frac{m_d^2 (n+d)}{m_{d-1} m_{d+1} (n+d-1)} \right) \left\langle \frac{\partial f}{\partial z_k}, \frac{\partial g}{\partial z_j} \right\rangle_{\mathscr{F}} d\bar{\xi}_j \right).$$

Proof. We write

$$K_d(z, w) = \sum_{g \in \mathscr{B}_d} g(z) \overline{g(w)}.$$

In view of Lemma 2.5 we have that

$$\begin{aligned} (S_N^* S_N)(f d\bar{z}_k)(\xi) &= \sum_{j=1}^{n} \langle f, \eta_{j,k}^d(\cdot, \xi) \rangle_{\mathscr{H}} \, d\bar{\xi}_j \\ &= \sum_{j=1}^{n} \int_\Omega \langle f, \overline{H_{\bar{z}_k}(K_d(z, \cdot))} \rangle_{\mathscr{H}} \overline{H_{\bar{z}_j}(K_d(z, \xi))} \, d\mu(z) \, d\bar{\xi}_j \\ &= \sum_{j=1}^{n} \int_\Omega H_{\bar{z}_k}(f) \overline{H_{\bar{z}_j}(K_d(z, \xi))} \, d\mu(z) \, d\bar{\xi}_j \\ &= \sum_{g \in \mathscr{B}_d} g(\xi) \sum_{j=1}^{n} \int_\Omega H_{\bar{z}_k}(f) \overline{H_{\bar{z}_j}(g)} \, d\mu(z) \, d\bar{\xi}_j. \end{aligned}$$

On the other hand, for j, k fixed and $g \in \mathscr{B}_d$ we have that

$$\begin{aligned} \int_\Omega H_{\bar{z}_k}(f) \overline{H_{\bar{z}_j}(g)} \, d\mu(z) &= \langle z_j f, z_k g \rangle_{L^2(\mu)} + \frac{m_d^2 \left\langle \frac{\partial f}{\partial z_k}, \frac{\partial g}{\partial z_j} \right\rangle_{L^2(\mu)}}{(n+d-1)^2 m_{d-1}^2} \\ &\quad - \frac{m_d}{(n+d-1) m_{d-1}} \left(\left\langle f, z_k \frac{\partial g}{\partial z_j} \right\rangle_{L^2(\mu)} + \left\langle z_j \frac{\partial f}{\partial z_k}, g \right\rangle_{L^2(\mu)} \right) \\ &= \frac{(n-1)! m_{d+1}}{(n+d)!} \langle z_j f, z_k g \rangle_{\mathscr{F}} + \frac{(n-1)! m_d^2 \left\langle \frac{\partial f}{\partial z_k}, \frac{\partial g}{\partial z_j} \right\rangle_{\mathscr{F}}}{m_{d-1} (n+d-1)! (n+d-1)} \\ &\quad - \frac{(n-1)! m_d^2 \left(\left\langle z_j \frac{\partial f}{\partial z_k}, g \right\rangle_{\mathscr{F}} + \left\langle f, z_k \frac{\partial g}{\partial z_j} \right\rangle_{\mathscr{F}} \right)}{(n+d-1)! (n+d-1) m_{d-1}} \end{aligned}$$

$$
\begin{aligned}
&= \frac{(n-1)!m_{d+1}}{(n+d)!}\langle z_j f, z_k g\rangle_{\mathscr{F}} - \frac{(n-1)!m_d^2\left\langle \frac{\partial f}{\partial z_k}, \frac{\partial g}{\partial z_j}\right\rangle_{\mathscr{F}}}{m_{d-1}(n+d-1)!(n+d-1)} \\
&= \frac{(n-1)!m_{d+1}}{(n+d)!}\left(1 - \frac{m_d^2(n+d)}{m_{d-1}m_{d+1}(n+d-1)}\right)\left\langle \frac{\partial f}{\partial z_k}, \frac{\partial g}{\partial z_j}\right\rangle_{\mathscr{F}} \\
&\quad + \frac{(n-1)!m_{d+1}}{(n+d)!}\delta_{j,k}\langle f, g\rangle_{\mathscr{F}},
\end{aligned}
$$

with the understanding that $\delta_{j,k} := 0$ for $j \neq k$ and $\delta_{j,j} := 1$. The latter equalities hold because the multiplication operator by z_j and the differentiation operator $\frac{\partial}{\partial z_j}$ are adjoint of each other with respect to the Fischer inner product. From this it now follows that

$$
\begin{aligned}
(S_N^* S_N)(f d\bar{z}_k)(\xi) = \frac{(n-1)!m_{d+1}}{(n+d)!}\sum_{g\in\mathscr{B}_d} g(\xi)\Bigg(&\sum_{j=1}^n \delta_{j,k}\langle f, g\rangle_{\mathscr{F}}\, d\bar{\xi}_j \\
&+ \sum_{j=1}^n \left(1 - \frac{m_d^2(n+d)}{m_{d-1}m_{d+1}(n+d-1)}\right)\left\langle \frac{\partial f}{\partial z_k}, \frac{\partial g}{\partial z_j}\right\rangle_{\mathscr{F}} d\bar{\xi}_j\Bigg).
\end{aligned}
$$

The proof of the lemma is now complete. □

Lemma 2.7. *There is a positive constant C such that*

$$
\sum_{k=1}^n \langle S_N^* S_N(f(\xi)\, d\bar{\xi}_k), f(\xi)\, d\bar{\xi}_k\rangle_{\mathscr{H}^{(0,1)}} \geqslant C\frac{m_{d+1}}{h_d(n+d)!}\left(d\left(1 - \frac{m_d^2}{m_{d-1}m_{d+1}}\right) + (n-1)\right)
$$

for all $d \geqslant N$ with $h_d > 0$ and all unit vectors $f \in \mathscr{H}_d$.

Proof. By Lemma 2.6 and (1.1) we see that

$$
\begin{aligned}
&\sum_{k=1}^n \langle S_N^* S_N(f(\xi)\, d\bar{\xi}_k), f(\xi)\, d\bar{\xi}_k\rangle_{\mathscr{H}^{(0,1)}} \\
&\quad = \frac{(n-1)!m_{d+1}}{(n+d)!}\sum_{k=1}^n \left[\|f\|_{\mathscr{F}}^2 + \left(1 - \frac{m_d^2(n+d)}{m_{d-1}m_{d+1}(n+d-1)}\right)\left\|\frac{\partial f}{\partial z_k}\right\|_{\mathscr{F}}^2\right] \\
&\quad = \|f\|_{\mathscr{F}}^2 \frac{(n-1)!m_{d+1}}{(n+d)!}\left[n + d - \frac{d(n+d)m_d^2}{m_{d-1}m_{d+1}(n+d-1)}\right] \\
&\quad \geqslant C_1 \frac{(n-1)!}{h_d(n+d-1)!}\frac{m_{d-1}m_{d+1}(n+d-1) - dm_d^2}{m_{d-1}(n+d-1)} \\
&\quad \geqslant C\frac{m_{d+1}}{h_d(n+d)!}\left(d\left(1 - \frac{m_d^2}{m_{d-1}m_{d+1}}\right) + (n-1)\right). \qquad \square
\end{aligned}
$$

Lemma 2.8. *There is a positive constant C such that*

$$\langle S_N^* S_N(u), u \rangle_{\mathscr{H}^{(0,1)}} \leqslant C \frac{m_{d+1}}{h_d (n+d)!} \left(d \left(1 - \frac{m_d^2}{m_{d-1} m_{d+1}} \right) + (n-1) \right)$$

for all $d \geqslant N$ *with* $h_d > 0$ *and all unit vectors* $u \in \mathscr{H}_d^{(0,1)}$.

Proof. As before, we consider the orthonormal basis $\mathscr{B}_d$ of $\mathscr{H}_d$ and write

$$u = \sum_{f \in \mathscr{B}_d, k=1,\dots,n} a_{f,k} f(\xi)\, d\bar{\xi}_k,$$

where the finite sequence $(a_{f,k})_{f,k}$ of complex numbers satisfies

$$\sum_{f \in \mathscr{B}_d, k=1,\dots,n} |a_{f,k}|^2 = 1. \tag{2.7}$$

Note by Cauchy–Schwarz inequality that the moments (m_d) satisfy the inequality

$$m_d^2 \leqslant m_{d-1} m_{d+1}, \quad \text{for all } d \geqslant 1. \tag{2.8}$$

If $n \geqslant 2$, then by Lemma 2.6, (2.7) and (2.8) we have

$$\begin{aligned} \langle S_N^* S_N(u), u \rangle_{\mathscr{H}^{(0,1)}} &= \sum_{f,f',k,k'} a_{f,k} \overline{a_{f',k'}} \langle S_N^* S_N(f(\xi)\, d\bar{\xi}_k), f'(\xi)\, d\bar{\xi}_{k'} \rangle_{\mathscr{H}^{(0,1)}} \\ &= \frac{(n-1)! m_{d+1}}{(n+d)!} \Bigg(\sum_{f,k} |a_{f,k}|^2 \|f\|_{\mathscr{F}}^2 \\ &\quad + \left(1 - \frac{m_d^2 (n+d)}{m_{d-1} m_{d+1} (n+d-1)} \right) \left\| \sum_{f,k} a_{f,k} \frac{\partial f}{\partial z_k} \right\|_{\mathscr{F}}^2 \Bigg) \\ &\leqslant \frac{(n-1)! m_{d+1}}{(n+d)!} \Bigg(\sum_{f,k} |a_{f,k}|^2 \|f\|_{\mathscr{F}}^2 \\ &\quad + \left(1 - \frac{m_d^2}{m_{d-1} m_{d+1}} \right) \left\| \sum_{f,k} a_{f,k} \frac{\partial f}{\partial z_k} \right\|_{\mathscr{F}}^2 \Bigg). \end{aligned}$$

Since $\| \sum_{f,k} a_{f,k} \frac{\partial f}{\partial z_k} \|_{\mathscr{F}}^2 \leqslant d \sum_{f,k} |a_{f,k}|^2 \|f\|_{\mathscr{F}}^2$, it follows that

$$\langle S_N^* S_N(u), u \rangle_{\mathscr{H}^{(0,1)}} \leqslant \frac{(n-1)! m_{d+1}}{(n+d)!} \left(1 + d \left(1 - \frac{m_d^2}{m_{d-1} m_{d+1}} \right) \right) \sum_{f,k} |a_{f,k}|^2 \|f\|_{\mathscr{F}}^2.$$

This, combined with (1.1) and (2.7), implies that

$$\langle S_N^* S_N(u), u \rangle_{\mathscr{H}^{(0,1)}} \leqslant C \frac{m_{d+1}}{(n+d)!h_d}\left((n-1) + d\left(1 - \frac{m_d^2}{m_{d-1}m_{d+1}}\right)\right),$$

for some positive constant C independent of d.

If $n=1$, then by Lemma 2.6, (1.1) and the preceding calculation we obtain

$$|\langle S_N^* S_N(f(\xi)\, d\bar{\xi}), f(\xi)\, d\bar{\xi} \rangle_{\mathscr{H}^{(0,1)}}| = \frac{m_{d+1}}{d!}||f||_{\mathscr{F}}^2 \left(1 - \frac{m_d^2}{m_{d-1}m_{d+1}}\right)$$
$$\simeq \frac{m_{d+1}}{h_d d!}\left(1 - \frac{m_d^2}{m_{d-1}m_{d+1}}\right)$$

and hence the lemma is proved. □

3. Proof of the main results

Using the direct sums in (2.5) and the corresponding projection in (2.6) and writing $S = S\pi_N + S_N(I - \pi_N)$, we see that S is bounded if and only if S_N is, S is compact if and only if S_N is and S is in a Schatten class if and only if S_N is. Therefore, we only need show the results when S is replaced by S_N.

Proof of Theorem A. To prove that $S : \mathscr{D}om(S) \to L^2(\mu)$ is bounded we only need to show that $S_N^* S_N$ is bounded. This in turn is equivalent to showing that

$$\sup_{d \geqslant N, h_d \neq 0} \; \sup_{u \in \mathscr{H}_d^{(0,1)}, ||u||_{\mathscr{H}^{(0,1)}} = 1} |\langle S_N^* S_N(u), u \rangle_{\mathscr{H}^{(0,1)}}| < \infty.$$

Theorem A now follows since by Lemmas 2.7 and 2.8, it is not hard to see that for $d \geqslant N$ such that $h_d > 0$,

$$\sup_{u \in \mathscr{H}_d^{(0,1)}, ||u||_{\mathscr{H}^{(0,1)}} = 1} |\langle S_N^* S_N(u), u \rangle_{\mathscr{H}^{(0,1)}}| \simeq \frac{m_{d+1}}{h_d (n+d)!}\left(d\left(1 - \frac{m_d^2}{m_{d-1}m_{d+1}}\right) + (n-1)\right).$$

Proof of Theorem B. It is well known that S_N is compact if and only if $S_N^* S_N$ is compact or equivalently for all $\varepsilon > 0$ there is a finite codimensional orthogonal projection Q such that $||QS_N^* S_N Q|| < \varepsilon$.

Assume that $S_N^* S_N$ is compact. Then we have

$$\lim_{d \to \infty, h_d > 0} |\langle S_N^* S_N(f(\xi)\, d\bar{\xi}_k), f(\xi)\, d\bar{\xi}_k \rangle_{\mathscr{H}^{(0,1)}}| = 0$$

for all unit vectors f in $\mathscr{H}_d$ and $k = 1, \dots, n$. By Lemma 2.7 this implies that

$$\lim_{d\to\infty} \frac{m_{d+1}}{h_d(n+d)!}\left(d\left(1 - \frac{m_d^2}{m_{d-1}m_{d+1}}\right) + (n-1)\right) = 0$$

and the "only if "part of Theorem B is proved. Conversely, suppose that

$$\lim_{d\to\infty} \frac{m_{d+1}}{h_d(n+d)!}\left(d\left(1 - \frac{m_d^2}{m_{d-1}m_{d+1}}\right) + (n-1)\right) = 0.$$

Let $\varepsilon > 0$. If $d \geqslant N$ and Q_d is the orthogonal projection from $\mathscr{H}^{(0,1)}$ onto $\oplus_{l\geqslant d}\mathscr{H}_l^{(0,1)}$, then by Lemma 2.8 it follows that for some positive constant C,

$$||Q_d S_N^* S_N Q_d|| \leqslant C \sup_{l\geqslant d} \frac{m_{l+1}}{h_l(n+l)!}\left(l\left(1 - \frac{m_l^2}{m_{l-1}m_{l+1}}\right) + (n-1)\right),$$

for d sufficiently large. This shows that $S_N^* S_N$ is compact and thereby Theorem B is proved. □

Proof of Theorem C. Suppose that $p > 0$ and (1.4) holds. Lemma 2.6 implies that for any $d \geqslant N$ the subspace $\mathscr{H}_d^{(0,1)}$ is invariant under the compact operator $S_N^* S_N$. Therefore, there are positive numbers $(\lambda_{j,d})_{1\leqslant j\leqslant \dim\mathscr{H}_d^{(0,1)}}$ and an orthonormal system $(u_{j,d})_{1\leqslant j\leqslant \dim\mathscr{H}_d^{(0,1)}}$ in $\mathscr{H}_d^{(0,1)}$ such that

$$S_N^* S_N(f) = \sum_{d=N}^{+\infty} \sum_{j=1}^{\dim\mathscr{H}_d^{(0,1)}} \lambda_{j,d} \langle u_{j,d}, f\rangle_{\mathscr{H}^{(0,1)}} u_{j,d},$$

for all $f \in \mathscr{H}^{(0,1)}$. Indeed, $\{\lambda_{j,d}\}$ is the sequence of eigenvalues of $S_N^* S_N$ and we have

$$\langle S_N^* S_N(u_{j,d}), u_{j,d}\rangle_{\mathscr{H}^{(0,1)}} = \lambda_{j,d}.$$

Lemma 2.8 yields

$$\begin{aligned}
\sum_{d=N}^{+\infty} \sum_{j=1}^{\dim\mathscr{H}_d^{(0,1)}} \lambda_{j,d}^{\frac{p}{2}} &= \sum_{d=N}^{+\infty} \sum_{j=1}^{\dim\mathscr{H}_d^{(0,1)}} \langle S^* S(u_{j,d}), u_{j,d}\rangle_{\mathscr{H}^{(0,1)}}^{\frac{p}{2}} \\
&\leqslant C \sum_{d=N}^{+\infty} n \dim \mathscr{H}_d \left[\frac{m_{d+1}}{h_d(n+d)!}\left(d\left(1 - \frac{m_d^2}{m_{d-1}m_{d+1}}\right) + (n-1)\right)\right]^{\frac{p}{2}} \\
&< \infty.
\end{aligned}$$

This implies that S_N is in the Schatten class $\mathscr{S}_p(\mathscr{R}_N^{(0,1)}, L^2(\mu))$ and hence S is in the Schatten class $\mathscr{S}_p(\mathscr{H}^{(0,1)}, L^2(\mu))$.

Suppose $n = 1$ and $p > 0$ and $S \in \mathscr{S}_p(\mathscr{H}^{(0,1)}, L^2(\mu))$. By Lemma 2.6, we see that, for $d \geqslant N$ such that $h_d > 0$, $\frac{z^d}{\|z^d\|} d\bar{z}$ is an eigenvector of $S_N^* S_N$. If λ_d is the corresponding eigenvalue, then Lemmas 2.7 and 2.8 give that

$$\lambda_d \simeq \frac{m_{d+1}}{h_d\, d!}\left(1 - \frac{m_d^2}{m_{d-1}m_{d+1}}\right)$$

and hence (1.4) follows.

Next, suppose that $n \geqslant 2$ and $p \geqslant 2$ and assume that the $\bar{\partial}$-canonical solution operator S is in the Schatten class $\mathscr{S}_p(\mathscr{H}^{(0,1)}, L^2(\mu))$. Consider an orthonormal basis $\mathscr{B}_d$ of $\mathscr{H}_d$ and denote by E the spectral measure of the operator $S_N^* S_N$. By the spectral theorem, we see that for all $k = 1, \ldots, n$ and any $f \in \mathscr{B}_d$, we have that

$$\begin{aligned} |\langle S_N^* S_N(f(\xi)\, d\bar{\xi}_k), f(\xi)\, d\bar{\xi}_k \rangle_{\mathscr{H}^{(0,1)}}|^{p/2} &= \left(\int_{\mathbb{R}} t\, d\langle E(t)(f(\xi)\, d\bar{\xi}_k), f(\xi)\, d\bar{\xi}_k \rangle_{\mathscr{H}^{(0,1)}}\right)^{p/2} \\ &\leqslant \int_{\mathbb{R}} t^{p/2}\, d\langle E(t)(f(\xi)\, d\bar{\xi}_k), f(\xi)\, d\bar{\xi}_k \rangle_{\mathscr{H}^{(0,1)}} \\ &= \langle (S_N^* S_N)^{p/2}(f(\xi)\, d\bar{\xi}_k), f(\xi)\, d\bar{\xi}_k \rangle_{\mathscr{H}^{(0,1)}}. \end{aligned}$$

This implies that

$$\sum_{f,k} \langle S_N^* S_N(f(\xi)\, d\bar{\xi}_k), f(\xi)\, d\bar{\xi}_k \rangle_{\mathscr{H}^{(0,1)}}^{p/2} < \infty.$$

This, combined with Lemma 2.7, gives that

$$\sum_{d=N, h_d>0}^{+\infty} \dim \mathscr{H}_d \left[\frac{m_{d+1}}{h_d(n+d)!}\left(d\left(1 - \frac{m_d^2}{m_{d-1}m_{d+1}}\right) + (n-1)\right)\right]^{\frac{p}{2}} < \infty.$$

Then (1.4) holds. The proof of Theorem C is now complete. □

4. Applications and concluding remarks

Let $d\mu_s(z) = (1 - |z|^2)^s\, dV(z)$, where $s > -1$ and dV is the normalized Lebesgue measure on the unit ball $\mathbb{B}$ in $\mathbb{C}^n$. If $m \geqslant 0$ consider the Hilbert space $\mathscr{H}_m$ consisting of all holomorphic functions $f = \sum_{\alpha \in \mathbb{N}_0^n} a_\alpha z^\alpha$ on $\mathbb{B}$, whose power series expansion satisfies

$$\sum_{\alpha \in \mathbb{N}_0^n} \frac{|a_\alpha|^2 \alpha!(|\alpha| + m)}{\Gamma(|\alpha| + m + 1)} < \infty,$$

equipped with its inner product $\langle f,g\rangle_{\mathscr{H}_m} = \sum_{\alpha\in\mathbb{N}_0^n} \frac{a_\alpha \overline{b_\alpha}\alpha!(|\alpha|+m)}{\Gamma(|\alpha|+m+1)}$, for $f = \sum_{\alpha\in\mathbb{N}_0^n} a_\alpha z^\alpha$ and $g = \sum_{\alpha\in\mathbb{N}_0^n} b_\alpha z^\alpha$ and let $\mathscr{H}_m^{(0,1)}$ be the corresponding Hilbert space of all $(0,1)$-forms with holomorphic coefficients in $\mathscr{H}_m$.

In this case, the sequence $\{h_d\}_{d\in\mathbb{N}_0}$ defined by

$$h_d := \frac{1}{\Gamma(d+m)}, \quad d\in\mathbb{N}_0$$

satisfies (1.1) with respect to the inner product of $\mathscr{H}_m$. The moments of the measure μ_s are given by

$$\begin{aligned} m_d &:= \int_{\mathbb{B}} |z|^{2d}(1-|z|^2)^s \, dV(z) \\ &= \int_0^1 r^{2n+2d-1}(1-r^2)^s \, dr \\ &= \frac{1}{2}\frac{\Gamma(s+1)(d+n-1)!}{\Gamma(d+n+s+1)}. \end{aligned}$$

In addition we have, for $d\in\mathbb{N}_0$,

$$\frac{m_{d+1}}{h_d(n+d)!} = \frac{\Gamma(d+m)}{\Gamma(d+n+s+2)}$$

and

$$1-\frac{m_d^2}{m_{d-1}m_{d+1}} = \frac{s+1}{(d+n)(d+n+s)}$$

so that by Stirling's formula we see that

$$\frac{m_{d+1}}{h_d(n+d)!}\left(d\left(1-\frac{m_d^2}{m_{d-1}m_{d+1}}\right)+(n-1)\right) \simeq d^{m-n-s-2}\left(\frac{1}{d}+(n-1)\right).$$

Therefore by Theorem A we see that when $n=1$ the $\bar{\partial}$-canonical solution operator S is bounded from $\mathscr{H}_m^{(0,1)}$ into $L^2(\mu_s)$ if and only if $m\leqslant s+4$, however when $n\geqslant 2$ the operator S is bounded from $\mathscr{H}_m^{(0,1)}$ into $L^2(\mu_s)$ if and only if $m\leqslant n+2+s$.

Also Theorem B yields that when $n=1$ the $\bar{\partial}$-canonical solution operator S is a compact operator from $\mathscr{H}_m^{(0,1)}$ into $L^2(\mu_s)$ if and only if $m<s+4$, however when $n\geqslant 2$ the operator S is compact from $\mathscr{H}_m^{(0,1)}$ into $L^2(\mu_s)$ if and only if $m<n+2+s$.

In addition, the reduction of Theorem C to this case gives the following.

Corollary 4.1. *Assume that $m\geqslant 0$ and $p>0$.*

(1) *If $n=1$ and $m<s+4$ then the $\bar{\partial}$-canonical solution operator S is in the Schatten class $\mathscr{S}_p(\mathscr{H}_m^{(0,1)}, L^2(\mu_s))$ if and only if*

$$p > \frac{2}{4+s-m}.$$

(2) *If* $n \geqslant 2, p \geqslant 2$ and $m < n+2+s$, *then the* $\bar{\partial}$*-canonical solution operator* S *is in the Schatten class* $\mathscr{S}_p(\mathscr{H}_m^{(0,1)}, L^2(\mu_s))$ *if and only if*

$$p(n - m + s + 2) > 2n.$$

We should point out that the parameters $m = n+1$, $m = n$ and $m = 0$, correspond, respectively, to the cases where $\mathscr{H}_m$ is the Bergman space, the Hardy space and the Möbius invariant space on $\mathbb{B}$.

We should remark that when $m = n+1$ and $s = 0$, the condition in Corollary 4.1 is equivalent to the fact that for $n \geqslant 2$, the Schatten class $\mathscr{S}_p(\mathscr{H}_{n+1}, L^2(\mathbb{B}))$ contains nonzero Hankel operators with antiholomorphic symbols. See [AFJP,HY,Zh1,Zh2]. We do not know whether the same is true when the parameter $m \neq n+1$.

Now, we consider a class of weighted Hilbert spaces $\mathscr{H}_\varphi$ consisting of the entire functions in $\mathbb{C}^n$ which are square integrable with respect to the measure μ_φ where μ_φ is the measure with weight $e^{-\varphi(|z|)}$ with respect to the Lebesgue measure on $\mathbb{C}^n$ and φ is a nonnegative function on $[0, +\infty[$ such that the moments

$$m_d(\varphi) \coloneqq \int_0^{+\infty} t^{2n+2d-1} e^{-\varphi(t)}\, dt$$

are finite for all nonnegative integers d. We have for $\alpha \in \mathbb{N}_0^n$ such that $|\alpha| = d$,

$$\begin{aligned} ||z^\alpha||^2_{\mathscr{H}_\varphi} &= \int_{\mathbb{C}^n} |z^\alpha|^2 e^{-\varphi(|z|)}\, dV(z) \\ &= \int_0^\infty r^{2n+2d-1} e^{-\varphi(r)}\, dr \int_{\mathbb{S}} |\zeta^\alpha|^2\, d\sigma(\zeta). \end{aligned}$$

In this case, the sequence $\{h_d\}_{d \in \mathbb{N}_0}$ is chosen to be

$$h_d(\varphi) \coloneqq \frac{(n-1)!}{(n+d-1)!} m_d(\varphi), \text{ for all } d \in \mathbb{N}_0. \tag{4.1}$$

Applying (4.1) and Theorems A, B and C to this example we obtain the following.

Corollary 4.2. *The* $\bar{\partial}$*-canonical solution operator* S *is bounded if and only if*

$$\sup_{d \in \mathbb{N}_0} \frac{m_{d+1}(\varphi)}{d m_d(\varphi)} \left(d \left(1 - \frac{m_d^2(\varphi)}{m_{d-1}(\varphi) m_{d+1}(\varphi)} \right) + (n-1) \right) < +\infty.$$

Corollary 4.3. *The* $\bar{\partial}$*-canonical solution operator* S *is compact if and only if*

$$\sup_{d \in \mathbb{N}_0} \frac{m_{d+1}(\varphi)}{d m_d(\varphi)} \left(d \left(1 - \frac{m_d^2(\varphi)}{m_{d-1}(\varphi) m_{d+1}(\varphi)} \right) + (n-1) \right) = 0.$$

Corollary 4.4. *If $p > 0$ and*

$$\sum_{d \in \mathbb{N}_0} d^{n-1} \left[\frac{m_{d+1}(\varphi)}{d m_d(\varphi)} \left(d(1 - \frac{m_d^2(\varphi)}{m_{d-1}(\varphi) m_{d+1}(\varphi)}) + (n-1) \right) \right]^{\frac{p}{2}} < +\infty \tag{4.2}$$

then the $\bar{\partial}$-canonical solution operator S is in the Schatten class $\mathscr{S}_p(\mathscr{H}_\varphi^{(0,1)}, L^2(\mu_\varphi)$.

Suppose that either $n = 1$ and $p > 0$ or $n \geqslant 2$ and $p \geqslant 2$. If the $\bar{\partial}$-canonical solution operator S is in the Schatten class $\mathscr{S}_p(\mathscr{H}_\varphi^{(0,1)}, L^2(\mu_\varphi)$, then (4.2) *holds.*

Examples of such weights can be taken to be $\varphi_m(t) = t^m$, where m is a positive real number. In this case, the moments of the measure are given by

$$m_d := \int_0^\infty t^{2n+2d-1} e^{-t^m} \, dt = \frac{1}{m} \Gamma\left(\frac{2n+2d}{m} \right).$$

Hence,

$$\frac{m_{d+1}}{d m_d} = \frac{\Gamma\left(\frac{2n+2d+2}{m} \right)}{d \Gamma\left(\frac{2n+2d}{m} \right)} \simeq \left(\frac{2n+2d}{m} \right)^{\frac{2}{m}-1}.$$

Using the following refined Stirling's formula in [MOS]

$$\Gamma(x) = \sqrt{2\pi} e^{-x} x^{x-\frac{1}{2}} \left(1 + \frac{1}{12x} + O\left(\frac{1}{x^2} \right) \right),$$

we obtain that

$$\frac{m_d^2}{m_{d+1} m_{d-1}} = \frac{\Gamma\left(\frac{2n+2d}{m} \right)^2}{\Gamma\left(\frac{2n+2n-2}{m} \right) \Gamma\left(\frac{2n+2d+2}{m} \right)} = 1 - \frac{6}{m(n+d)} + O\left(\frac{1}{d^2} \right).$$

This shows that

$$\frac{m_{d+1}}{d m_d} \left(d\left(1 - \frac{m_d^2}{m_{d-1} m_{d+1}} \right) + (n-1) \right) \simeq d^{\frac{2}{m}-1}.$$

From this and Corollary 4.2 it follows that S is bounded if and only if $m \geqslant 2$. Corollary 4.3 shows that S is compact if and only if $m > 2$ and finally Corollary 4.3 implies that S is in the Schatten class $\mathscr{S}_p(\mathscr{H}_\varphi^{(0,1)}, L^2(\mathbb{C}^n, \mu_\varphi))$ if and only if $p > \frac{2mn}{m-2}$.

In particular, if $m = 2$, then S is bounded but not compact this gives as a consequence Theorem 2 in [Ha4].

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