Logical Bilattices and Inconsistent Data

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Abstract

The notion of a bilattice was first proposed by Ginsberg as a general framework for many applications. This notion was further investigated and applied for various goals by Fitting. In the present paper we develop proof systems, which correspond to bilattices in an essential way. We then show how to use those bilattices for efficient inferences from possibly inconsistent data. For this we incorporate certain ideas of Kifer and Lozinskii concerning inconsistencies, which happen to suit well the framework of bilattices. The outcome is a paraconsistent logic with a lot of desirable properties.

1 Introduction

When using multiple-valued logics, it is usual to order the truth values in a lattice structure, where its partial order, $\leq_t$, describes intuitively differences in the "measure of truth" that the lattice elements are supposed to represent. However, these elements (the "truth values") can be ordered differently. Another reasonable ordering, $\leq_k$, reflects (again, intuitively) differences in the amount of the *knowledge* or in the amount of *information* that each one of these elements exhibits. Ginsberg introduced (in [Gins]) the notion of *bilattices*, which are algebraic structures that contain two such partial orders simultaneously (see definition 2.1). His motivation was to present a general framework for many applications, like truth maintenance systems and default inferences. This notion was further investigated and applied for various properties by Fitting (see [Fit1]–[Fit6]).

The present paper has two main goals: The first is to develop proof systems, which correspond to bilattices in an essential way. For this purpose we have found it useful to introduce and investigate the notion of a *logical* bilattice. (All the bilattices which were actualy proposed for applications in the literature fall under this category). The general logic of these bilattices has indeed a very nice proof theory.

Our second goal is to use logical bilattices in a more specific way for efficient inferences from inconsistent data (this was also the original purpose of Belnap, who had introduced the first bilattice in [Bel1],[Bel2]). For this we incorporate certain ideas from [KiLo]. We show (so we believe) that bilattices provide a better framework for applying these ideas than the one used in the original paper. The outcome is a paraconsistent [dCos] logic with a lot of desirable properties.

Due to the lack of space, some of the proofs are omitted, and others are given in outlines. Full proofs, as well as a more detailed presentation, will be given in the full paper.

2 Logical bilattices

2.1 Bilattices - General background

Definition 2.1 A *bilattice* [Gins] is a structure $\mathcal{B} = (\mathrm{B}, \leq_t, \leq_k, \neg)$ such that B is a non empty set containing at least two elements; $(\mathrm{B}, \leq_t)$, $(\mathrm{B}, \leq_k)$ are complete lattices; and $\neg$ is a unary operation on B that has the following properties:
if $a \leq_t b$, then $\neg a \geq_t \neg b$.
if $a \leq_k b$, then $\neg a \leq_k \neg b$.
$\neg\neg a = a$.

Notations: Following Fitting, we shall use $\wedge$ and $\vee$ for the lattice operations which correspond to $\leq_t$, and $\otimes$, $\oplus$ for those that correspond to $\leq_k$. f and t will denote, respectively, $inf_{\leq_t}(\mathrm{B})$ and $sup_{\leq_t}(\mathrm{B})$, while $\bot$ and $\top$ – $inf_{\leq_k}(\mathrm{B})$ and $sup_{\leq_k}(\mathrm{B})$. Obviously, $f \neq t$ and $\bot \neq \top$.

While $\wedge$ and $\vee$ can be associated with their usual intuitive meanings of "and" and "or", one may understand $\otimes$ and $\oplus$ as the "consensus" and the "guillibility" ("accept all") operators, respectivelly. A practical application of $\otimes$ and $\oplus$ is provided, for example, in an implementation of a logic programming language designed for distributed knowledgebases (see [Fit4] for more details).

Note that negation is order preserving w.r.t $\leq_k$. This reflects the intuition that $\leq_k$ corrsponds to differences in our *knowledge* about formulae and not to their truth values. (see [Gins] for further discusion).

Definition 2.2 A bilattice is called *distributive* [Gins] if all the twelve possible distributive laws concerning $\wedge$, $\vee$, $\otimes$, and $\oplus$ hold. It is called *interlaced* [Fit1] if each one of $\wedge$, $\vee$, $\otimes$, and $\oplus$, is monotonic with respect to both $\leq_t$ and $\leq_k$.

Lemma 2.3 [Fit1] Every distributive bilattice is interlaced.

Example 2.4 The bilattices *FOUR* and *NINE* (figure 1) are both distributive bilattices [1], while Ginsberg's *DEFAULT* [Gins] (figure 2) is not even interlaced.

Definition 2.5 [Gins] Let $(L,\leq)$ be a complete lattice. The structure $L\odot L=(L\times L,\leq_t,\leq_k,\neg)$ is defined as follows:
$(y_1,y_2)\geq_t(x_1,x_2)$ iff $y_1\geq x_1$ and $y_2\leq x_2$.
$(y_1,y_2)\geq_k(x_1,x_2)$ iff $y_1\geq x_1$ and $y_2\geq x_2$.
$\neg(x_1,x_2)=(x_2,x_1)$.

$L\odot L$ was introduced in [Gins], and later used by Fitting as a general mechanizm for constructing bilattices. A truth value $(x,y)\in L\odot L$ may intuitively be understood as simultaneously representing the degree of belief *for* an assertion, and the degree of belief *against* it.

Lemma 2.6
a) [Fit3] $L\odot L$ is an interlaced bilattice.
b) [Gins] If L is distributive, then so is $L\odot L$.

Example 2.7 Denote {0,1} by *TWO*. Then *FOUR* is isomorphic to $TWO\odot TWO$. Similarly, *NINE* is isomorphic to $\{-1,0,1\}\odot\{-1,0,1\}$.

[1] *FOUR* is due to Belnap (see [Bel1], [Bel2])

2.2 Bifilters and logicality

One of the most important component in a many-valued logic is the subset of the *designated* truth values. This subset is used for defining validity of formulae and a consequence relation. Frequently, in an algebraic treatment of the subject, the set of designated valued forms a filter, or even a prime (ultra-) filter, relative to some natural ordering of the truth values. Natural analogues for bilattices of filters, prime-filters, and set of designated values in genetal, are the following:

Definition 2.8
a) A *bifilter* of a bilattice $\mathcal{B}$ is a nonempty set $\mathcal{F}\subset$ B, $\mathcal{F}\neq$ B, such that:
$a\wedge b\in\mathcal{F}$ iff $a\in\mathcal{F}$ and $b\in\mathcal{F}$
$a\otimes b\in\mathcal{F}$ iff $a\in\mathcal{F}$ and $b\in\mathcal{F}$
b) A bifilter $\mathcal{F}$ is called *prime*, if it satisfies also:
$a\vee b\in\mathcal{F}$ iff $a\in\mathcal{F}$ or $b\in\mathcal{F}$
$a\oplus b\in\mathcal{F}$ iff $a\in\mathcal{F}$ or $b\in\mathcal{F}$

Example 2.9 *FOUR* and *DEFAULT* contain exactly one bifilter, $\{\top,t\}$, which is prime in both. $\{\top,t\}$ is also the only bifilter of *FIVE* [Gins] (figure 3), but it is *not* prime there: $d\top\vee\bot\in\mathcal{F}$, while $d\top\notin\mathcal{F}$, and $\bot\notin\mathcal{F}$. *NINE* contains two bifilters: $\{\top,ot,t\}$, as well as $\{\top,ot,of,t,d\top,dt\}$; both are prime.

Since every bifilter $\mathcal{F}$ is necessarily upward-closed w.r.t $\leq_t$ and $\leq_k$, $\{x\mid x\geq_k t\}$ and $\{x\mid x\geq_t\top\}$ are subsets of $\mathcal{F}$. On the other hand, $f\notin\mathcal{F}$, and $\bot\notin\mathcal{F}$, since $\mathcal{F}\neq$ B.

Definition 2.10 A *logical bilattice* is a pair $(\mathcal{B},\mathcal{F})$, in which $\mathcal{B}$ is a bilattice, and $\mathcal{F}$ is a prime bifilter on $\mathcal{B}$.

In the next section we shall use logical bilattices for defining logics in a way which is completely analogous to the way Boolean algebras and ultrafilters are used in classical logic. The role which *TWO* has among Boolean algebras is taken here by *FOUR*:

Theorem 2.11 Let $(\mathcal{B},\mathcal{F})$ be a logical bilattice. Then there exists a unique homomorphism $h:\mathcal{B}\rightarrow FOUR$, such that $h(b)\in\{\top,t\}$ iff $b\in\mathcal{F}$.

Outline of Proof: Define $h(b)=\top$ if $b\in\mathcal{F}$ and $\neg b\in\mathcal{F}$, $h(b)=t$ if $b\in\mathcal{F}$ and $\neg b\notin\mathcal{F}$, $h(b)=f$ if $\neg b\in\mathcal{F}$ and $b\notin\mathcal{F}$, and $h(b)=\bot$ if $b\notin\mathcal{F}$ and $\neg b\notin\mathcal{F}$. $\Box$

We next discuss the existence of bifilters and prime bifilters, concentrating on an important special case:

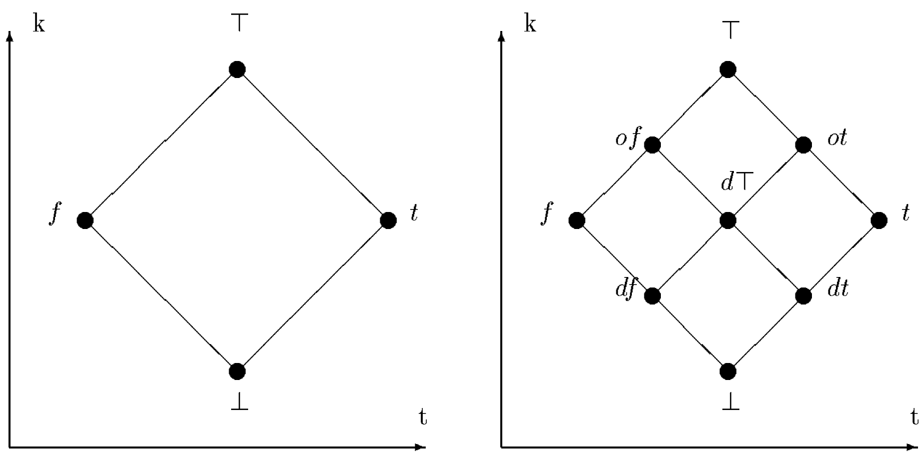

Figure 1: *FOUR* and *NINE*

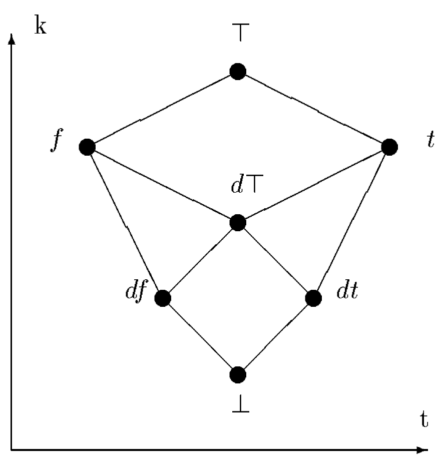

Figure 2: *DEFAULT*

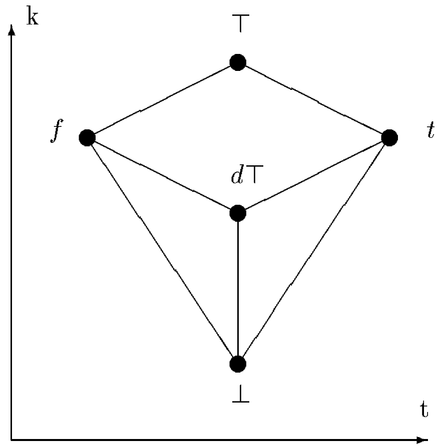

Figure 3: *FIVE*

Definition 2.12 Let $\mathcal{B}$ be a bilattice. Define:
• $\mathcal{D}_k(\mathcal{B}) \stackrel{\text{def}}{=} \{ x \mid x \geq_k t \}$
• $\mathcal{D}_t(\mathcal{B}) \stackrel{\text{def}}{=} \{ x \mid x \geq_t \top \}$

Intuitively, each element of $\mathcal{D}_k(\mathcal{B})$ represents a truth value which is known to be "at least true" ([Bel2], p.36). Hence it seems that $\mathcal{D}_k(\mathcal{B})$ is a particulary natural candidate to play the role of the set of the designated values of $\mathcal{B}$.

Example 2.13
a) $\mathcal{D}_k(FOUR) = \mathcal{D}_t(FOUR) = \{\top, t\}$.
b) $\mathcal{D}_k(DEFAULT) = \mathcal{D}_t(DEFAULT) = \{\top, t\}$.
c) $\mathcal{D}_k(FIVE) = \mathcal{D}_t(FIVE) = \{\top, t\}$.
d) $\mathcal{D}_k(NINE) = \mathcal{D}_t(NINE) = \{\top, ot, t\}$.
e) $\mathcal{D}_k(L \odot L) = \mathcal{D}_t(L \odot L) = \{ (sup(L), x) \mid x \in L \}$.

Proposition 2.14 Let $\mathcal{B}$ be an interlaced bilattice. Then $\mathcal{D}_k(\mathcal{B}) = \mathcal{D}_t(\mathcal{B})$, and it is the smallest bifilter (i.e.: it is contained in any other bifilter). Moreover, $\{b, \neg b\} \subseteq \mathcal{D}_k(\mathcal{B})$ iff $b = \top$.

It follows that if $\mathcal{B}$ is interlaced, then $(\mathcal{B}, \mathcal{D}_k(\mathcal{B}))$ is a logical bilattice iff $\mathcal{D}_k(\mathcal{B})$ is prime. In fact, $(\mathcal{B}, \mathcal{D}_k(\mathcal{B}))$ is logical bilattice in all the exapmles which were actually used in the literature for constructive purposes. This is true even for *DEFAULT*, although it is not interlaced.

We next provide a sufficient and neccessary conditions for $\mathcal{D}_k(\mathcal{B})$ to be prime in one particularly important case:

Proposition 2.15 If L is a complete lattice, then $(L \odot L, \mathcal{D}_k(L \odot L))$ is a logical bilattice iff $sup(L)$ is join irreducible (i.e.: if $a \vee b = sup(L)$, then $a = sup(L)$ or $b = sup(L)$).

3 The basic logic of logical bilattices

For simplicity, we treat here only the propositional case; the extension to full predicate logic is in most cases straightforward.

3.1 The basic consequence relation

Definition 3.1
a) The language *BL* (Bilattice-based Language) is the standard propositional language over $\{\wedge, \vee, \neg, \otimes, \oplus\}$.
b) $BL(4)$ is BL enriched with the constants $\{f, t, \bot, \top\}$.
c) Let B=$(\mathcal{B}, \mathcal{F})$ be a logical bilattice. $BL(B)$ is BL enriched with a propositional constant for each element in B.

Given a bilattice $\mathcal{B}$, the semantic notion of valuations in B is defined in the obvious way. The associated logics are also defined naturally:

Definition 3.2
a) $\Gamma \models_{BL(B)} \Delta$ iff for every valuation ν such that $\nu(\psi) \in \mathcal{F}$ for all $\psi \in \Gamma$, we have that $\nu(\phi) \in \mathcal{F}$ for some $\phi \in \Delta$.
b) $\Gamma \models_{BL} \Delta$ ($\Gamma \models_{BL(4)} \Delta$), where Γ and Δ are finite sets of formulae in BL (in $BL(4)$), iff $\Gamma \models_{BL(B)} \Delta$ for every logical bilattice B.

Proposition 3.3
a) $\models_{BL(B)}$ is paraconsistent: $p, \neg p \not\models_{BL(B)} q$.
b) $\models_{BL}$ has no tautologies. [2]

Our next theorem is an easy consequence of theorem 2.11. It shows that in order to check consequence

[2] In $BL(4)$, however, t and $\top$ *are* tautologies.

in any logical bilattice, it is sufficient to check it in *FOUR*.

Theorem 3.4 Let Γ and Δ be finite sets of formulae in *BL* (in *BL*(4)). For every B, $\Gamma \models_{BL(B)} \Delta$ iff $\Gamma \models_{BL(FOUR)} \Delta$.

3.2 A Gentzen-type proof system

Since $\models_{BL}$ does not have valid formulae, it cannot have a Hilbert-type representation. However, there is a nice Gentzen-type formulation, which we shall call *GBL* (*GBL*(4)):

The system ***GBL***

Axioms:

$$\Gamma, \psi \Rightarrow \Delta, \psi$$

Rules:

Exchange, Contraction, and the following logical rules:

$$\frac{\Gamma, \psi, \phi \Rightarrow \Delta}{\Gamma, \psi \wedge \phi \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow \Delta, \psi \quad \Gamma \Rightarrow \Delta, \phi}{\Gamma \Rightarrow \Delta, \psi \wedge \phi}$$

$$\frac{\Gamma, \neg\psi \Rightarrow \Delta \quad \Gamma, \neg\phi \Rightarrow \Delta}{\Gamma, \neg(\psi \wedge \phi) \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow \Delta, \neg\psi, \neg\phi}{\Gamma \Rightarrow \Delta, \neg(\psi \wedge \phi)}$$

$$\frac{\Gamma, \psi \Rightarrow \Delta \quad \Gamma, \phi \Rightarrow \Delta}{\Gamma, \psi \vee \phi \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow \Delta, \psi, \phi}{\Gamma \Rightarrow \Delta, \psi \vee \phi}$$

$$\frac{\Gamma, \neg\psi, \neg\phi \Rightarrow \Delta}{\Gamma, \neg(\psi \vee \phi) \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow \Delta, \neg\psi \quad \Gamma \Rightarrow \Delta, \neg\phi}{\Gamma \Rightarrow \Delta, \neg(\psi \vee \phi)}$$

$$\frac{\Gamma, \psi, \phi \Rightarrow \Delta}{\Gamma, \psi \otimes \phi \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow \Delta, \psi \quad \Gamma \Rightarrow \Delta, \phi}{\Gamma \Rightarrow \Delta, \psi \otimes \phi}$$

$$\frac{\Gamma, \neg\psi, \neg\phi \Rightarrow \Delta}{\Gamma, \neg(\psi \otimes \phi) \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow \Delta, \neg\psi \quad \Gamma \Rightarrow \Delta, \neg\phi}{\Gamma \Rightarrow \Delta, \neg(\psi \otimes \phi)}$$

$$\frac{\Gamma, \psi \Rightarrow \Delta \quad \Gamma, \phi \Rightarrow \Delta}{\Gamma, \psi \oplus \phi \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow \Delta, \psi, \phi}{\Gamma \Rightarrow \Delta, \psi \oplus \phi}$$

$$\frac{\Gamma, \neg\psi \Rightarrow \Delta \quad \Gamma, \neg\phi \Rightarrow \Delta}{\Gamma, \neg(\psi \oplus \phi) \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow \Delta, \neg\psi, \neg\phi}{\Gamma \Rightarrow \Delta, \neg(\psi \oplus \phi)}$$

$$\frac{\Gamma, \psi \Rightarrow \Delta}{\Gamma, \neg\neg\psi \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \neg\neg\psi}$$

Note: The positive rules for $\wedge$ and $\otimes$ are identical. Both behave as classical conjunction. The difference is with respect to the negations of $p \wedge q$ and $p \otimes q$. Unlike the conjunction of classical logic, the negation of $p \otimes q$ is equivalent to $\neg p \otimes \neg q$. This follows from the fact that $p \leq_k q$ iff $\neg p \leq_k \neg q$. The difference between $\vee$ and $\oplus$ is similar.

Definition 3.5 Δ follows from Γ (notation: $\Gamma \vdash_{GBL} \Delta$) if $\Gamma \Rightarrow \Delta$ is provable in *GBL*.

Theorem 3.6 (*Soundness and Completeness*) $\Gamma \models_{BL} \Delta$ iff $\Gamma \vdash_{GBL} \Delta$.

Theorem 3.7 (*Cut Elimination*) If $\Gamma_1 \vdash_{GBL} \Delta_1, \psi$ and $\Gamma_2, \psi \vdash_{GBL} \Delta_2$, then $\Gamma_1, \Gamma_2 \vdash_{GBL} \Delta_1, \Delta_2$.

Outline of Proofs: The two theorems are proved together by showing, using induction on compexity of sequents and the fact that all the rules are reversible, that every sequent has either a cut free proof or a counter-model. □

Theorem 3.8 (*Monotonicity and Compactness*) Let Γ, Δ be arbitrary sets of formulae in *BL* (not necessarily finite). Then $\Gamma \models_{BL} \Delta$ iff there exist finite sets Γ', Δ' such that $\Gamma' \subseteq \Gamma$, $\Delta' \subseteq \Delta$ and $\Gamma' \models_{BL} \Delta'$ (iff $\Gamma' \vdash_{GBL} \Delta'$). The same is true for $\models_{BL(4)}$.

Outline of Proof: Suppose that Γ,Δ are sets for which no such Γ',Δ' exist. Extend the pair (Γ, Δ) to a maximal pair (Γ^*, Δ^*) with the same property. Using Γ^* and Δ^* construct a refuting ν in *FOUR* in a way which is similar to the construction of h in the proof of 2.11. □

Notes:

1. The $\{\wedge, \vee, \neg\}$-fragment was called "the basic $\{\wedge, \vee, \neg\}$-system" in [Avr1], and was introduced there following a different motivation. It had generally been known as the system of "first degree entailments" in relevance logic (see [AnBe], [Dunn]).
2. In [Avr1] it is shown that if we add $\Gamma, \neg\psi, \psi \Rightarrow \Delta$ as an axiom to the $\{\wedge, \vee, \neg\}$ (or $\{\wedge, \vee, \neg, f, t\}$)-fragment of *GBL*, we get a sound and complete system for Kleene 3-valued logic, while if we add $\Gamma \Rightarrow \Delta, \psi, \neg\psi$ we get one of the basic three-valued paraconsistent logics [3]. By adding both we get classical logic.

3.3 Implication connectives

3.3.1 Weak implication

The language *BL*, rich as it is, lacks an appropriate general implication connective (relative to $\models_{BL}$). Defining $\psi \rightarrow \phi$ as $\neg\psi \vee \phi$ is not adequate, since both modus ponens and the deduction theorem fail for this connective. Instead we follow [Avr1] by looking for

[3] Also known as J_3 - see, e.g., chapter IX of [Epst] as well as [OtdC],[Otta],[Avr3],[Rozo].

an internal implication $\supset$ that satisfies what is called there the *symmetry conditions* for implication. Such an implication can be defined in every logical bilattice $(\mathcal{B}, \mathcal{F})$ as follows: [4]

$$a \supset b \stackrel{\text{def}}{=} \begin{cases} b & \text{if } a \in \mathcal{F} \\ t & \text{if } a \notin \mathcal{F} \end{cases}$$

We enrich now the languages BL, $BL(4)$, and $BL(B)$), with the connective $\supset$. The various consequence relations are extended accordingly. The following facts hold:

Proposition 3.9 We still have that $\models_{BL(4)} = \models_{BL(FOUR)} = \models_{BL(B)}$.

Proof: Similar to that of theorem 2.11. $\Box$

Proposition 3.10 Both modus ponens and the deduction theorem are valid for $\supset$ under $\models_{BL}$ ($\models_{BL(4)}$, etc).

Theorem 3.11 Extend the systems above with the following rules:

$$\frac{\Gamma \Rightarrow \psi, \Delta \quad \Gamma, \phi \Rightarrow \Delta}{\Gamma, \psi \supset \phi \Rightarrow \Delta} \qquad \frac{\Gamma, \psi \Rightarrow \phi, \Delta}{\Gamma \Rightarrow \psi \supset \phi, \Delta}$$

$$\frac{\Gamma, \psi, \neg\phi \Rightarrow \Delta}{\Gamma, \neg(\psi \supset \phi) \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow \psi, \Delta \quad \Gamma \Rightarrow \neg\phi, \Delta}{\Gamma \Rightarrow \neg(\psi \supset \phi), \Delta}$$

The soundness, completeness, and cut elimination theorems hold for the extended systems as well.

Proof: Similar to that of theorems 3.6 and 3.7. $\Box$

Unlike the previous case, once we have $\supset$, the language does have valid sentences, hence it is possible to give a Hilbert-type axiomatization, which we will denote by *HBL*. *HBL* can be obtained from what was called in [Avr1] "the basic Hilbert-type system" by adding as axioms the counterparts of the rules for $\otimes$ and $\oplus$: [5]

The system *HBL*

Defined connective:

$$\psi \equiv \phi \stackrel{\text{def}}{=} (\psi \supset \phi) \wedge (\phi \supset \psi)$$

Inference rule:

$$\frac{\psi \quad \psi \supset \phi}{\phi}$$

Axioms:

$$\psi \supset \phi \supset \psi$$

$$(\psi \supset \phi \supset \varphi) \supset (\psi \supset \phi) \supset (\psi \supset \varphi)$$

$$((\psi \supset \phi) \supset \psi) \supset \psi$$

$$\psi \wedge \phi \supset \psi \qquad \psi \wedge \phi \supset \phi$$

$$\psi \supset \phi \supset \psi \wedge \phi$$

$$\psi \otimes \phi \supset \psi \qquad \psi \otimes \phi \supset \phi$$

$$\psi \supset \phi \supset \phi \otimes \phi$$

$$\psi \supset \psi \vee \phi \qquad \phi \supset \psi \vee \phi$$

$$(\psi \supset \varphi) \supset (\phi \supset \varphi) \supset (\psi \vee \phi \supset \varphi)$$

$$\psi \supset \psi \oplus \phi \qquad \phi \supset \psi \oplus \phi$$

$$(\psi \supset \varphi) \supset (\phi \supset \varphi) \supset (\psi \oplus \phi \supset \varphi)$$

$$\neg(\psi \wedge \phi) \equiv \neg\psi \vee \neg\phi$$

$$\neg(\psi \vee \phi) \equiv \neg\psi \wedge \neg\phi$$

$$\neg(\psi \otimes \phi) \equiv \neg\psi \otimes \neg\phi$$

$$\neg(\psi \oplus \phi) \equiv \neg\psi \oplus \neg\phi$$

$$\neg(\psi \supset \phi) \equiv \psi \wedge \neg\phi$$

$$\neg\neg\psi \equiv \psi$$

Note that the $\{\wedge, \vee, \supset\}$-fragment of these systems is identical to the classical one. The critical connective is, therefore, negation.

3.3.2 Strong implication

The implication connective $\supset$ has two drawbacks: the main one is that even in case that $\psi \supset \phi$ and $\phi \supset \psi$ are both valid, ψ and ϕ might not be equivalent (in the sense that one can be substituted for the other in any context). For example, if $\psi = \neg(\varphi \supset \rho)$ and $\phi = \varphi \wedge \neg\rho$, then both $\psi \supset \phi$ and $\phi \supset \psi$ are valid, but $\neg\psi \supset \neg\phi$ is not. The second disadvantage is that $\psi \supset \phi$ may be true, its conclusion false, without this entailing that the premise is false (for example: $\bot \supset f = t$). As is always the case when we have an internal implication which satisfies the symmetry conditions, we can introduce however a stronger implication which does not have these disadvantages (see [Avr1]):

Definition 3.12 (strong implication)
$\psi \rightarrow \phi \stackrel{\text{def}}{=} (\psi \supset \phi) \wedge (\neg\phi \supset \neg\psi)$

[4] It is not difficult to show that in *FOUR* this is the only possible definition.

[5] In the formulae below the associations of nested implication should be taken to the right.

The connective $\rightarrow$ has a lot of similarities with Girard's linear implication (see [Gira]). All the basic axioms concerning that implication (see [Avr2]) are valid for $\rightarrow$, while the contraction axiom and the weakening axiom are not. On the other hand, on $\{t, f, \bot\}$, $\rightarrow$ is exactly Lukasiewicz implication ([Luka],[Urqu]), while on $\{t, f, \top\}$ it is Sobocinski's implication ([Sobo]), which is the implication of RM_3 - the strongest logic in the family of relevance logics (see also [AnBe], [Dunn], and [Hind]).

Notes:
1. Using $\rightarrow$ we can sometimes translate "annotated atomic formulae" from Subrahmanian's annotated logic (see [CHLS],[Sub1],[Sub2],[KiLo],[KiSu]). Thus, the translation to $BL(4)$ of $\psi : b$ when $b \in FOUR$, and the partial order is $\leq_t$, is simply $b \rightarrow \psi$.
2. In $FOUR$, $\nu(\psi \rightarrow \phi) \in \mathcal{D}_k(FOUR)$ iff $\nu(\psi) \leq_t \nu(\phi)$. Moreover, $\psi \supset \phi$ is equivalent to $\phi \vee (\psi \rightarrow \psi \rightarrow \phi)$.

The next example demonstrates the potential use of $\models_{BL}$ as well as of the various implication connectives. We shall use in it $\leadsto$ to denote the implication of the classical calculus (i.e: $\psi \leadsto \phi = \neg\psi \vee \phi$).

Example 3.13 Consider the following knowledge base:

$bird(tweety) \leadsto fly(tweety)$

$penguin(tweety) \supset bird(tweety)$

$penguin(tweety) \rightarrow \neg fly(tweety)$

$bird(tweety)$

Note that we are using different implication connectives according to the strength we attach to each entailment: Penguins *never* fly. This is a characteristic feature of penguins, and there are no exceptoins to that, hence we use the strongest implication ($\rightarrow$) in the third asertion in order to express this fact. The second asertion states that every penguin is a bird. Again, there are no exceptions to that fact. Still, penguins are not *typical* birds, thus they shouldn't inherit all the properties we expect birds to have. The use of a weaker implication ($\supset$) forces us, indeed, to infer that something is a bird whenever we know that it is a penguin, but it does not forces us to infer that it has every property of a bird. Finally, the first assertion states only a default feature of birds, hence we attach the weakest implication ($\leadsto$) to it. Indeed, since from ψ and $\psi \leadsto \phi$ we cannot infer ϕ (by $\models_{BL}$) without more information, the first assertion does not cause automatic inference of flying abilities just from the fact that something is a bird. It does give, however, strong connection between these two facts.

The above knowledge-base does not allow us to infer whether tweety is a penguin or not (as it should be), and if it can fly or not (which is less satisfactory; we shall return to it in the next section). However, if we add to the knowledge-base an exstra assumption, $penguin(tweety)$, we can infer $\neg fly(tweety)$ but we still can *not* infer $fly(tweety)$, as should be expected.

4 A more subtle consequence relation

$\models_{BL}$ should be taken as a first approximation of what can be safely inferred when we have a classically inconsistent knowledge-base; this safety is its main advantage. The disadvantage is that $\models_{BL}$ is somewhat "over cautious". Thus, in the last example we would have liked to be able to infer $fly(tweety)$ from the original knowledge-base, before the new information is added to it. We can't, of course, since $\models_{BL}$ is monotonic [6].

To overcome this difficulty we adapt an idea of Kifer and Lozinskii (see [KiLo]). Their idea, basically, is to order models of a given knowledge-base in a way that somehow reflects their degree of consistency, and then take into account only the models which are maximal w.r.t this order. The main difference is that they were using just ordinary (semi)lattices, in which the partial order relation corresponds, intuitively, to our $\leq_k$. Hence, no direct interpretation of the standard logical connectives ($\wedge, \vee$) was available to them. They were forced, therefore, to use an unnatural language, in which the *atomic* formulae are of the form $p : b$ (where p is an atomic formula of the basic language, and b – a value from the semilattice). $\psi : b$ is meaningless, however, for nonatomic ψ. The use of bilattices allows us to give the standard logical language a direct interpretation, and so gives a meaning to every annotated formula. On the other hand, by using $\mathcal{F}$ we can dispense with annotated formulae altogether, as we do below [7].

Definition 4.1 Let $\mathcal{B} = (\mathrm{B}, \leq_t, \leq_k, \neg)$ be a bilattice. A subset $\mathcal{I}$ of B is called an *inconsistency set*, if it has the following properties:

[6] Another disadvantage is, perhaps, that $\models_{BL}$ is basically just the logic of $FOUR$.

[7] Despite the fact that this method of using "annotated" atomic formulae is quite common, it is still artificial from a logical point of view, since semantic notions interfere within the syntax.

a) $b \in \mathcal{I}$ iff $\neg b \in \mathcal{I}$.
b) $b \in \mathcal{F} \cap \mathcal{I}$ iff $b \in \mathcal{F}$ and $\neg b \in \mathcal{F}$.

Notes:

1. From (b), always $\top \in \mathcal{I}$. Also, from (b), $t \notin \mathcal{I}$, and so, from (a), $f \notin \mathcal{I}$.

2. As for $\bot$, both $\mathcal{I} \cup \{\bot\}$ and $\mathcal{I} \setminus \{\bot\}$ are inconsistency sets in case $\mathcal{I}$ is. On one hand, in every bilattice, $\neg\bot = \bot$, so $\bot$ has some features that may be associated with inconsistent elements. Now, on the other hand, $\bot$ intuitively reflects no knowledge at all about the assertions it represents; in particular, one might not take such assertions to be inconsistent.

Example 4.2
a) $\mathcal{I}_1 = \{b \mid b \in \mathcal{F} \wedge \neg b \in \mathcal{F}\}$
b) $\mathcal{I}_2 = \{b \mid b = \neg b\}$
c) $\mathcal{I}_3 = \{b \mid b = \neg b\} \setminus \{\bot\}$
$\mathcal{I}_1$ is the minimal possible inconsistency set in every logical bilattice. In case that $\mathcal{B}$ is interlaced, and $\mathcal{F} = \mathcal{D}_k(\mathcal{B})$, $\mathcal{I}_1$ is just $\{\top\}$. $\mathcal{I}_2$ and $\mathcal{I}_3$ are always inconsistency sets in case that $\mathcal{B}$ is interlaced, and $\mathcal{F} = \mathcal{D}_k(\mathcal{B})$. There are, however, other cases in which $\mathcal{I}_2$ and $\mathcal{I}_3$ are inconsistency sets; for example, in *DEFAULT*.

We fix henceforth some logical bilattice $B = (\mathcal{B}, \mathcal{F})$, and an inconsistency subset $\mathcal{I}$ of B. All the definitions below will be relative to B and $\mathcal{I}$. $\mathcal{A}(\Gamma)$ will denote the set of the atomic formulae that appear in some formula of Γ.

Definition 4.3 Let Γ and Δ be two sets of formulae, M, N – models of Γ.
a) M is *more consistent* than N ($N <_{con} M$), if the set of the atomic formulae in $\mathcal{A}(\Gamma)$ that are assigned under M values from $\mathcal{I}$ is properly contained in the corresponding set of N.
b) M is a *most consistent* model of Γ (mcm), if there is no other model of Γ which is more consistent than M.
c) $\Gamma \models_{con} \Delta$ if every mcm of Γ is a model of some formula of Δ.

Example 4.4 Let's return to the knowledge-base KB of example 3.13. This knowledge-base has exactly one mcm, which takes values in $\{t, f\}$. Hence, $KB \models_{con} \psi$, iff ψ follows classically from KB. So, *unlike the case of* $\models_{BL}$:
$KB \models_{con} fly(tweety)$, $KB \models_{con} \neg penguin(tweety)$, $KB \not\models_{con} \neg fly(tweety)$, $KB \not\models_{con} penguin(tweety)$.
Now, consider again what happens when we add "$penguin(tweety)$" to KB: the new knowledge-base, KB' has two mcms, M_1 and M_2, where:

$$M_1(bird(tweety)) = t,$$
$$M_1(penguin(tweety)) = \top,$$
$$M_1(fly(tweety)) = \top,$$

and

$$M_2(bird(tweety)) = \top,$$
$$M_2(penguin(tweety)) = t,$$
$$M_2(fly(tweety)) = f.$$

This time, therefore:
$KB' \models_{con} penguin(tweety)$, $KB' \models_{con} \neg fly(tweety)$, $KB' \not\models_{con} \neg penguin(tweety)$, $KB' \not\models_{con} fly(tweety)$.
It follows that $\models_{con}$ is a nonmonotonic consequence relation, which seems to behave according to our expectations.

Some important properties of $\models_{con}$ are summerized below. We formulate them for *BL*, but with the exception of propositions 4.8 and 4.10, they are true for all the other languages as well.

Proposition 4.5 If $\Gamma \models_{BL} \Delta$, then $\Gamma \models_{con} \Delta$.

Proposition 4.6 $\models_{con}$ is non-monotonic.

Proof: Consider, e.g., $\Gamma = \{p, \neg p \vee q\}$. $\Gamma \models_{con} q$, but $\Gamma, \neg p \not\models_{con} q$. □

Proposition 4.7 $\models_{con}$ is paraconsistent:
$p, \neg p \not\models_{con} q$

Proof: Consider a valuation that assigns p the value $\top$, and assigns q the value f. □

Proposition 4.8 If Γ is classically consistent set in the basic language (i.e, without $\supset$), and ψ is a clause that does not contain any pair of an atomic formula and its negation, then $\Gamma \models_{con} \psi$ iff ψ follows classically from Γ (Hence the difference here between $\models_{con}$ and classical logic is only with respect to inconsistent theories).

Definition 4.9 [Lehm]: A *plausibility logic* is a logic that satisfies the following conditions:

Inclusion:
$\Gamma, \psi \Rightarrow \psi$.

Right Monotonicity:
If $\Gamma \Rightarrow \Delta$, then $\Gamma \Rightarrow \psi, \Delta$.

Cautious Left Monotonicity:
If $\Gamma \Rightarrow \psi$ and $\Gamma \Rightarrow \Delta$, then $\Gamma, \psi \Rightarrow \Delta$ [8].

[8] This rule was first proposed in [Gabb].

Cautious Cut:
If $\Gamma, \psi_1, \ldots, \psi_n \Rightarrow \Delta$ and $\Gamma \Rightarrow \psi_i, \Delta$ for $i = 1 \ldots n$, then $\Gamma \Rightarrow \Delta$.

Proposition 4.10 If $\mathcal{B}$ is interlaced bilattice and $\bot \notin \mathcal{I}$, then $\models_{con}$, limited to the basic language, is a plausibility logic.

Outline of Proof: We summarize the proof of the most difficult case – that of Cautious Cut. Well, given an mcm M of Γ, we construct another model M' of Γ so that $M'(p) = M(p)$ if $p \in \Gamma$, and $\bot$ otherwise. We then show that M' is a model of some formula in Δ. But $M(p) \geq_k M'(p)$ for every p. Since $\mathcal{B}$ is interlaced, this is true for *every* formula ψ. Hence M is also a model of some formula in Δ. □

Proposition 4.11 All the rules of *GBL* are valid for $\models_{con}$.

Thus, $\models_{con}$ has a lot of desirable properties. We should mention, however, one disadvantage: $\models_{con}$ is *not* closed under substitutions. In other words: it is sensitive to the choice of the atomic formulae. Thus, although $\neg p, p \vee q \models_{con} q$ when p and q are *atomic*, it is not true in general (take, e.g., $p = \neg(\neg r \wedge r)$). This, however, is unavoidable when one wants to achieve both propositions 4.7 and 4.8 above.

5 Conclusion

Bilattices have had an extensive use in several areas, most notably in logic programming, but their role so far was almost algebraic in nature. We develop a real notion of logic based on bilattices, giving two associated consequence relations and coresponding proof systems. These consequence relations are strongly related to non-monotonic reasoning, and especially to reasoning in the presence off inconsistent data.

The next natural step is to investigate how the resulting logics are affected by the choice of the bilattice under consideration, the truth values that are taken to be designated, and the choice of the inconsistency subsets.

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