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Peter A. Loeb

*The American Mathematical Monthly*, Vol. 98, No. 3. (Mar., 1991), pp. 242-244.

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*The American Mathematical Monthly* is currently published by Mathematical Association of America.

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expansion of  $\pi \cot \pi \zeta$  in partial fractions (see [5, p. 113], [2, p. 207]). As for (6),

$$f(\theta) = -(\theta + \pi), \quad \text{for } -\pi < \theta < 0,$$

(5) gives the corresponding expansion for  $\pi \operatorname{cosec} \pi \zeta$  (see [5, p. 113]).

As a second example, we let  $0 \leq \lambda < \mu \leq \pi$ ,

$$f(\theta) = 1, \quad \lambda < \theta < \mu,$$

$f(\theta)$  be 0 otherwise in  $(0, 2\pi)$ , and  $f$  have period  $2\pi$ . Then we find, by (4) and (5), that

$$\frac{e^{i\mu\zeta} - e^{i\lambda\zeta}}{\zeta[e^{2\pi i\zeta} - 1]} = \frac{\mu - \lambda}{2\pi\zeta} + \sum'_{n=-\infty}^{\infty} \frac{1}{2\pi in} [e^{-in\lambda} - e^{-in\mu}] \frac{1}{\zeta + n},$$

$$\frac{e^{i\mu\zeta} - e^{i\lambda\zeta}}{2\pi i\zeta \sin \pi\zeta} = \frac{\mu - \lambda}{2\pi\zeta} + \sum'_{n=-\infty}^{\infty} \frac{(-1)^n}{2\pi in} [e^{-in\lambda} - e^{-in\mu}] \frac{1}{\zeta + n}.$$

Clearly we can do an unlimited number of examples of this type. Both (4) and (5) can be subsumed under a more general formula if we integrate over  $[\alpha, \alpha + 2\pi]$ .

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## A Note on Dixon's Proof of Cauchy's Integral Theorem

PETER A. LOEB\*

*Department of Mathematics, University of Illinois, Urbana, IL 61801*

In this note, we establish an elementary result in complex function theory and apply that result to simplify John Dixon's elegant proof [2] (also see [1] and [3]) of the general Cauchy integral theorem and formula. Dixon's proof uses local Cauchy theory. It is based on the fact that if  $f$  is analytic on a region  $G$  and  $\gamma$  is a closed rectifiable curve in  $G$ , then the integral

$$\int_{\gamma} \frac{f(w) - f(z)}{w - z} dw$$

(with the integrand replaced by  $f'(w)$  when  $z = w$ ) is an analytic function of  $z$  on all of  $G$ . A simple proof of this fact is given here, and we outline its application in Dixon's proof of Cauchy's integral formula and theorem.

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\*The author's work was supported in part by a grant from the U.S. National Science Foundation (DMS 87-02064).

Throughout this note,  $G$  will be an open set in the complex plane  $\mathbb{C}$ , and  $\gamma$  will be a closed rectifiable curve in  $G$ . We write  $f \in H(G)$  if  $f$  is holomorphic, i.e., analytic, on  $G$ . We use the notation  $D(z, r)$  for the disk  $\{w \in \mathbb{C}: |w - z| < r\}$ ; we write  $C(z, r)$  for the circle  $\{w \in \mathbb{C}: |w - z| = r\}$ . The trace of the curve  $\gamma$  in the complex plane is denoted by  $\{\gamma\}$ . We begin with our elementary result; it is an application of the Cauchy integral theorem for disks. The author is indebted to Professor John B. Conway for a simplification of its proof.

**PROPOSITION 1.** *Given  $z \in \{\gamma\}$ , there is a closed curve  $\sigma$  in  $G$  with  $z \notin \{\sigma\}$  such that  $\int_\gamma f = \int_\sigma f$  for all  $f \in H(G)$ .*

*Proof.* We assume that there is a point  $\zeta \neq z$  with  $\zeta \in \{\gamma\}$ ; otherwise the result is trivial. Pick  $r > 0$  so that  $D(z, 2r) \subset G$  and  $\zeta \notin D(z, r)$ . We will assume that  $\gamma$  is given by  $\gamma(t)$  for  $t \in [0, 1]$  and  $\gamma(0) = \gamma(1) = \zeta$ . By the uniform continuity of the mapping  $\gamma$ ,  $\exists \delta > 0$  such that if  $s, t \in [0, 1]$  and  $|t - s| < \delta$ , then  $|\gamma(t) - \gamma(s)| < r$ . Let  $U$  be the inverse image under  $\gamma$  of the disk  $D(z, r)$ . Each component of  $U$  is an open interval in  $(0, 1)$ . If  $(a, b)$  is a component of  $U$ , then by continuity,  $\gamma(a)$  and  $\gamma(b)$  are points on  $C(z, r)$ . Moreover, if  $(a, b)$  contains a point  $t$  such that  $\gamma(t) = z$ , then  $b - a \geq 2\delta$  since  $|\gamma(b) - \gamma(t)| = r$  and  $|\gamma(t) - \gamma(a)| = r$ . Thus, there are only a finite number of components  $(a_i, b_i), 1 \leq i \leq n$ , of  $U$  containing points  $t$  with  $\gamma(t) = z$ . We replace the curve  $\gamma$  on the closure  $[a_i, b_i]$  of each such component with the arc on the circle  $C(z, r)$  going from  $\gamma(a_i)$  to  $\gamma(b_i)$  in the positive direction. By Cauchy's integral theorem, applied to the disk  $D(z, 2r)$ , this replacement does not change the value of the integral for any  $f \in H(G)$ . The new path  $\sigma$  avoids  $z$ .  $\square$

The next result, Proposition 2, contains the application of Proposition 1 to Dixon's proof: Unlike published proofs of this result, the proof here does not need the joint continuity of the function  $\varphi$  defined below nor does it employ Fubini's theorem.

**PROPOSITION 2.** *Let  $f$  be analytic on  $G$ , and let  $\varphi$  be the mapping of  $G \times G$  into  $\mathbb{C}$  defined by setting  $\varphi(w, z) = (f(w) - f(z))/(w - z)$  for  $w \neq z$  and  $\varphi(w, w) = f'(w)$ . Then for each  $z \in G, \varphi(\cdot, z)$  is analytic on  $G$ . Moreover, the function  $g$  defined by setting  $g(z) = \int_\gamma \varphi(w, z) dw$  is analytic on  $G$ .*

*Proof.* Given  $z \in G$ , the function  $\varphi(\cdot, z)$  is analytic on  $G - \{z\}$  and continuous at  $z$ . From local Cauchy theory (e.g., [3], Theorem 10.14), it follows that  $\varphi(\cdot, z) \in H(G)$ . If  $z \notin \{\gamma\}$ , then

$$g(z) = \int_\gamma \varphi(w, z) dw = \int_\gamma \frac{f(w) - f(z)}{w - z} dw = \int_\gamma \frac{f(w)}{w - z} dw - f(z) \cdot \int_\gamma \frac{1}{w - z} dw.$$

It is easy to see that  $g$  is analytic on  $G - \{z\}$ . Now fix a point  $z \in \{\gamma\}$ . By Proposition 1, we can replace  $\gamma$  with another curve  $\sigma$  that misses  $z$ , and therefore an open disk  $D(z, \varepsilon)$  about  $z$ , without changing the values of  $g$  on  $D(z, \varepsilon)$ . It follows that  $g$  is analytic on  $D(z, \varepsilon)$ , and thus on  $G$ .  $\square$

To outline Dixon's proof of Cauchy's integral theorem and formula, we need to recall that the index  $n(\gamma, a)$  of  $\gamma$  at any complex number  $a \notin \{\gamma\}$  is given by  $n(\gamma, a) = (1/2\pi i) \int_\gamma 1/(w - a) dw$ . The index is an integer valued function that is constant on the components of the complement of  $\{\gamma\}$  and 0 on the unbounded component. A cycle  $\Gamma$  is a finite set of closed, rectifiable curves  $\{\gamma_i: 1 \leq i \leq n\}$ . The trace of  $\Gamma, \{\Gamma\}$ , is  $\cup_i \{\gamma_i\}$ . For any function  $f$  continuous on  $\{\Gamma\}$ , the integral

$\int_{\Gamma} f = \sum_i \int_{\gamma_i} f$ . Moreover, for  $a \notin \{\Gamma\}$ ,  $n(\Gamma, a) = \sum_i n(\gamma_i, a)$ . We say that  $\Gamma$  is a cycle in an open set  $G$  if  $\{\Gamma\} \subset G$ .

**THEOREM.** *Let  $G$  be an open subset of  $\mathbb{C}$ , and let  $\Gamma$  be a cycle in  $G$  such that  $n(\Gamma, a) = 0$  for each point  $a \notin G$ . Then we have Cauchy's Integral Formula*

$$\forall f \in H(G), \quad \forall z \in G - \{\Gamma\}, \quad f(z) \cdot n(\Gamma, z) = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(w)}{w - z} dw,$$

and Cauchy's Integral Theorem

$$\forall f \in H(G), \quad \int_{\Gamma} f(w) dw = 0.$$

*Proof.* For the Integral Formula, let  $f \in H(G)$ , and let  $\varphi$  and  $g$  be the mappings of Proposition 2. Let  $H = \{z \in \mathbb{C} - \{\Gamma\}: n(\Gamma, z) = 0\}$ . Then  $H$  is an open subset of  $\mathbb{C}$  such that  $G \cup H = \mathbb{C}$ . Let  $h$  be the analytic function defined on  $H$  by setting

$$h(z) = \int_{\Gamma} \frac{f(w)}{w - z} dw \quad \forall z \in H.$$

At all points  $z \in G \cap H$ , we have

$$g(z) = h(z) - \int_{\Gamma} \frac{f(z)}{w - z} dw = h(z) - f(z) \cdot 2\pi i \cdot n(\Gamma, z) = h(z).$$

Therefore, we may extend the function  $g$  to an entire function by setting  $g(z) = h(z)$  for  $z \notin G$ . Since  $\lim_{z \rightarrow \infty} g(z) = 0$ ,  $g \equiv 0$  on  $\mathbb{C}$  by Liouville's theorem. It follows that for all points  $z \in G - \{\Gamma\}$ ,

$$f(z) \cdot n(\Gamma, z) = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(z)}{w - z} dw = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(w)}{w - z} dw.$$

To establish Cauchy's Integral Theorem, we fix  $a \in G - \{\Gamma\}$  and note that

$$\int_{\Gamma} f(w) dw = \int_{\Gamma} \frac{f(w) \cdot (w - a)}{w - a} dw = 2\pi i \cdot n(\Gamma, a) \cdot f(a) \cdot (a - a) = 0. \quad \square$$

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