expansion of $\pi \cot \pi\zeta$ in partial fractions (see [5, p. 113], [2, p. 207]). As for (6),

$$f(\theta) = -(\theta + \pi), \quad \text{for } -\pi < \theta < 0,$$

(5) gives the corresponding expansion for $\pi \operatorname{cosec} \pi\zeta$ (see [5, p. 113]).

As a second example, we let $0 \leqslant \lambda < \mu \leqslant \pi$,

$$f(\theta) = 1, \qquad \lambda < \theta < \mu,$$

$f(\theta)$ be 0 otherwise in $(0, 2\pi)$, and $f$ have period $2\pi$. Then we find, by (4) and (5), that

$$\frac{e^{i\mu\zeta} - e^{i\lambda\zeta}}{\zeta[e^{2\pi i\zeta} - 1]} = \frac{\mu - \lambda}{2\pi\zeta} + \sum_{n=-\infty}^{\infty}{}' \frac{1}{2\pi i n}[e^{-in\lambda} - e^{-in\mu}]\frac{1}{\zeta + n},$$

$$\frac{e^{i\mu\zeta} - e^{i\lambda\zeta}}{2\pi i\zeta \sin \pi\zeta} = \frac{\mu - \lambda}{2\pi\zeta} + \sum_{n=-\infty}^{\infty}{}' \frac{(-1)^n}{2\pi i n}[e^{-in\lambda} - e^{-in\mu}]\frac{1}{\zeta + n}.$$

Clearly we can do an unlimited number of examples of this type. Both (4) and (5) can be subsumed under a more general formula if we integrate over $[\alpha, \alpha + 2\pi]$.

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# A Note on Dixon's Proof of Cauchy's Integral Theorem

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In this note, we establish an elementary result in complex function theory and apply that result to simplify John Dixon's elegant proof [2] (also see [1] and [3]) of the general Cauchy integral theorem and formula. Dixon's proof uses local Cauchy theory. It is based on the fact that if $f$ is analytic on a region $G$ and $\gamma$ is a closed rectifiable curve in $G$, then the integral

$$\int_\gamma \frac{f(w) - f(z)}{w - z}\, dw$$

(with the integrand replaced by $f'(w)$ when $z = w$) is an analytic function of $z$ on all of $G$. A simple proof of this fact is given here, and we outline its application in Dixon's proof of Cauchy's integral formula and theorem.

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Throughout this note, $G$ will be an open set in the complex plane $\mathbb{C}$, and $\gamma$ will be a closed rectifiable curve in $G$. We write $f \in H(G)$ if $f$ is holomorphic, i.e., analytic, on $G$. We use the notation $D(z,r)$ for the disk $\{w \in \mathbb{C}: |w - z| < r\}$; we write $C(z,r)$ for the circle $\{w \in \mathbb{C}: |w - z| = r\}$. The trace of the curve $\gamma$ in the complex plane is denoted by $\{\gamma\}$. We begin with our elementary result; it is an application of the Cauchy integral theorem for disks. The author is indebted to Professor John B. Conway for a simplification of its proof.

PROPOSITION 1. *Given $z \in \{\gamma\}$, there is a closed curve $\sigma$ in $G$ with $z \notin \{\sigma\}$ such that $\int_\gamma f = \int_\sigma f$ for all $f \in H(G)$.*

*Proof.* We assume that there is a point $\zeta \neq z$ with $\zeta \in \{\gamma\}$; otherwise the result is trivial. Pick $r > 0$ so that $D(z,2r) \subset G$ and $\zeta \notin D(z,r)$. We will assume that $\gamma$ is given by $\gamma(t)$ for $t \in [0,1]$ and $\gamma(0) = \gamma(1) = \zeta$. By the uniform continuity of the mapping $\gamma$, $\exists \delta > 0$ such that if $s, t \in [0,1]$ and $|t - s| < \delta$, then $|\gamma(t) - \gamma(s)| < r$. Let $U$ be the inverse image under $\gamma$ of the disk $D(z,r)$. Each component of $U$ is an open interval in $(0,1)$. If $(a,b)$ is a component of $U$, then by continuity, $\gamma(a)$ and $\gamma(b)$ are points on $C(z,r)$. Moreover, if $(a,b)$ contains a point $t$ such that $\gamma(t) = z$, then $b - a \geqslant 2\delta$ since $|\gamma(b) - \gamma(t)| = r$ and $|\gamma(t) - \gamma(a)| = r$. Thus, there are only a finite number of components $(a_i, b_i)$, $1 \leqslant i \leqslant n$, of $U$ containing points $t$ with $\gamma(t) = z$. We replace the curve $\gamma$ on the closure $[a_i, b_i]$ of each such component with the arc on the circle $C(z,r)$ going from $\gamma(a_i)$ to $\gamma(b_i)$ in the positive direction. By Cauchy's integral theorem, applied to the disk $D(z,2r)$, this replacement does not change the value of the integral for any $f \in H(G)$. The new path $\sigma$ avoids $z$. $\square$

The next result, Proposition 2, contains the application of Proposition 1 to Dixon's proof: Unlike published proofs of this result, the proof here does not need the joint continuity of the function $\varphi$ defined below nor does it employ Fubini's theorem.

PROPOSITION 2. *Let $f$ be analytic on $G$, and let $\varphi$ be the mapping of $G \times G$ into $\mathbb{C}$ defined by setting $\varphi(w,z) = (f(w) - f(z))/(w - z)$ for $w \neq z$ and $\varphi(w,w) = f'(w)$. Then for each $z \in G$, $\varphi(\cdot, z)$ is analytic on $G$. Moreover, the function $g$ defined by setting $g(z) = \int_\gamma \varphi(w,z)\,dw$ is analytic on $G$.*

*Proof.* Given $z \in G$, the function $\varphi(\cdot, z)$ is analytic on $G - \{z\}$ and continuous at $z$. From local Cauchy theory (e.g., [3], Theorem 10.14), it follows that $\varphi(\cdot, z) \in H(G)$. If $z \notin \{\gamma\}$, then

$$g(z) = \int_\gamma \varphi(w,z)\,dw = \int_\gamma f(w)/(w - z)\,dw - f(z) \cdot \int_\gamma 1/(w - z)\,dw.$$

It is easy to see that $g$ is analytic on $G - \{\gamma\}$. Now fix a point $z \in \{\gamma\}$. By Proposition 1, we can replace $\gamma$ with another curve $\sigma$ that misses $z$, and therefore an open disk $D(z,\varepsilon)$ about $z$, without changing the values of $g$ on $D(z,\varepsilon)$. It follows that $g$ is analytic on $D(z,\varepsilon)$, and thus on $G$. $\square$

To outline Dixon's proof of Cauchy's integral theorem and formula, we need to recall that the index $n(\gamma, a)$ of $\gamma$ at any complex number $a \notin \{\gamma\}$ is given by $n(\gamma,a) = (1/2\pi i)\int_\gamma 1/(w - a)\,dw$. The index is an integer valued function that is constant on the components of the complement of $\{\gamma\}$ and 0 on the unbounded component. A cycle $\Gamma$ is a finite set of closed, rectifiable curves $\{\gamma_i: 1 \leqslant i \leqslant n\}$. The trace of $\Gamma$, $\{\Gamma\}$, is $\bigcup_i\{\gamma_i\}$. For any function $f$ continuous on $\{\Gamma\}$, the integral

$\int_\Gamma f = \Sigma_i \int_{\gamma_i} f$. Moreover, for $a \notin \{\Gamma\}$, $n(\Gamma, a) = \Sigma_i n(\gamma_i, a)$. We say that $\Gamma$ is a cycle in an open set $G$ if $\{\Gamma\} \subset G$.

THEOREM. *Let $G$ be an open subset of $\mathbb{C}$, and let $\Gamma$ be a cycle in $G$ such that $n(\Gamma, a) = 0$ for each point $a \notin G$. Then we have Cauchy's Integral Formula*

$$\forall f \in H(G), \qquad \forall z \in G - \{\Gamma\}, \qquad f(z) \cdot n(\Gamma, z) = \frac{1}{2\pi i} \int_\Gamma \frac{f(w)}{w - z}\, dw,$$

*and Cauchy's Integral Theorem*

$$\forall f \in H(G), \qquad \int_\Gamma f(w)\, dw = 0.$$

*Proof.* For the Integral Formula, let $f \in H(G)$, and let $\varphi$ and $g$ be the mappings of Proposition 2. Let $H = \{z \in \mathbb{C} - \{\Gamma\}: n(\Gamma, z) = 0\}$. Then $H$ is an open subset of $\mathbb{C}$ such that $G \cup H = \mathbb{C}$. Let $h$ be the analytic function defined on $H$ by setting

$$h(z) = \int_\Gamma \frac{f(w)}{w - z}\, dw \quad \forall z \in H.$$

At all points $z \in G \cap H$, we have

$$g(z) = h(z) - \int_\Gamma \frac{f(z)}{w - z}\, dw = h(z) - f(z) \cdot 2\pi i \cdot n(\Gamma, z) = h(z).$$

Therefore, we may extend the function $g$ to an entire function by setting $g(z) = h(z)$ for $z \notin G$. Since $\lim_{z \to \infty} g(z) = 0$, $g \equiv 0$ on $\mathbb{C}$ by Liouville's theorem. It follows that for all points $z \in G - \{\Gamma\}$,

$$f(z) \cdot n(\Gamma, z) = \frac{1}{2\pi i} \int_\Gamma \frac{f(z)}{w - z}\, dw = \frac{1}{2\pi i} \int_\Gamma \frac{f(w)}{w - z}\, dw.$$

To establish Cauchy's Integral Theorem, we fix $a \in G - \{\Gamma\}$ and note that

$$\int_\Gamma f(w)\, dw = \int_\Gamma \frac{f(w) \cdot (w - a)}{w - a}\, dw = 2\pi i \cdot n(\Gamma, a) \cdot f(a) \cdot (a - a) = 0. \quad \square$$

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