Locally symmetric spaces and K-theory of number fields

Abstract

We discuss which flat bundles over locally symmetric spaces give rise to nontrivial elements in the K-theory of number fields.

1 Introduction

Meanwhile elements in topological K-theory $K_*(X)$ are, by definition, represented by (virtual) vector bundles over the space X, it is less evident what the topological meaning of elements in algebraic K-theory $K_*(A)$, for a ring A, may be. An approach, which can be found e.g. in the appendix of [18], is to associate elements in $K_*(A)$ to any flat $GL(A)$-bundle over an aspherical homology sphere M. Namely, let $\rho : \pi_1 M \to GL(A)$ be the monodromy of the flat bundle, then $(B\rho)^+ : M^+ \to BGL^+(A)$ can, in view of $M^+ \simeq \mathbb{S}^n$, be considered as an element in algebraic K-theory $K_n(A) := \pi_n BGL^+(A)$. It was proved by Hausmann and Vogel (see [15]) that, for $n \geq 5$ or $n = 3$, all elements in $K_n(A)$ arise from such a construction.

If the aspherical manifold M is not a homology sphere, but still possesses a fundamental class $[M] \in H_n(M;\mathbb{Z})$, one can still consider $(B\rho)_n [M]_{\mathbb{Q}}$ as an element of $H_n(BGL(A);\mathbb{Q}) \cong K_n(A) \otimes \mathbb{Q}$.

An interesting special case, which has been studied by Dupont-Sah and others, is $K_3(\mathbb{C})$. By a theorem of Suslin, $K_3(\mathbb{C})$ is, up to 2-torsion, isomorphic to the Bloch group $B(\mathbb{C})$. On the other hand, each ideally triangulated hyperbolic manifold yields, in a very natural way, an element in $B(\mathbb{C})$, the Bloch invariant. By [22], this element does not depend on the chosen ideal triangulation.

A generalization to higher-dimensional hyperbolic manifolds was provided by Goncharov. He associated to an odd-dimensional hyperbolic manifold and flat bundles coming from the half-spinor representations, an element in $K_*\left(\overline{\mathbb{Q}}\right) \otimes \mathbb{Q}$, and proved its nontriviality by showing that application of the Borel regulator yields (a fixed multiple of) the volume.

It thus arises as a natural question, how much of algebraic K-theory in higher (odd) degrees can be represented by locally symmetric spaces and representations of their fundamental groups.

In section 2, we generalize the argument in [12] to the extent that, for a compact locally symmetric space $M^{2n+1} = \Gamma \backslash G/K$ of noncompact type and a representation $\rho : G \to GL(N,\mathbb{C})$, nontriviality of the associated elements in K-theory $K_{2n-1}(\mathbb{C}) \otimes \mathbb{Q}$ is (independently of Γ) equivalent to nontriviality of the Borel class $\rho^* b_{2n+1}$.

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Theorem. *For each symmetric space G/K of noncompact type and odd dimension n, and to each representation $\rho : G \to GL(N, \mathbb{C})$ with $\rho^* b_n \neq 0$, there exists a constant $c_\rho \neq 0$, such that the following holds: to each compact, orientable, locally symmetric space $M = \Gamma \backslash G/K$, with $\rho(\Gamma) \subset GL(N, \mathbb{C})$, there exists an element*

$$\gamma(M) \in K_n(\mathbb{C}) \otimes \mathbb{Q}$$

such that the Borel regulator $r_n : K_n(\mathbb{C}) \otimes \mathbb{Q} \to \mathbb{R}$ fulfills

$$r_n(\gamma(M)) = c_\rho vol(M).$$

If $\rho^* b_{2n+1} \neq 0$, then it follows that manifolds of $\mathbb{Q}$-independent volume give $\mathbb{Q}$-independent elements in $K_*(\mathbb{C}) \otimes \mathbb{Q}$.

The construction actually yields elements in $K_{2n+1}(\overline{\mathbb{Q}}) \otimes \mathbb{Q}$, due to the fact that compact locally symmetric spaces of finite volume are defined over $\overline{\mathbb{Q}}$. Accordingly locally symmetric spaces defined over some number field $F \subset \overline{\mathbb{Q}}$ will give elements in $K_{2n+1}(F) \otimes \mathbb{Q}$.

In section 3, we work out the list of representations $\rho : G \to GL(N, \mathbb{C})$ for which $\rho^* b_{2n+1} \neq 0$ holds true. (It is pretty clear from the definitions that $\rho^* b_{2n+1} \neq 0$ is always true if $2n+1 \equiv 3 \ mod \ 4$. We work out, for which representations $\rho^* b_{2n+1} \neq 0$ holds in the case $2n+1 \equiv 1 \ mod \ 4$.) The proof uses only standard Lie algebra and representation theory. The result reads as follows.

Theorem. *The following is a complete list of non-exceptional symmetric spaces G/K and irreducible representations $\rho : G \to GL(N, \mathbb{C})$ with $\rho^* b_n \neq 0$ for $n := dim(G/K)$.*
- $SL_l(\mathbb{R})/SO_l, l \equiv 0, 3, 4, 7 \ mod \ 8$, any irreducible representation,
- $SL_l(\mathbb{C})/SU_l, l \equiv 0 \ mod \ 2$, any irreducible representation,
- $SL_{2l}(\mathbb{H})/Sp_l, l \equiv 0 \ mod \ 2$, any irreducible representation,
- $Spin_{p,q}/(Spin_p \times Spin_q), p, q \equiv 1 \ mod \ 2, p \not\equiv q \ mod \ 4$, any irreducible representation,
- $Spin_{p,q}/(Spin_p \times Spin_q), p, q \equiv 1 \ mod \ 2, p \equiv q \ mod \ 4$, tensor products of copies of positive and/or negative half-spinor representations,
- $Spin_l(\mathbb{C})/Spin_l, l \equiv 3 \ mod \ 4$, tensor products of copies of the spinor representation and/or its conjugate,
- $Spin_l(\mathbb{C})/Spin_l, l \equiv 2 \ mod \ 4$, any irreducible representation,
- $Sp_l(\mathbb{C})/Sp_l, l \equiv 3 \ mod \ 4$, any irreducible representation.

The only exceptional symmetric space which may possess representations with $\rho^* b_n \neq 0$ is $E_7(\mathbb{C})/E_7$. In this case we have checked that indeed Adams' representation ([1], section 8) satisfies $\rho^* b_n \neq 0$.

In section 4 we show that also (not necessarily compact) odd-dimensional locally symmetric spaces M of finite volume give rise to classes $\gamma(M) \in K_*(\mathbb{C}) \otimes \mathbb{Q}$. (This is not the case for arbitrary manifolds with boundary.) Unfortunately, the proof of $\gamma(M) \neq 0$ for noncompact finite-volume manifolds is not as natural as in the compact case, but requires more elaborate topological arguments.

Section 5 discusses a few examples.

2 Preparations

The results of this section are fairly straightforward generalizations of the results in [12] from hyperbolic manifolds to locally symmetric spaces of noncompact type. We will define a notion of representations with nontrivial Borel class and will, mimicking the arguments in [12], show that representations with nontrivial Borel class give rise to nontrivial elements in algebraic K-theory of number fields. The problem of constructing representations with nontrivial Borel class will be tackled in the section 3.

2.1 Construction of elements in algebraic K-theory

In this paper, rings A will always be commutative rings with unit. (Mainly we are interested in subrings of $\mathbb{C}$.)

Assume that M is a closed, orientable n-manifold with $\Gamma := \pi_1 M$. Assume that we are given a ring A and a representation $\rho : \Gamma \to GL(A)$, where $GL(A)$ denotes the increasing union of $GL(N, A)$ over all $N \in \mathbb{N}$.

Assume that M is aspherical, that is, $M \simeq B\Gamma$, then we get an induced map

$$B\rho : M \simeq B\Gamma \to BGL(A).$$

Throughout this paper $BGL(A)$ resp. $BSL(A)$ will mean the classifying space for the group with the discrete topology. Thus $\pi_1 BSL(A) = SL(A)$.

Quillen's plus construction (see [24]) provides us with a map

$$(B\rho)^+ : M^+ \to BGL^+(A).$$

If M happens to be a homology sphere, then M^+ is homotopy equivalent to $\mathbb{S}^n$ and one gets a map

$$\mathbb{S}^n \simeq M^+ \to BGL^+(A)$$

which may be considered as representative of an element

$$\left[(B\rho)^+\right] \in K_n(A) := \pi_n(BGL(A)).$$

It was actually shown by Hausmann and Vogel (cf. [15] or [14]) that, for $n \geq 5$ or $n = 3$, each element in $K_n(A)$ for a finitely generated commutative ring A can be constructed by some homology sphere M and some representation ρ.

If M is not a homology sphere, but closed, orientable and aspherical, then we will not construct an element in $K_n(A)$ but rather in $K_n(A) \otimes \mathbb{Q}$, as follows. We do get a map

$$(B\rho)_* : H_n(M; \mathbb{Q}) \to H_n(BGL(A); \mathbb{Q}).$$

Since M is a closed, orientable n-manifold, we have a fundamental class in $H_n(M; \mathbb{Q})$, which is the image of a generator of $H_n(M; \mathbb{Z})$ under the change-of-rings homomorphism associated to the inclusion $\mathbb{Z} \to \mathbb{Q}$. We may consider the image of the fundamental class $[M] \in H_n(M; \mathbb{Q})$

$$(B\rho)_* [M] \in H_n(BGL(A), \mathbb{Q}) \cong H_n\left(BGL(A)^+, \mathbb{Q}\right).$$

By the Milnor-Moore Theorem, the Hurewicz homomorphism $K_n(A) := \pi_n\left(BGL(A)^+\right) \rightarrow H_n\left(BGL(A)^+; \mathbb{Z}\right)$ gives, after tensoring with $\mathbb{Q}$, an injective homomorphism

$$I_n : K_n(A) \otimes \mathbb{Q} = \pi_n\left(BGL(A)^+\right) \otimes \mathbb{Q} \rightarrow H_n\left(BGL(A)^+, \mathbb{Q}\right) = H_n(BGL(A), \mathbb{Q}).$$

Again by the Milnor-Moore Theorem, its image consists of the subgroup of indecomposable elements, which we denote by $P_n(BGL(A), \mathbb{Q})$.

If n is even and A is a ring of integers in any number field, then $K_n(A) \otimes \mathbb{Q} = 0$. Therefore one is only interested in the case that n is odd, $n = 2m - 1$.

We note that there is a canonical projection $pr_n : H_n(BGL(A), \mathbb{Q}) \rightarrow P_n(BGL(A), \mathbb{Q})$. Namely, the kernel of the Borel regulator $r_n : H_n(BGL(A); \mathbb{Q}) \rightarrow \mathbb{R}$ (defined by pairing with the Borel class b_n, see section 2.3.) is complementary to $P_n(BGL(A), \mathbb{Q})$, thus we may define pr_n as the projection along the kernel of the Borel regulator.

The element $\gamma(M) \in K_{2m-1}(A) \otimes \mathbb{Q}$ that we are going to consider in this paper is then defined as

$$\gamma(M) := I_{2m-1}^{-1} pr_{2m-1} (B\rho)_* [M].$$

If M is a (compact, orientable) manifold with *nonempty* boundary, then there is no general construction of an element in algebraic K-theory. However, we will show in section 4 that for special cases of finite-volume locally symmetric spaces one can generalize the above construction.

2.2 The volume class in $H_c^n\left(Isom\left(\widetilde{M}\right)\right)$

Let $\widetilde{M} = G/K$ be a symmetric space of noncompact type, with isometry group G. It is well-known that $\widetilde{M}$ has nonpositive sectional curvature.

Volume class. We recall that the continuous cohomology $H_c^*(G; \mathbb{R})$ is defined as the homology of the complex $\left(C_c\left(G^{*+1}, \mathbb{R}\right)^G, d\right)$, where $C_c\left(G^{*+1}, \mathbb{R}\right)^G$ stands for the *continuous* G-invariant mappings from G^{*+1} to $\mathbb{R}$ and d is the usual boundary operator. In particular, the group cohomology of G is the continuous cohomology for G with the discrete topology. There is the obvious comparison map $H_c^*(G; \mathbb{R}) \rightarrow H^*(G; \mathbb{R})$. In particular, elements of $H_c^*(G; \mathbb{R})$ can be evaluated on $H_*(G; \mathbb{R})$.

The volume class

$$v_n \in H_c^n(G; \mathbb{R})$$

is defined as follows. Fix an arbitrary point $\tilde{x}_0 \in \widetilde{M} = G/K$. Then we define an n-cochain v_n on G by

$$v_n(g_0, \ldots, g_n) := \text{signed volume of the straight simplex with vertices } g_0\tilde{x}_0, \ldots, g_n\tilde{x}_0.$$

(Note that in simply connected spaces of nonpositive sectional curvature each ordered $n+1$-tuple of vertices determines a unique straight n-simplex.) Since volume is invariant under isometries (and isometries map straight simplices to straight simplices), this cochain is G-invariant. Its cohomology class does not depend on $\tilde{x}_0$. From the additivity of the volume, it is easy to see that this cochain is a cocycle. Thus we have defined a cohomology class v_n.

Theorem 1 *: Let $M = \Gamma \backslash G/K$ be a (closed, orientable) locally symmetric space of noncompact type, and $j : \Gamma \to G$ the inclusion of $\Gamma = \pi_1 M$. Let $j_* : H_n(M;\mathbb{R}) \cong H_n(\Gamma;\mathbb{R}) \to H_n(G;\mathbb{R})$ be the induced homomorphism, and denote $[M] \in H_n(M;\mathbb{R})$ the fundamental class of M. Then*

$$vol(M) = < v_n, j_*[M] > .$$

Proof: Fix a point $x_0 \in M$ and a lift $\tilde{x}_0 \in \widetilde{M}$. Let $C_*^{str,x_0}(M)$ be the chain complex of straight simplices with all vertices in x_0. Each $\sigma \in C_*^{str,x_0}(M)$ is uniquely determined by the homotopy classes of the n edges γ_j between its vertices $\sigma(v_0)$ and $\sigma(v_j)$ for $j = 1, \ldots, n$. Thus

$$C_*^{str,x_0}(M) \cong C_*(\Gamma)$$

for $\Gamma = \pi_1(M, x_0)$, where the isomorphism maps σ to $(1, \gamma_1, \ldots, \gamma_n)$.

Moreover, inclusion $C_*^{str,x_0}(M) \to C_*(M)$ induces an isomorphism in homology. Indeed, each cycle in $C_*(M)$ can first be homotoped such that all vertices are in x_0, and then be straightened. Straightening a simplex σ means to chose the unique geodesic simplex whose edges represent the same element of $\pi_1(M, x_0)$ as the corresponding edges of σ. It is well-known that straightening all simplices of a cycle yields a cycle in the same homology class. (Remark: This construction uses that M is a manifold of nonpositive sectional curvature. It was actually shown in [10] that for each aspherical space $C_*^{str,x_0}(M) \to C_*(M)$ induces an isomorphism in homology.)

Let $\sum_{i=1}^r a_i\sigma_i$ be a representative of the fundamental class. (One may choose e.g. a triangulation $\sigma_1 + \ldots + \sigma_r$.) Then $vol(M) = \sum_{i=1}^r a_i vol(\sigma_i)$. The cycle $\sum_{i=1}^r a_i\sigma_i$ is homologous to some $\sum_{i=1}^r a_i\tau_i \in C_*^{str,x_0}(M)$. (Possibly after straightening some simplices overlap, so we do not get a triangulation. However, it will be sufficient to have a fundamental cycle consisting of geodesic simplices.) By Stokes Theorem, $vol(M) = \sum_{i=1}^r a_i vol(\tau_i)$. The isomorphism $C_*^{str,x_0}(M) \cong C_*(\Gamma)$ maps each τ_i to $\left(1, \gamma_1^i, \ldots, \gamma_n^i\right) \in \Gamma^{n+1}$, where $\gamma_j^i \in \Gamma$ is the class of the (closed) edge from $\tau_i(v_0)$ to $\tau_i(v_j)$. Thus

$$j_*[M] \in H_n(G;\mathbb{R})$$

is represented by

$$\left(1, \gamma_1^i, \ldots, \gamma_n^i\right) \in G^{n+1}.$$

But $< v_n, \left(1, \gamma_1^i, \ldots, \gamma_n^i\right) >$ is the volume of the straight simplex with vertices $\tilde{x}_0, \gamma_1^i\tilde{x}_0, \ldots, \gamma_n^i\tilde{x}_0$, i.e. of the lift of τ_i to $\widetilde{M}$ with first vertex $\tilde{x}_0$. Hence $< v_n, \left(1, \gamma_1^i, \ldots, \gamma_n^i\right) >= vol(\tau_i)$, which implies

$$< v_n, \sum_{i=1}^r a_i \left(1, \gamma_1^i, \ldots, \gamma_n^i\right) >= \sum_{i=1}^r a_i vol(\tau_i) = vol(M).$$

□

2.3 Borel classes

Let $\widetilde{M} = G/K$ be a symmetric space of noncompact type. Then G is a semisimple, noncompact Lie group and K is a maximal compact subgroup.

Let $\mathfrak{g}$ be the Lie algebra of G, and $\mathfrak{k} \subset \mathfrak{g}$ be the Lie algebra of K. There is a decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ with $[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}, [\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}, [\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$. It is a well-known fact that the Killing form $B(X, Y) = Tr(ad(X) \circ ad(Y))$ is negatively definite on $\mathfrak{k}$ and positively definite on $\mathfrak{p}$.

The dual symmetric space is G_u/K, where G_u is the simply connected Lie group with Lie algebra $\mathfrak{g}_u = \mathfrak{k} \oplus i\mathfrak{p} \subset \mathfrak{g} \otimes \mathbb{C}$. The Killing form on $\mathfrak{g}_u$ is negatively definite, thus G_u/K is a compact symmetric space.

The Lie algebra cohomology $H^*(\mathfrak{g})$ is the cohomology of the complex $(\Lambda^* \mathfrak{g}, d)$ with $d\phi(X_0, \ldots, X_n) = \sum_{i<j} \phi(-1)^{i+j} \phi\left([X_i, X_j], X_0, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_n\right)$. The relative Lie algebra cohomology $H^*(\mathfrak{g}, \mathfrak{k})$ is the cohomology of the subcomplex $(C^*(\mathfrak{g}, \mathfrak{k}), d)$ with $C^*(\mathfrak{g}, \mathfrak{k}) = \{\phi \in \Lambda^* \mathfrak{g} : X \lrcorner \phi = 0, ad(X)\phi = 0 \ \ \forall X \in \mathfrak{k}\}$. If G/K is a symmetric space of noncompact type, and G_u/K its compact dual, then there is an obvious isomorphism $H^*(\mathfrak{g}, \mathfrak{k}) = H^*(\mathfrak{g}_u, \mathfrak{k})$, dual to the obvious linear map $\mathfrak{k} \oplus i\mathfrak{p} \to \mathfrak{k} \oplus \mathfrak{p}$. Moreover, $H^*(\mathfrak{g}, \mathfrak{k})$ is the cohomology of the complex of G-invariant differential forms on $H^*(G/K)$. Since G_u is compact and connected, there is an isomorphism (defined by averaging) $H^*(\mathfrak{g}_u, \mathfrak{k}) \simeq H^*(G_u/K)$.

For example,

$$H^*(spin(n,1), spin(n)) \cong H^*(Spin(n+1)/Spin(n)) = H^*(\mathbb{S}^n).$$

Dualizing representations. Let $\rho : G \to GL(N, \mathbb{C})$ be a representation. ρ can be conjugated such that K is mapped to $U(N)$. We will henceforth always assume that ρ has been fixed such that ρ sends K to $U(N)$.

Definition 1 *: Let $\widetilde{M} = G/K$ be a symmetric space of noncompact type. Let $\rho : (G, K) \to (GL(N, \mathbb{C}), U(N))$ be a smooth representation. We denote*

$$D_e\rho : (\mathfrak{g}, \mathfrak{k}) \to (gl(N, \mathbb{C}), u(N))$$

the associated Lie-algebra homomorphism, and, with $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}, \mathfrak{g}_u := \mathfrak{k} \oplus i\mathfrak{p}$,

$$D_e\rho_u : (\mathfrak{g}_u, \mathfrak{k}) \to (u(N) \oplus u(N), u(N))$$

the induced homomorphism on $\mathfrak{k} \oplus i\mathfrak{p}$. The corresponding Lie group homomorphism

$$\rho_u : (G_u, K) \to (U(N) \times U(N), U(N))$$

will be called the dual homomorphism to ρ.

Here $\mathfrak{g}_u$, $\mathfrak{k}$ and $i\mathfrak{p}$ are to be understood as subsets of the complexification $\mathfrak{g} \otimes \mathbb{C}$, and G_u is the simply connected Lie group with Lie algebra $\mathfrak{g}_u$. In particular, the complexification of $gl_N\mathbb{C}$ is isomorphic to $gl_N\mathbb{C} \oplus gl_N\mathbb{C}$, and $i\mathfrak{p} \simeq u(N)$ in this case. We emphasize that ρ_u sends K to the first factor of $U(N) \times U(N)$, and not to the diagonal subgroup as has been claimed in [12].

Van Est Theorem. The van Est theorem states that there is a natural isomorphism

$$H_c^*(G;\mathbb{R}) = H^*(\mathfrak{g},\mathfrak{k}).$$

If $\rho : G \to GL(N,\mathbb{C})$ is a representation, sending K to $U(N)$, then we conclude that there exists the following commutative diagram, where all vertical arrows are isomorphisms

$$\begin{array}{ccc}
H_c^*(GL(N,\mathbb{C})) & \xrightarrow{\rho^*} & H_c^*(G) \\
\cong\uparrow & & \cong\uparrow \\
H^*(gl(N,\mathbb{C}),u(N)) & \xrightarrow{D_e\rho^*} & H^*(\mathfrak{g},\mathfrak{k}) \\
\cong\uparrow & & \cong\uparrow \\
H^*(u(N)\oplus u(N),u(N)) & \xrightarrow{D_e\rho_u^*} & H^*(\mathfrak{g}_u,\mathfrak{k}) \\
\cong\uparrow & & \cong\uparrow \\
H^*(U(N)) & \xrightarrow{\rho_u^*} & H^*(G_u/K)
\end{array}$$

If $dim(G/K) = n$, then G_u/K is an n-dimensional, compact, orientable manifold and we have $H_c^n(G,\mathbb{R}) \cong H^n(G_u/K,\mathbb{R}) \cong \mathbb{R}$. Thus the volume class $[v_n]$ is the (up to multiplication by real numbers) unique nontrivial continuous cohomology class in degree n.

Corollary 1 *: The volume class $[v_n] \in H_c^n(G)$ corresponds (under the van Est isomorphism) to the (de Rham) cohomology class of the volume form $[dvol] \in H^n(G_u/K,\mathbb{R})$.*

Proof: According to [8], Prop. 1.5, v_n corresponds to the class of the volume form in $H^n(\mathfrak{g},\mathfrak{k})$. It is obvious that the isomorphism $H^n(\mathfrak{g},\mathfrak{k}) \simeq H^n(\mathfrak{g}_u,\mathfrak{k})$ maps the volume form of G/K to the volume form of G_u/K. □

Borel classes

Let G be a *compact* Lie group. Let $I_S(G)$ resp. $I_A(G)$ be the *ad*-invariant symmetric resp. antisymmetric multilinear forms on $\mathfrak{g}$. We have the isomorphism $I_A^n(G) \to H^n(G;\mathbb{R})$. Moreover, we remind that there is the Chern-Weil isomorphism $I_S^n(G) \to H^{2n}(BG;\mathbb{R})$, where BG means the classifying space for G with its Lie group topology. In particular, if $G = U(N)$, then the invariant polynomial $\frac{1}{(2\pi i)^n} Tr(A^n)$ is mapped to $C_n \in H^{2n}(BU(N);\mathbb{Z})$, the n-th component of the universal Chern character.

There is a fibration $G \to EG \to BG$ and an associated 'transgression map' τ which maps a subspace of $H^{2n}(BG;\mathbb{Z})$ (the so-called transgressive elements) to a quotient of $H^{2n-1}(G;\mathbb{Z})$, cf. [3], p.410. If $G = U(N)$, then, according to Borel([3], p.412), one has

$$H^*(U(N);\mathbb{Z}) \cong \Lambda_{\mathbb{Z}}(b_1,\ldots,b_{2N-1}),$$

for transgressive elements $b_{2n-1} \in H^{2n-1}(U(N);\mathbb{Z})$ which satisfy

$$\tau(b_{2n-1}) = C_n.$$

The classes b_{2n-1} are called Borel classes. By definition, Borel classes exist only in odd degrees. (We will not distinguish between b_{2n-1} and its image in $H^{2n-1}(U(N);\mathbb{R})$.)

According to Cartan ([6]), there is a homomorphism

$$R : I_S^n(G) \to I_A^{2n-1}(G),$$

whose image corresponds (after the isomorphism $I_A^{2n-1}(G) \cong H^*(G;\mathbb{R})$) precisely to the transgressive elements. It is explicitly given by
$R(f)(X_1, \ldots, X_{2n-1}) = \sum_{\sigma \in S_{2n-1}} f\left(X_{\sigma(1)}, \left[X_{\sigma(2)}, X_{\sigma(3)}\right], \ldots, \left[X_{\sigma(2n-2)}, X_{\sigma(2n-1)}\right]\right)$.

In particular, for $H^*(U(N);\mathbb{R}) \cong H^*(u(N))$, a representative of $b_{2n+1} \in H^{2n+1}(u(N))$ is given by the cocycle

$$b_{2n+1}(X_0, \ldots, X_{2n}) = \frac{1}{(2\pi i)^{n+1}} \sum_{\sigma \in S_{2n+1}} Tr\left(X_{\sigma(0)}\left[X_{\sigma(1)}, X_{\sigma(2)}\right] \ldots \left[X_{\sigma(2n-1)}, X_{\sigma(2n)}\right]\right).$$

It will be clear from the context whether we consider the Borel classes as elements of $H^*(u(N)) \simeq H^*(U(N);\mathbb{R})$ or as the (under the van Est isomorphism) corresponding elements of $H_c^*(GL(N,\mathbb{C});\mathbb{R})$.
We note that stabilization $H^*(U(N+1);\mathbb{R}) \to H^*(U(N);\mathbb{R})$ preserves b_{2n-1}, thus b_{2n-1} may also be considered as an element of $H^*(U;\mathbb{R}) \cong H_c^*(GL(\mathbb{C});\mathbb{R})$.

Let $m \in \mathbb{N}$. We consider $K_{2m-1}(\mathbb{C}) \otimes \mathbb{Q}$ as a subgroup of $H_{2m-1}(GL(\mathbb{C});\mathbb{Q})$, as in section 2.1. The **Borel regulator**

$$r_{2m-1} : K_{2m-1}(\mathbb{C}) \otimes \mathbb{Q} \to \mathbb{R}$$

is, for $x \in K_{2m-1}(\mathbb{C}) \otimes \mathbb{Q}$, defined as applying the Borel class

$$b_{2m-1} \in H_c^{2m-1}(GL(\mathbb{C});\mathbb{R})$$

to $pr_{2m-1}(x)$, where $pr_{2m-1} : H_{2m-1}(GL(\mathbb{C});\mathbb{Q}) \to K_{2m-1}(\mathbb{C}) \otimes \mathbb{Q}$ is the projection defined in section 2.2.

Let A be a subring of $\mathbb{C}$, for example $\overline{\mathbb{Q}}$, any number field or its ring of integers. Inclusion induces a homomorphism $K_{2m-1}(A) \otimes \mathbb{Q} \to K_{2m-1}(\mathbb{C}) \otimes \mathbb{Q}$, thus the Borel regulator also defines a homomorphism

$$r_{2m-1} : K_{2m-1}(A) \otimes \mathbb{Q} \to \mathbb{R}.$$

Borel class of representations.

Definition 2 *: Let $\widetilde{M} = G/K$ be a symmetric space of noncompact type of odd dimension $2n-1$. We say that a (continuous) representation $\rho : G \to GL(N,\mathbb{C})$ has nonvanishing Borel class if $\rho^* b_n \neq 0 \in H_c^{2n-1}(G;\mathbb{R})$.*

Lemma 1 *: Let G/K be a symmetric space of noncompact type, of odd dimension $2n-1$. A representation $\rho : G \rightarrow GL(N,\mathbb{C})$ has nonvanishing Borel class if and only if $\rho_u^* b_n \neq 0 \in H^{2n-1}(G_u/K;\mathbb{R})$, and the latter holds if and only if*

$$< b_n, (\rho_u)_* [G_u/K] > \neq 0.$$

Proof: The first equivalence follows from naturality of the van Est isomorphism. The second equivalence follows from $H_{2n-1}(G_u/K;\mathbb{R}) \simeq \mathbb{R}$. ($G_u/K$ is orientable, because it is simply connected.) □

2.4 Compact locally symmetric spaces and K-theory

In this subsection, we finally show that to each representation of nontrivial Borel class, and each compact, orientable, locally symmetric space of noncompact type we can find a nontrivial element in the algebraic K-theory of the algebraic numbers tensored with $\mathbb{Q}$.

Theorem 2 *: For each symmetric space G/K of noncompact type and odd dimension n, and to each representation $\rho : G \rightarrow GL(N,\mathbb{C})$ with $\rho^* b_n \neq 0$, there exists a constant $c_\rho \neq 0$, such that the following holds: to each compact, orientable, locally symmetric space $M = \Gamma\backslash G/K$, with $\rho(\Gamma) \subset GL(N,A)$ for a subring $A \subset \mathbb{C}$, there exists an element*

$$\gamma(M) \in K_n(A) \otimes \mathbb{Q}$$

such that the Borel regulator $r_n : K_n(A) \otimes \mathbb{Q} \rightarrow \mathbb{R}$ fulfills

$$r_n(\gamma(M)) = c_\rho vol(M).$$

Proof:

In Theorem 1 we have produced an element $j_*[M] \in H_n(G;\mathbb{Z})$. Applying $(B\rho)_*$ we get an element

$$(B\rho)_* j_* [M] \in H_n(GL(N,\mathbb{C});\mathbb{Z}).$$

Since $(B\rho)_* j$ maps Γ to $GL(N,A)$, we have

$$(B\rho)_* j_* [M] \in H_n(GL(N,A);\mathbb{Z}).$$

By assumption $\rho^* b_n \neq 0$. Since $H_c^n(G)$ is one-dimensional, this implies $\rho^* b_n = c_\rho v_n$ for some real number $c_\rho \neq 0$.
As in section 2.1., we consider

$$\gamma(M) := I_n^{-1} pr_n (B\rho)_n j_n [M] \in K_n(A) \otimes \mathbb{Q}.$$

Since the Borel regulator is defined by pairing with b_n, and pr_n is the projection along the kernel of the Borel regulator, we get

$$r_n(\gamma(M)) =< b_n, \gamma(M) >=< b_n, pr_n (B\rho)_n j_n [M] >=< b_n, (B\rho)_n j_n [M] >=< \rho^* b_n, j_* [M] >$$

$$=< c_\rho v_n, j_* [M] >= c_\rho vol(M)$$

where the last equality is true by Theorem 1. □

Corollary 2 *: For each symmetric space G/K of noncompact type and odd dimension n, and to each representation $\rho : G \to GL(N, \mathbb{C})$ with $\rho^* b_n \neq 0$, there exists a constant $c_\rho \neq 0$, such that the following holds: to each compact, orientable, locally symmetric space $M = \Gamma \backslash G/K$ there exists an element*

$$\gamma(M) \in K_n\left(\overline{\mathbb{Q}}\right) \otimes \mathbb{Q}$$

such that the Borel regulator $r_n : K_n\left(\overline{\mathbb{Q}}\right) \otimes \mathbb{Q} \to \mathbb{R}$ fulfills

$$r_n(\gamma(M)) = c_\rho vol(M).$$

Proof: By Weil's rigidity theorem all finite-volume locally symmetric manifolds can be defined over $\overline{\mathbb{Q}}$, that is, they are of the form $M = \Gamma \backslash G/K$ with $\Gamma \subset G\left(\overline{\mathbb{Q}}\right)$.

For each irreducible representation $\rho : G \to GL(N, \mathbb{C})$, $G\left(\overline{\mathbb{Q}}\right)$ is mapped to $GL\left(N, \overline{\mathbb{Q}}\right)$. This follows from the classification of irreducible representations of Lie groups, see [11].

Thus we can apply Theorem 2. □

Corollary 3 *: Assume a representation of $\rho : G \to GL(N, \mathbb{C})$ is given with $\rho^* b_n \neq 0$. Then compact, orientable, locally symmetric n-manifolds of rationally independent volumes yield rationally independent elements in $K_n\left(\overline{\mathbb{Q}}\right) \otimes \mathbb{Q}$.*

Remark: *In [12] it was claimed that for hyperbolic manifolds one can construct an element $\gamma(M) \in K_n\left(\overline{\mathbb{Q}}\right) \otimes \mathbb{Q}$ such that $r_n(\gamma(M)) = vol(M)$. However, since $\rho^* b_n$ is an integer cohomology class, c_ρ is rational if and only if v_n is a rational cohomology class, and this is equivalent to $vol(M) = < v_n, [M] > \in \mathbb{Q}$. Since, conjecturally, all hyperbolic manifolds have irrational volumes, one can probably not get rid of the factor c_ρ in Theorem 2.*

If M is defined over some subring $A \subset \overline{\mathbb{Q}}$, then we do get an element in $K_n(A) \otimes \mathbb{Q}$. The proof is an obvious modification. We will state and prove the general case of (possibly noncompact) finite-volume symmetric spaces in section 4.

In conclusion, we are left with the problem of finding representations of nontrivial Borel class, which will be solved in section 3.

The Matthey-Pitsch-Scherer construction. The following construction gives a somewhat stronger invariant under the assumption that M is stably parallelizable. Assume that $M^d \to \mathbb{R}^n$ is an embedding with trivial normal bundle νM. Let U be a regular neighborhood. Then there is the composition

$$\mathbb{S}^n \to \overline{U}/\partial U \to \overline{U}/\partial U \wedge M_+ = Th(\nu M) \wedge M_+ = \Sigma^{n-d} M_+ \wedge M_+ \to \mathbb{S}^{n-d} \wedge M_+$$

giving an element $\gamma(M) \in \pi_d^s(M)$. By [21], if M is a closed hyperbolic 3-manifold and $\rho : M \to BSL$ is the map given by the stable trivialization, then $\rho_*(\gamma(M))$ is the Bloch invariant.

An analogous construction works for locally symmetric spaces, as long as they are stably parallelizable.

It is known by a Theorem of Deligne and Sullivan that each hyperbolic manifold M admits a finite covering $\widehat{M}$ which is stably parallelizable. Let k be the degree of this covering. Then, rationally, we can define $\gamma(M) := \frac{1}{k}\gamma\left(\widehat{M}\right) \in \pi_d^s(M) \otimes \mathbb{Q}$, and thus get a finer invariant which gives back $\gamma(M) \in K_d(\mathbb{C}) \otimes \mathbb{Q}$. We will not pursue further this approach in this paper.

3 Existence of representations of nontrivial Borel class

3.1 Invariant polynomials

Lemma 2 *: Let G/K be a symmetric space of noncompact type, of dimension $2n-1$. Let $\mathfrak{t}$ be a Cartan subalgebra of $\mathfrak{g}$.*

Then a representation $\rho : G \to GL(N, \mathbb{C})$ has nonvanishing Borel class if and only if $Tr\left(D\rho(t)^n\right) \neq 0$ for some $t \in \mathfrak{t}$.

Proof: As in section 2.3, we consider the dual representation

$$\rho_u : G_u/K \to U(N) \times U(N)/U(N) = U(N).$$

We know that ρ has nonvanishing Borel class if and only if

$$\rho_u^* b_{2n-1} \neq 0 \in H^{2n-1}(G_u/K).$$

The projection $p : G_u \to G_u/K$ induces an injective map $p^* : H^*(G_u/K) \to H^*(G_u)$, because a left inverse to p^* is given by averaging differential forms over the compact group K. Hence, $\rho_u^* b_{2n-1} \neq 0$ if and only if its image in $H^{2n-1}(G_u)$ does not vanish. The latter equals

$$(\pi_2\rho_u)^* b_{2n-1},$$

where $\pi_2 : U(N) \times U(N) \to U(N)$ is projection to the second factor.

We identify $H^{2n-1}(G_u)$ with $I_A^{2n-1}(G_u)$ and $H^{2n}(BG_u)$ with $I_S^n(G_u)$. Let

$$R : I_S^n(G_u) \to I_A^{2n-1}(G_u)$$

be Cartan's homomorphism (see section 2.3). According to [6], the image of R are the transgressive elements and one has

$$R \circ \tau = id$$

for the universal transgression map τ. (In particular $b_{2n-1} = R(C_n)$.),

In view of naturality of the transgression map, $C_n = \tau(b_{2n-1})$ implies that

$$(\pi_2\rho_u)^* C_n = \tau\left((\pi_2\rho_u)^* b_{2n-1}\right),$$

thus $(\pi_2\rho_u)^* b_{2n-1} \neq 0$ is implied by

$$(\pi_2\rho_u)^* C_n \neq 0 \in H^{2n}(BG_u).$$

Moreover, $R \circ \tau = id$ implies injectivity of τ, hence $(\pi_2\rho_u)^* C_n \neq 0$ is also a necessary condition for $(\pi_2\rho_u)^* b_{2n-1} \neq 0$.

Recall that C_n corresponds to

$$\frac{1}{(2\pi i)^n} Tr\left(A^n\right)$$

under the isomorphism $H^{2n}(BU(N)) \simeq I_S^n(u(N))$. Hence it suffices to check that the invariant polynomial

$$Tr\left(\pi_2 D\rho_u(A)^n\right)$$

is not trivial on $\mathfrak{g}_u$.

Let $\mathfrak{t}_u$ be the Cartan subalgebra of $\mathfrak{g}_u$, which corresponds to $\mathfrak{t}$ under the canonical bijection $\mathfrak{k} \oplus \mathfrak{p} \simeq \mathfrak{k} \oplus i\mathfrak{p}$. There is an action of the Weyl group W on $\mathfrak{t}_u$, we denote its space of invariant polynomials by $S_*^W(\mathfrak{t}_u)$. By Chevalley's theorem (see [4]), restriction induces an isomorphism

$$I_S^*(\mathfrak{g}_u) \cong S_*^W(\mathfrak{t}_u).$$

In particular, it suffices to check that $Tr\left(\pi_2 D\rho_u(.)^n\right)$ is not trivial on $\mathfrak{t}_u$.

We note that the Cartan algebra $\mathfrak{t}$ can be conjugated into be a subspace of $\mathfrak{p}$. Since the conclusion of Lemma 2 is invariant under conjugation, we can w.l.o.g. assume that $\mathfrak{t} \subset \mathfrak{p}$ and thus $\mathfrak{t}_u \subset i\mathfrak{p}$. This implies that, for $t \in \mathfrak{t}_u$, $D\rho_u(t)$ belongs to the second factor of $u(N) \oplus u(N)$, and thus $\pi_2 D\rho_u(t) = D\rho_u(t)$ on $\mathfrak{t}_u$. Finally we note that, for $t \in \mathfrak{p}$, $Tr\left(D\rho(t)^n\right)$ and $Tr\left(D\rho_u(it)^n\right)$ coincide up to a power of i. The claim follows. □

Example: Spinor representations. We consider the spinor representations of $so(m, \mathbb{C})$ because their Borel classes have been computed in [12] and the computation given there appears not to be correct. (The main point of confusion seems to be that [12] computes a supertrace rather than a trace. This seems to be related to the wrong assertion that $K = U(N)$ embeds as a diagonal subgroup into $G_u = U(N) \times U(N)$. However, K corresponds to the first factor of $U(N) \times U(N)$, see section 2.3.) We use the description of the spinor representations as they can be found in [11].

m even. Let $V = \mathbb{C}^{2m}$ with $\mathbb{C}$-basis $e_1, \ldots, e_{2m}$, and Q the quadratic form given by $Q(e_i, e_{m+i}) = Q(e_{m+i}, e_i) = 1$ and $Q(e_i, e_j) = 0$ else. There is an injective homomorphism $so(Q) \to Cl(Q)^{even}$ which maps, in particular, $E_{i,i} - E_{m+i,m+i}$ to $\frac{1}{2}(e_i e_{m+i} - 1)$ (see [11], pp.303-305). Let W be the $\mathbb{C}$-subspace of V spanned by $e_1, \ldots, e_m$, W' the subspace spanned by $e_{m+1}, \ldots, e_{2m}$.

$Cl(Q)$ acts on $\Lambda^* W$ as follows: e_i sends $v \in W$ to $e_i \wedge v$ and e_{m+i} sends $v \in W$ to $2v - 2Q(v, e_{m+i}) e_i$, for $i = 1, \ldots, m$. (This follows from the proof of [11], Lemma 20.9.) This action extends in the obvious way to an action of $Cl(Q)$ on $\Lambda^* W$. In particular $\frac{1}{2}(e_i e_{m+i} - 1)$ acts by sending $v \in W$ to $e_i \wedge v - \frac{1}{2} v$. Thus it maps e_i to $\frac{1}{2} e_i$ and e_j to $-\frac{1}{2} e_j$ for $j \neq i$.

This action gives rise to an isomorphism $Cl(Q)^{even} \cong End\left(\Lambda^{even} W\right) \oplus End\left(\Lambda^{odd} W\right)$ (see [11], p.305). The induced actions of $so(Q)$ on $\Lambda^{even} W$ resp. $\Lambda^{odd} W$ are the positive resp. negative half-spinor representations. We will denote them by S^+ and S^-.

Thus $E_{i,i} - E_{m+i,m+i}$ acts on $e_{i_1} \wedge \ldots \wedge e_{i_k}$ by multiplication with $\frac{2-k}{2}$ if $i \in \{i_1, \ldots, i_k\}$ and by multiplication with $-\frac{k}{2}$ if $i \notin \{i_1, \ldots, i_k\}$.

As a Cartan-algebra we choose the algebra of diagonal matrices $diag(h_1, \ldots, h_m, -h_1, \ldots, -h_m)$. Let $\{A_i : i = 1, \ldots, m\}$ be a basis, where $A_i = E_{i,i} - E_{m+i,m+i}$.

For the positive half-spinor representation we have

$$Tr\left(S^{+}\left(A_i\right)^n\right)=\sum_{k\ even}\sum_{i\in\{i_1,\ldots,i_k\}}\left(1-\frac{k}{2}\right)^n+\sum_{i\notin\{i_1,\ldots,i_k\}}\left(-\frac{k}{2}\right)^n$$

$$=\sum_{k\ even}\left(\begin{array}{c}m-1\\k-1\end{array}\right)\left(1-\frac{k}{2}\right)^n+\left(\begin{array}{c}m-1\\k\end{array}\right)\left(-\frac{k}{2}\right)^n.$$

In particular, we have $Tr\left(S^{+}\left(A_i\right)^n\right)<0$ for n odd and $Tr\left(S^{+}\left(A_i\right)^n\right)>0$ for n even. For the negative half-spinor representation we have

$$Tr\left(S^{-}\left(A_i\right)^n\right)=\sum_{k\ odd}\sum_{i\in\{i_1,\ldots,i_k\}}\left(1-\frac{k}{2}\right)^n+\sum_{i\notin\{i_1,\ldots,i_k\}}\left(-\frac{k}{2}\right)^n$$

$$=\sum_{k\ odd}\left(\begin{array}{c}m-1\\k-1\end{array}\right)\left(1-\frac{k}{2}\right)^n+\left(\begin{array}{c}m-1\\k\end{array}\right)\left(-\frac{k}{2}\right)^n.$$

In particular, we have $Tr\left(S^{-}\left(A_i\right)^n\right)<0$ for n odd and $Tr\left(S^{-}\left(A_i\right)^n\right)>0$ for n even.

m odd. Let $V=\mathbb{C}^{2m+1}$ with $\mathbb{C}$-basis $e_1,\ldots,e_{2m+1}$, and Q the quadratic form given by $Q\left(e_{2m+1},e_{2m+1}\right)=1, Q\left(e_i,e_{m+i}\right)=Q\left(e_{m+i,e_i}\right)=1$ and $Q\left(e_i,e_j\right)=0$ else. As in the case m even, we have $so\left(Q\right)\rightarrow Cl\left(Q\right)^{even}$ which maps $E_{i,i}-E_{m+i,m+i}$ to $\frac{1}{2}\left(e_ie_{m+i}-1\right)$ Let W be the $\mathbb{C}$-subspace of V spanned by $e_1,\ldots,e_m$, W' the subspace spanned by $e_{m+1},\ldots,e_{2m}$.

It follows from the proof of [11], Lemma 20.16., that $Cl\left(Q\right)$ acts on Λ^*W as follows: the action of e_i and e_{m+i}, for $i=1,\ldots,m$ is defined as in the case m even, and e_{2m+1} acts as multiplication by 1 on $\Lambda^{even}W$ and as multiplication by -1 on $\Lambda^{odd}W$. In particular, we have again that $\frac{1}{2}\left(e_ie_{m+i}-1\right)$ acts by sending e_i to $\frac{1}{2}e_i$ and e_j to $-\frac{1}{2}e_j$ for $j\neq i$.

This action gives rise to an isomorphism $Cl\left(Q\right)^{even}\cong End\left(\Lambda W\right)$ (see [11], p.306). The induced action of $so\left(Q\right)$ on ΛW is the spinor representation S.

As a Cartan-algebra we choose the algebra of diagonal matrices $diag\left(h_1,\ldots,h_m,-h_1,\ldots,-h_m,0\right)$. Let $\{A_i:i=1,\ldots,m\}$ be a basis, where $A_i=E_{i,i}-E_{m+i,m+i}$. A_i acts on $e_{i_1}\wedge\ldots\wedge e_{i_k}$ by multiplication with $\frac{2-k}{2}$ if $i\in\{i_1,\ldots,i_k\}$ and by multiplication with $-\frac{k}{2}$ if $i\notin\{i_1,\ldots,i_k\}$. Thus we have

$$Tr\left(S\left(A_i\right)^n\right)=\sum_{k}\sum_{i\in\{i_1,\ldots,i_k\}}\left(1-\frac{k}{2}\right)^n+\sum_{i\notin\{i_1,\ldots,i_k\}}\left(-\frac{k}{2}\right)^n$$

$$=\sum_{k}\left(\begin{array}{c}m-1\\k-1\end{array}\right)\left(1-\frac{k}{2}\right)^n+\left(\begin{array}{c}m-1\\k\end{array}\right)\left(-\frac{k}{2}\right)^n.$$

In particular, we have $Tr\left(S\left(A_i\right)^n\right)<0$ for n odd and $Tr\left(S\left(A_i\right)^n\right)>0$ for n even.

3.2 Borel class of Lie algebra representations

Let $\mathfrak{g}$ be a semisimple Lie algebra and $R\left(\mathfrak{g}\right)$ its (real) representation ring, with addition $\oplus$ and multiplication $\otimes$. Let $\mathfrak{t}$ be a Cartan subalgebra of $\mathfrak{g}$.

In this section we consider, for $n \in \mathbb{N}$, the map

$$b_{2n-1} : R(\mathfrak{g}) \to \mathbb{Z}[\mathfrak{t}]$$

given by

$$b_{2n-1}(\pi)(t) = Tr\left(\pi(t)^n\right).$$

It is obvious that $b_{2n-1}(\pi_1 \oplus \pi_2) = b_{2n-1}(\pi_1) + b_{2n-1}(\pi_2)$ holds for representations π_1, π_2. Therefore b_{2n-1} is uniquely determined by its values for irreducible representations. Moreover, $b_{2n-1}(\pi_1 \otimes \pi_2) = b_{2n-1}(\pi_1)\, b_{2n-1}(\pi_2)$ for representations π_1, π_2.

Complex-linear representations. First we consider complex semisimple Lie algebras and the ring $R_{\mathbb{C}}(\mathfrak{g}) \subset R(\mathfrak{g})$ of their $\mathbb{C}$-linear representations.

$\mathfrak{g} = sl(l+1, \mathbb{C})$.
Let $V = \mathbb{C}^{l+1}$ be the standard representation, with basis $e_1, \ldots, e_{l+1}$. Then $R_{\mathbb{C}}(\mathfrak{g}) = \mathbb{Z}[A_1, \ldots, A_l]$ with A_k the induced representation on $\Lambda^k V$, cf. [11]. In particular, irreducible representations correspond to monomials with coefficient 1, i.e. to tensor products of A_k's. Thus it suffices to compute b_{2n-1} on A_k.

A basis of $\Lambda^k V$ is given by $\{e_{i_1} \wedge \ldots \wedge e_{i_k} : 1 \leq i_1 < \ldots < i_k \leq l+1\}$. As Cartan-subalgebra we may choose the diagonal matrices $diag(h_1, \ldots, h_l, h_{l+1})$ with $h_1 + \ldots + h_{l+1} = 0$. $diag(h_1, \ldots, h_l, h_{l+1})$ acts on $e_{i_1} \wedge \ldots \wedge e_{i_k}$ by multiplication with $h_{i_1} + \ldots + h_{i_k}$. Hence

$$b_{2n-1}(A_k) \begin{pmatrix} h_1 & 0 & \ldots & 0 \\ 0 & h_2 & \ldots & 0 \\ . & . & \ldots & . \\ 0 & 0 & \ldots & h_{l+1} \end{pmatrix} = \sum_{1 \leq i_1 < \ldots < i_k \leq l+1} \left(h_{i_1} + \ldots + h_{i_k}\right)^n.$$

For $n = 1$, the sum is a multiple of $h_1 + \ldots + h_{l+1} = 0$. For $l = k = 1$ and n odd, we have that $b_n(A_1) = h_1^n + h_2^n$ is a multiple of $h_1 + h_2 = 0$. In all other cases, i.e. for $l \geq 2, n > 1$ or $l = k = 1, n$ even, the sum is not divisible by $h_1 + \ldots + h_{l+1}$ and thus not trivial. This is obvious for even n. In the case of odd $n > 1$ and $l \geq 2$, it follows for example from the computation $b_{2n-1}(A_k)\, diag(2, -1, -1, 0, \ldots, 0) = (2^n - 1)\left(\begin{pmatrix} l-2 \\ k-1 \end{pmatrix} - \begin{pmatrix} l-1 \\ k-1 \end{pmatrix}\right) \neq 0$. Thus, fundamental representations have nontrivial b_{2n-1}, for all $l \geq 2, n > 1$ or $l = k = 1, n$ even.

$\mathfrak{g} = sp(l, \mathbb{C})$.
Let $V = \mathbb{C}^{2l}$ be the standard representation. Then $R_{\mathbb{C}}(\mathfrak{g}) = \mathbb{Z}[B_1, \ldots, B_l]$ with B_k the induced representation on $\Lambda^k V$. Thus it suffices to compute b_n on B_k.

$sp(l, \mathbb{C})$ consists of matrices $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$, such that the lxl-blocks A, B, C, D satisfy $B^T = B, C^T = C, A^T = D$. As Cartan-subalgebra we may choose the diagonal matrices $diag(h_1, \ldots, h_l, -h_1, \ldots, -h_l)$.

Let $\{e_1, \ldots, e_l, f_1, \ldots, f_l\}$ be a basis of $\mathbb{C}^{2l}$ for the standard representation. A basis of $\Lambda^k V$ is given by $\left\{e_{i_1} \wedge \ldots \wedge e_{i_p} \wedge f_{j_1} \wedge \ldots \wedge f_{j_{k-p}} : 0 \leq p \leq k, 1 \leq i_1 < \ldots < i_p \leq l, 1 \leq j_1 < \ldots < j_{k-p} \leq l\right\}$.

$diag\,(h_1,\ldots,h_l,-h_1,\ldots,-h_l)$ acts on $e_{i_1}\wedge\ldots\wedge e_{i_p}\wedge f_{j_1}\wedge\ldots\wedge f_{j_{k-p}}$ by multiplication with $h_{i_1}+\ldots+h_{i_p}-h_{j_1}-\ldots-h_{j_{k-p}}$. Hence

$$b_{2n-1}\,(B_k)\begin{pmatrix} h_1 & 0 & \ldots & 0 & 0 & \ldots \\ 0 & h_2 & \ldots & 0 & 0 & \ldots \\ \ldots & & & \ldots & & \\ 0 & 0 & \ldots & -h_1 & 0 & \ldots \\ 0 & 0 & \ldots & 0 & -h_2 & \ldots \\ \ldots & & & \ldots & & \end{pmatrix} = \sum_{1\le i_1<\ldots<i_p\le l,1\le j_1<\ldots<j_{k-p}\le l}\left(h_{i_1}+\ldots+h_{i_p}-h_{j_1}-\ldots-h_{j_{k-p}}\right)$$

If n is even, we clearly get a nonvanishing polynomial without cancellations. If n is odd, then the permutation which transposes i_r and j_r simultaneously for all r multiplies the sum by -1 but on the other hand preserves the sum. Thus $b_{2n-1}\,(B_k)=0$ if n is odd.

$\mathfrak{g}=so\,(2l+1,\mathbb{C})$.
Let $V=\mathbb{C}^{2l+1}$ be the standard representation. Then $R_{\mathbb{C}}\,(\mathfrak{g})=\mathbb{Z}\,[C_1,\ldots,C_{l-1},S]$ with C_k the induced representation on Λ^kV, and S the spin representation.

As Cartan-subalgebra we may choose the diagonal matrices $diag\,(h_1,\ldots,h_l,-h_1,\ldots,-h_l,0)$. Then the computation of b_{2n-1} on C_k is exactly the same as for $sp\,(l,\mathbb{C})$, in particular $b_{2n-1}\,(C_k)\neq 0$ for n even and $b_{2n-1}\,(C_k)=0$ for n odd.
Moreover, we have computed in section 3.1 that $b_{2n-1}\,(S)\neq 0$ for all n.

$\mathfrak{g}=so\,(2l,\mathbb{C})$.
Let $V=\mathbb{C}^{2l}$ be the standard representation. Then $R_{\mathbb{C}}\,(\mathfrak{g})=\mathbb{Z}\,[D_1,\ldots,D_{l-2},S^+,S^-]$ with D_k the induced representation on Λ^kV, and $S^\pm$ the half-spinor representations.

As Cartan-subalgebra we may choose the diagonal matrices $diag\,(h_1,\ldots,h_l,-h_1,\ldots,-h_l)$. Again the same computation as for $sp\,(l,\mathbb{C})$ shows that $b_{2n-1}\,(D_k)\neq 0$ for n even and $b_{2n-1}\,(D_k)=0$ for n odd. Moreover, we have computed in section 3.1 that $b_{2n-1}\,(S^\pm)\neq 0$ for all n.

Exceptional Lie groups.
We will see in the next section that we will only be interested in Lie groups which admit a symmetric space of odd dimension. The only exceptional Lie group admitting an odd-dimensional symmetric space is E_7 with $dim\,(E_7/E_7\,(\mathbb{R}))=163=2.82-1$. The fact that $163\equiv\ 3\ mod 4$, i.e. $n=82$ even, implies automatically that $\rho^*b_{163}\neq 0$ holds for each irreducible representation ρ. However, for completeness we also show, at least for a specific representation, that $\rho^*b_{2n-1}\neq 0$ holds for each $n>1$.

Namely, we consider the representation $\rho: E_7\to GL\,(56,\mathbb{C})$, which has been constructed in [1], Corollary 8.2, and we are going to show that this representation satisfies $\rho^*b_{2n-1}\neq 0$ for each $n>1$, in particular for $n=82$.

By [1], chapter 7/8, there is a monomorphism $Spin\,(12)\times SU\,(2)\,/\mathbb{Z}_2\to E_7$ and the Cartan-subalgebra of e_7 coincides with the Cartan-subalgebra t of $spin\,(12)\oplus su\,(2)$. According to [1], Corollary 8.2, the restriction of ρ to $Spin\,(12)\times SU\,(2)$ is $\lambda_{12}^1\otimes\lambda_1\oplus S^-\otimes 1$, where λ_{12}^1 resp. λ_1 are the standard representations and S^- is the negative spinor representation.

For even n, we know that $\rho^*b_{2n-1}\neq 0$. If n is odd then, for the derivative π_1 of the standard representation λ_1 of $SU\,(2)$ we have $Tr\,(\pi_1\,(h)^n)=0$, whenever $h\in\mathfrak{t}\cap su\,(2)$

belongs to the Cartan-subalgebra of $su(2)$, because the latter are the diagonal 2x2-matrices of trace 0. Thus the first direct summand $\lambda_{12}^1 \otimes \lambda_1$ does not contribute to $Tr(\pi(h)^n)$. Hence , for $h = (h_{spin}, h_{su}) \in \mathfrak{t} \subset spin(12) \oplus su(2)$, we have $Tr(\pi(h)^n) = Tr(S^-(h_{spin})^n)$. But the nontriviality of the latter has already been shown in section 3.1.

Noncomplex Lie algebras. Let $\pi : \mathfrak{g} \to gl(N, \mathbb{C})$ be an $\mathbb{R}$-linear representation of a simple Lie-algebra $\mathfrak{g}$ which is not a complex Lie algebra. Then $\mathfrak{g} \otimes \mathbb{C}$ is a simple complex Lie algebra and π is the restriction of some $\mathbb{C}$-linear representation $\mathfrak{g} \otimes \mathbb{C} \to gl(N, \mathbb{C})$. Let $\mathfrak{t}$ be a Cartan subalgebra of $\mathfrak{g}$. Then it is obvious that an element $t \in \mathfrak{t} \otimes \mathbb{C}$ with $Tr(\pi(t)^n) \neq 0$ exists if and only if such an element exists in $\mathfrak{t}$. Thus π has nontrivial Borel class if and only if $\pi \otimes \mathbb{C}$ has nontrivial Borel class. Hence we can use the results for complex-linear representations.

Real representations of complex Lie algebras. If $\mathfrak{g}$ is a simple complex Lie algebra, then each $\mathbb{R}$-linear representation $\pi : \mathfrak{g} \to gl(N, \mathbb{C})$ is of the form $\pi = \pi_1 \otimes \overline{\pi_2}$ for $\mathbb{C}$-linear representations π_1, π_2. We have $Tr(\pi(t)^n) = Tr(\pi_1(t)^n) Tr(\overline{\pi_2}(t)^n)$. In particular, real representations with nontrivial b_{2n-1} can only exist if there are complex representations of nontrivial b_{2n-1}.

3.3 Conclusion

In this section, we discuss, for which symmetric spaces G/K (irreducible, of noncompact type, of dimension $2n-1$) and which representations $\rho : G \to GL(N, \mathbb{C})$ the inequality $\rho^* b_{2n-1} \neq 0$ holds.

Definition 3 *: We say that a Lie algebra representation $\pi : \mathfrak{g} \to gl(N, \mathbb{C})$ has nontrivial Borel class if $b_{2n-1}(\pi) \not\equiv 0$, for $b_{2n-1} : R[\mathfrak{g}] \to \mathbb{Z}[\mathfrak{t}]$ defined in section 3.2.*

Proposition 1 *: Let $\rho : G \to GL(N, \mathbb{C})$ be a representation of a Lie group G, and $\pi : \mathfrak{g} \to gl(N, \mathbb{C})$ the associated Lie algebra representation $\pi = D_e\rho$. Then ρ has nontrivial Borel class if and only if π has nontrivial Borel class.*

Proof: This is precisely the statement of Lemma 2. □

We use the classification of symmetric spaces as it can be read off table 4 in [23]. Of course, we are only interested in symmetric spaces of odd dimension. A simple inspection shows that all odd-dimensional irreducible symmetric spaces of noncompact type are given by the following list:
$SL_l(\mathbb{R})/SO_l, l \equiv 0, 3\ mod\ 4$,
$SL_{2l}(\mathbb{H})/Sp_l, l \equiv 0\ mod\ 2$,
$Spin_{p,q}/(Spin_p \times Spin_q), p, q \equiv 1\ mod\ 2$,
$SL_l(\mathbb{C})/SU_l, l \equiv 0\ mod\ 2$,
$SO_l(\mathbb{C})/SO_l, l \equiv 2, 3\ mod\ 4$,
$Sp_l(\mathbb{C})/Sp_l, l \equiv 1\ mod\ 2$,
$E_7(\mathbb{C})/E_7$.

First we note that for n even all representations satisfy $\rho^* b_{2n-1} \neq 0$. This applies to locally symmetric spaces of dimension $\equiv 3 \; mod \; 4$. In the above list this are the following symmetric spaces:
$SL_l(\mathbb{R})/SO_l, l \equiv 0, 7 \; mod \; 8$,
$SL_{2l}(\mathbb{H})/Sp_l, l \equiv 2 \; mod \; 4$,
$SO_{p,q}/(SO_p \times SO_q), p, q \equiv 1 \; mod \; 2, p \not\equiv q \; mod \; 4$,
$SL_l(\mathbb{C})/SU_l, l \equiv 0 \; mod \; 2$,
$SO_l(\mathbb{C})/SO_l, l \equiv 2 \; mod \; 4$,
$Sp_l(\mathbb{C})/Sp_l, l \equiv 3 \; mod \; 4$,
$E_7(\mathbb{C})/E_7$.

We now analyze the locally symmetric spaces of dimension $\equiv 1 \; mod \; 4$. For those, whose corresponding Lie algebra $\mathfrak{g}$ is not a complex Lie algebra (this concerns the first 3 cases), we can, as observed at the end of section 3.2, directly apply the results for the complexifications. Thus we get :
- for $SL_l(\mathbb{R})/SO_l, l \equiv 3, 4 \; mod \; 8$, every irreducible representation yields a nontrivial element,
- for $SL_{2l}(\mathbb{H})/Sp_l, l \equiv 0 \; mod \; 4$, every irreducible representation yields a nontrivial element,
- for $Spin_{p,q}/(Spin_p \times Spin_q), p, q \equiv 1 \; mod \; 2, p \equiv q \; mod \; 4$, tensor products of copies of positive and/or negative half-spinor representations are the only irreducible representations yielding nontrivial elements.
For those locally symmetric spaces whose corresponding Lie algebra $\mathfrak{g}$ is a complex Lie algebra, we use the fact that each real representation is of the form $\rho_1 \otimes \overline{\rho_2}$. We get:
- for $SO_l(\mathbb{C})/SO_l, l \equiv 3 \; mod \; 4$, tensor products of copies of the spinor representation and its conjugate are the only irreducible representations yielding nontrivial elements,
- for $Sp_l(\mathbb{C})/Sp_l, l \equiv 1 \; mod \; 4$, no representation yields nontrivial elements.

Example (Goncharov): Hyperbolic space $\mathbb{H}^m$ is the symmetric space $\mathbb{H}^m = Spin_{m,1}/(Spin_m \times Spin_1)$. Let $m = 2n-1$ be odd. It was shown in [12] that the positive and negative spinor representation have nontrivial Borel class. The question was raised ([12], p.587) whether these are the only fundamental representations of $Spin_{m,1}$ with this property. As a special case of the above results we see that for $m \equiv 3 \; mod \; 4$ each irreducible representation has nontrivial Borel class, meanwhile for $m \equiv 1 \; mod \; 4$ the positive and negative spinor representation are the only fundamental representations with this property.

3.4 Some clues on computation

So far we have only discussed how to decide $\rho^* b_{2n-1} \neq 0$, which is in view of Lemma 2 easier then computing $\rho^* b_{2n-1}$. The aim of this subsection is only to give some clues to computation of $\rho^* b_{2n-1}$. Its results are not needed throughout the paper, except for the explicit values of the Borel regulator in section 5.

For each Lie-algebra-cocycle $P \in C^n(\mathfrak{g}_u, \mathfrak{k})$, we denote by $\omega_P \in \Omega^n(G_u/K)$ the corresponding G_u-invariant differential form. Then we have the following obvious observation. ($[\omega_P]$ denotes the cohomology class of ω_P, and $[G_u/K]^v \in H^n(G_u/K, \mathbb{R})$ denotes the dual of the fundamental class $[G_u/K]$. The Riemannian metric is given by the negative of the Killing form.)

Lemma 3 *: Let $X_1, \ldots, X_n$ be an ON-basis for $i\mathfrak{p}$ (w.r.t. -B). Then, for each $P \in I^n(\mathfrak{g}_u, \mathfrak{k})$, we have*

$$[\omega_P] = [G_u/K]^v \, vol\,(G_u/K)\, P(X_1, \ldots, X_n).$$

Corollary 4 *: $[\omega_P] \neq 0$ iff $P(X_1, \ldots, X_n)$ for some (hence any) basis of $i\mathfrak{p}$.*

We will apply this to the Borel class $b_n \in H^{2n-1}(u(N) \oplus u(N), u(N))$ which is given by the cocycle

$$P(X_1, \ldots, X_{2n-1}) = \frac{1}{(2\pi i)^n} \frac{1}{(2n-1)!} \sum_{\sigma \in S_{2n-1}} Tr\left(X_{\sigma(1)} \left[X_{\sigma(2)}, X_{\sigma(3)}\right] \ldots \left[X_{\sigma(2n-2)}, X_{\sigma(2n-1)}\right]\right).$$

Example: Hyperbolic 3-manifolds.

Let $G = SL(2, \mathbb{C})$. Then

$$i\mathfrak{p} = \left\{ iA \in Mat(2, \mathbb{C}) : Tr(A) = 0, A = \overline{A}^T \right\}.$$

An ON-basis of $i\mathfrak{p}$ is given by $\frac{1}{2\sqrt{2}}H, \frac{1}{2\sqrt{2}}X, \frac{1}{2\sqrt{2}}Y$, with

$$H = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, X = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, Y = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

We have

$$[H, X] = -2Y, [H, Y] = -2X, [X, Y] = 2H.$$

Thus, for each representation $\rho : Sl(2, \mathbb{C}) \to GL(m+1, \mathbb{C})$ with associated Lie algebra representation $\Pi : sl(2, \mathbb{C}) \to Mat(m+1, \mathbb{C})$ we have

$$(2\pi i)^2 6\rho^* b_3(H, X, Y) = 2Tr(\Pi H [\Pi X, \Pi Y]) + 2Tr(\Pi X [\Pi Y, \Pi H]) + 2Tr(\Pi Y [\Pi H, \Pi X])$$

$$= 4Tr\left((\Pi H)^2\right) + 4Tr\left((\Pi X)^2\right) + 4Tr\left((\Pi Y)^2\right).$$

By the classification of irreducible representations of $sl(2, \mathbb{C})$, each $m+1$-dimensional irreducible representation is equivalent to π_m given by

$$\pi_m(H) = \begin{pmatrix} im & 0 & 0 & \ldots & 0 \\ 0 & i(m-2) & 0 & \ldots & 0 \\ 0 & 0 & i(m-4) & \ldots & 0 \\ . & . & . & \ldots & . \\ 0 & 0 & . & \ldots & -im \end{pmatrix},$$

$$\pi_m(X) = \begin{pmatrix} 0 & -i & 0 & \ldots & 0 \\ -im & 0 & -2i & \ldots & 0 \\ 0 & -i(m-1) & 0 & \ldots & 0 \\ . & . & . & \ldots & -im \\ 0 & 0 & 0 & ..-i & 0 \end{pmatrix},$$

$$\pi_m(Y) = \begin{pmatrix} 0 & 1 & 0 & ... & 0 \\ -m & 0 & 2 & ... & 0 \\ 0 & -(m-1) & 0 & ... & 0 \\ . & . & . & ... & m \\ 0 & 0 & 0 & ..-1 & 0 \end{pmatrix}.$$

Therefore, the diagonal entries of $\pi_m(H)^2$ are $\left(-m^2, -(m-2)^2, \ldots, 0, \ldots, -(m-2)^2, -m^2\right)$, the diagonal entries of $\pi_m(X)^2$ resp. $\pi_m(Y)^2$ are both equal to $(-m, -m-2(m-1), ...)$. Therefore $Tr\left(\pi_m(X)^2\right) = Tr\left(\pi_m(Y)^2\right)$ and we conclude

$$\rho_m^* b_3\left(\frac{1}{2\sqrt{2}}H, \frac{1}{2\sqrt{2}}X, \frac{1}{2\sqrt{2}}Y\right) = -\frac{1}{48\pi^2}Tr\left((\pi_m H)^2\right) - \frac{1}{24\pi^2}Tr\left((\pi_m X)^2\right)$$

$$= \frac{1}{48\pi^2}\sum_{k=0}^{m}(m-2k)^2 + \frac{1}{24\pi^2}\sum_{k=0}^{m}(k+1)(m+k) + (k+2)(m+k+1).$$

If $m = 1$, we get $\rho_1^* b_3\left(\frac{1}{2\sqrt{2}}H, \frac{1}{2\sqrt{2}}X, \frac{1}{2\sqrt{2}}Y\right) = \frac{1}{8\pi^2}$, that is the Borel regulator is $\frac{1}{8\pi^2}$ times the volume.

Example: $SL(3,\mathbb{R})/SO(3)$.

Let $\rho : SL(3,\mathbb{R}) \to GL(3,\mathbb{C})$ be the inclusion. Since $SL(3,\mathbb{R})/SO(3)$ is 5-dimensional, we wish to compute $\rho^* b_5$.

Let

$$H_1 = \begin{pmatrix} i & 0 & 0 \\ 0 & -i & 0 \\ 0 & 0 & 0 \end{pmatrix}, X_1 = \begin{pmatrix} 0 & i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, Y_1 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

We will use the convention that, for $A \in \{H, X, Y\}$ if A_1 is defined (in a given basis), then A_2 is obtained via the base change $e_1 \to e_2, e_2 \to e_3, e_3 \to e_1$ and A_3 is obtained via the base change $e_1 \to e_3, e_3 \to e_2, e_2 \to e_1$.

We have $[H_1, H_2] = 0, [H_1, X_1] = 2Y_1, [H_1, X_2] = -Y_2, [H_1, X_3] = -Y_3, [X_1, X_2] = iY_3$ and more relations are obtained out of these ones by base changes.

A basis of $i\mathfrak{p}$ is given by H_1, H_2, X_1, X_2, X_3. The formula for $\rho^* b_5(H_1, H_2, X_1, X_2, X_3)$ contains 120 summands. (24 of them contain $[H_1, H_2] = 0$ or $[H_2, H_1] = 0$.)

Each summand appears four times beacuse, for example, $H_1[H_2, X_1][X_2, X_3]$ also shows up as $-H_1[X_1, H_2][X_2, X_3], -H_1[H_2, X_1][X_3, X_2]$ and $H_1[X_1, H_2][X_3, X_2]$. Thus one has to add 30 summands (6 of them zero), and multiply their sum by 4.

We note that all summands of the form $H_1[H_2, .][., .]$ give after base change corresponding elements of the form $H_2[H_1, .][., .]$, which are summed with the opposite sign. Thus these terms cancel each other. The same cancellation occurs between summands of the form $X_2[., .][., .]$ and $X_3[., .][., .]$. Thus we only have to sum up summands of the form $X_1[., .][., .]$ and we get

$$(2\pi i)^3 5! \rho^* b_5(H_1, H_2, X_1, X_2, X_3) =$$

$$4Tr\left(X_1\left[H_1,H_2\right]\left[X_2,X_3\right]\right)+4Tr\left(X_1\left[X_2,X_3\right]\left[H_1,H_2\right]\right)+4Tr\left(X_1\left[H_1,X_2\right]\left[X_3,H_2\right]\right)+4Tr\left(X_1\left[X_3,H_2\right]\left[H_1,X_2\right]\right)+$$
$$+4Tr\left(X_1\left[H_1,X_3\right]\left[H_2,X_2\right]\right)+4Tr\left(X_1\left[H_2,X_2\right]\left[H_1,X_3\right]\right)$$
$$=0+0+4Tr\left(X_1Y_2Y_3\right)+4Tr\left(X_1Y_3Y_2\right)+4Tr\left(-2X_1Y_3Y_2\right)+4Tr\left(-2X_1Y_2Y_3\right)$$
$$=0+0+4i+4i-8i-8i=-8i.$$

We note that H_1, H_2, X_1, X_2, X_3 are orthogonal of norm $2\sqrt{3}$. Dividing each of them by $2\sqrt{3}$ gives an ON-basis, on which evaluation of $\rho^* b_5$ gives $\frac{1}{(2\sqrt{3})^5}\frac{1}{5!}\frac{1}{(2\pi i)^3}(-8i) = \frac{1}{34560\sqrt{3}\pi^3}$.

4 The cusped case

4.1 Preparations

Let G be a connected, semisimple Lie group with maximal compact subgroup K. Then G/K is a symmetric space of noncompact type. G acts on $\widetilde{M}=G/K$ and thus on the ideal boundary $\partial_\infty \widetilde{M}$. A group $P\subset G$ is maximal parabolic if there is some $x\in\partial_\infty\widetilde{M}$ with $P=\{g\in G: gx=x; gy\neq y\ \forall\ y\neq x\}$.

Let $\rho: G\to GL(N,\mathbb{C})$ be a representation. We assume that ρ maps K to $U(N)$, which can be achieved upon conjugation. We note that connected, semisimple Lie groups are perfect, hence ρ has image in $SL(N,\mathbb{C})$ and maps K to $SU(N)$.

The induced map $\rho: G/K\to SL(N,\mathbb{C})/SU(N)$ maps geodesics in G/K to geodesics in $SL(N,\mathbb{C})/SU(N)$, because geodesics are of the form $t\to gexp(tX)K$ for some $X\in\mathfrak{g}$.

Thus one gets a map of ideal boundaries $\partial_\infty\widetilde{M}\to\partial_\infty(SL(N,\mathbb{C})/SU(N))$ which is ρ-equivariant for the actions of G resp. $SL(N,\mathbb{C})$. In particular, maximal parabolic subgroups are mapped to maximal parabolic subgroups.

We note that for $SL(N,\mathbb{C})/SU(N)$ a maximal parabolic subgroup is given by the group B of upper triangular matrices with all diagonal entries equal. This follows from the well-known description of $\partial_\infty(SL(N,\mathbb{C})/SU(N))$ as a flag manifold. Moreover, each maximal parabolic subgroup B' is conjugate to B

Relative classifying spaces.

Let M be a manifold with boundary such that $int(M)=M-\partial M$ admits a locally symmetric Riemannian metric of finite volume. Let M_+ be the quotient space obtained by identifying points in respectively each boundary component. In particular $H_n(M_+)$ has a fundamental class.

First, we briefly discuss the approach via relative classifying spaces, which works exactly as in [7]. Let M be a manifold with boundary such that $int(M)=M-\partial M$ admits a locally symmetric Riemannian metric of finite volume. Let M_+ be the quotient space obtained by identifying points in respectively each boundary component. In particular $H_n(M_+)$ has a fundamental class.

Let $P\subset G$ and $B'\subset SL(N;\mathbb{C})$ be maximal parabolic subgroups, such that $\rho: (G,K)\to(SL(N,\mathbb{C}),SU(N))$ sends P to B'. To a locally symmetric space $\Gamma\backslash G/K$ of finite volume and dimension n, we can consider $G/K\cup C$, where C denotes the set of

parabolic fixed points in $\partial_\infty G/K$ and get as in [7], section 3, an (up to Γ-equivariant homotopy unique) Γ-equivariant map $G/K \cup C \to E(G, \mathcal{F}(P))$. The quotient of $G/K \cup C$ by Γ is homeomorphic to M_+. In particular, $H_n(\Gamma \backslash (G/K \cup C))$ has a fundamental class, and we get as in [7] an element in $H_n(B(G, \mathcal{F}(P)))$, which by the representation ρ is pushed forward to an element in $H_n(B(GL(N, \mathbb{C}), \mathcal{F}(B')))$.
In the rest of this section we will discuss the more interesting question of lifting this invariant to $K(\mathbb{C}) \otimes \mathbb{Q}$. The construction of this lift in section 4.2. is close to the work of Cisneros-Molina and Jones. However, proving nontriviality of the constructed invariant $\gamma(M)$ needs more arguments. For hyperbolic 3-manifolds, nontriviality was proved in [7] by showing that $\gamma(M)$ can be pushed forward to a certain nontrivial invariant $\alpha(M)$ in the Bloch group. This approach does not seem to generalize to other locally symmetric spaces. Therefore we will prove in section 4.3. that application of the volume cocycle to $\gamma(M)$ gives $vol(M)$. This implies in particular nontriviality of $\gamma(M)$. The proof in section 4.3. is unfortunately not so canonical like the proof for the compact case in section 2.5. but requires more technical arguments.

Relative group cohomology.

We discuss the approach via relative group cohomology, which (for hyperbolic manifolds) has been used in [12]. (The approach in [12] implicitly assumes that ∂M is connected. However it is possible to extend it to the general case.)

Let $M = \Gamma \backslash G/K$ be a locally symmetric space of finite volume. For each cusp let $\tilde{x}_r \in \partial_\infty \widetilde{M}$ be a lift of the cusp and $P_r = \{g \in G : g\tilde{x}_r = \tilde{x}_r; gy \neq y \ \forall \ y \neq x\}$. ρ maps P_r to some $B_r' \subset GL(N, \mathbb{C})$, where B_r' is conjugate to the group of upper triangular matrices.

Let $Cone(\partial M \to M)$ be the mapping cone of the inclusion. (It is homotopy equivalent to M_+.) We use the well-known isomorphism $H_*(M, \partial M) \simeq H_*(Cone(\partial M \to M))$. In particular, we consider the fundamental class $[M, \partial M]$ as a homology class on $Cone(\partial M \to M)$.

M and each connnected component of ∂M are classifying spaces for their fundamental groups. If ∂M is connected, then we get a homotopy equivalence $Cone(\partial M \to M) \simeq Cone(B\pi_1 \partial M \to B\pi_1 M)$. In the general class we can, after performing a homotopy equivalence h, assume that the images $h(\partial_r M)$ of all connected components $\partial_r M$ of the boundary ∂M intersect in exactly one point p. If we choose this point as a base point for $\pi_1 M$ as well as $\pi_1 h(\partial_r M)$ for each connected component $\partial_r M$, we get a homotopy equivalence $Cone(h(\partial M) \to M) \simeq Cone\left(\bigcup_{r=1}^k B\pi_1 \partial_r M \to B\pi_1 M\right)$. (We note that $vol(M) < \infty$ implies $\pi_1 h(\partial_r M) \cap \pi_1 h(\partial_s M) = \{1\}$, because otherwise the corresponding cusps would have infinite volume.)

By Weil rigidity we have $\Gamma \subset G\left(\overline{\mathbb{Q}}\right)$. Let $j : \pi_1 M = \Gamma \to G\left(\overline{\mathbb{Q}}\right)$ be the inclusion and $\rho \mid_{G(\overline{\mathbb{Q}})}: G\left(\overline{\mathbb{Q}}\right) \to GL\left(\overline{\mathbb{Q}}\right)$ the restriction of ρ. We note that $\pi_1 h(\partial_r M) = P_r \cap \Gamma$, thus $\rho(\pi_1 h(\partial_r M)) \subset B_r'\left(\overline{\mathbb{Q}}\right)$.

We get a map

$$B\rho \mid_{G(\overline{\mathbb{Q}})} Bj : Cone(\partial M \to M) \simeq Cone(B\pi_1 h(\partial M) \to B\pi_1 M) \to Cone\left(\bigcup_{r=1}^k BB_r'\left(\overline{\mathbb{Q}}\right) \to BGL\left(N, \overline{\mathbb{Q}}\right)\right).$$

We recall that

$$e_* : H_*\left(GL\left(N,\overline{\mathbb{Q}}\right);\mathbb{Q}\right) \to H_*\left(GL\left(N,\overline{\mathbb{Q}}\right), B_r'\left(\overline{\mathbb{Q}}\right);\mathbb{Q}\right)$$

is surjective for each r (see [12], Thm.2.12.). From this one can derive that we find an element $\gamma(M) \in H_n\left(GL\left(N,\overline{\mathbb{Q}}\right);\mathbb{Q}\right)$ with $e_*\gamma(M) = (B\rho)_*(Bj)_*[M,\partial M]$. The problem is to show that indeed

$$r_n(\gamma(M)) = vol(M).$$

Possibly this can be shown by similar arguments as in [12] (although the existence of several cusps makes it unclear how to define $\overline{v}_n$ in formula (16) on p.579.) Here we will use another construction of $\gamma(M)$, which is inspired by [7], and show by explicit topological arguments that $r_n(\gamma(M)) = vol(M)$ holds true.

4.2 Generalized Cisneros-Molina-Jones construction

We start with some algebraic preparations. For a ring A, let $St(N,A)$ be the Steinberg group and $\Phi : St(N,A) \to SL(N,A)$ the canonical homomorphism. We use the notation of [24]: x_{ij}^a, with $1 \le i,j \le n$ and $a \in A$, are the generators of $St(N,A)$, and $e_{ij}^a = \Phi\left(x_{ij}^a\right)$ are the elementary matrices, which generate $SL(N,A)$.

Lemma 4 *: Let A be a ring. Let B_0 be the subgroup of $SL(N,A)$ consisting of upper triangular matrices with all diagonal entries equal to 1. Then there exists a homomorphism $\Pi : B_0 \to St(N,A)$ with $\Phi\Pi = id$.*

Proof: Let $U \subset St(N,A)$ be the subgroup generated by all x_{ij}^a with $i < j, a \in A$. The restriction of Φ to U is obviously a surjective morphism onto B_0, because B_0 is generated by all e_{ij}^a with $i < j$. Moreover $\Phi\mid_U$ is injective by [24], Lemma 4.2.3. Thus $\Phi\mid_U : U \to B$ is an isomorphism and we may define Π as its inverse. □

Lemma 5 *: Let $N \in \mathbb{N}$. Let $A \subset \mathbb{C}$ be a subring with*
i) either $A = \mathbb{C}$,
ii) or A contains no N-th root of unity $\xi \neq 1$.
Let B be the subgroup of $SL(N,A)$ consisting of upper triangular matrices with all diagonal entries equal. Then there exists a homomorphism $\Pi : B \to St(N,A)$ with $\Phi\Pi = id$.

Proof: In case ii) we have $B = B_0$, thus the claim is the same as Lemma 4.

We consider case i). We fix some primitive N-th root of unity $\xi \neq 1$. By surjectivity of Φ, we can fix some $\Pi(\xi\mathbb{I})$ with $\Phi\Pi(\xi\mathbb{I}) = \xi\mathbb{I}$. (For example, if $N = 2, \xi = -1$, one can choose $\Pi(-1) = x_{12}^1 x_{21}^{-2} x_{12}^1 x_{21}^{-2}$.)

Let $b \in B$. Since all diagonal entries are equal, $b \in B$ must be (uniquely) of the form $b = \eta b_0$ for some $b_0 \in B_0$ and some N-th root of unity η, which thus must be of the form $\eta = \xi^k$ for some $k \in \mathbb{Z}$. Then we define $\Pi(b) = \Pi(\xi\mathbb{I})^k\,\Pi(b_0)$, where $\Pi(b_0)$ is defined by Lemma 4.

We have to show that the so-defined Π is a homomorphism. Let $b_1 = \xi^{k_1}\Pi e_{ij}^a$ and $b_2 = \xi^{k_2}\Pi e_{kl}^b$. Then $\Pi(b_1)\Pi(b_2) = \Pi(b_1 b_2)$ is equivalent to the statement that $\Pi(\xi\mathbb{I})^{k_2}$ commutes with $\Pi\left(\Pi e_{ij}^a\right)$.

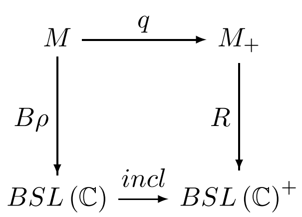
Let $A = \xi^{k_2}$ and $B = \Pi e_{ij}^a$. Then $A * B = [\Pi(A), \Pi(B)] \in K_2(\mathbb{C})$ (see e.g. [7], p.337/8 for the definition and the properties of $A * B$). We have $A^k = 1$, thus $A^k * B = 0$. But $A^k * B = (A * B)^k$ by bilinearity. By the Bass-Tate Theorem, $(K_2(F), *)$ is uniquely divisible for algebraically closed fields F. Thus $(A * B)^k = 0$ implies $A * B = 0$, that is, $\Pi(\xi\mathbb{I})^{k_2}$ and $\Pi\left(\Pi e_{ij}^a\right)$ commute. Hence Π is a homomorphism.

This implies in particular $\Phi\Pi\left(\xi^k b_0\right) = \Phi\Pi(\xi\mathbb{I})^k \Phi\Pi(b_0) = \xi^k b_0$, that is, $\Phi\Pi = id$. $\square$

Corollary 5 *: Let $B' \subset SL(N;\mathbb{C})$ be some maximal parabolic subgroup. Then there exists a homomorphism $\Pi' : B' \to St(N, \mathbb{C})$ with $\Phi\Pi' = id$.*

Proof: B' is conjugate to B: $B' = \gamma B \gamma^{-1}$ for some $\gamma \in SL(N;\mathbb{C})$. Fix some $y \in St(N;\mathbb{C})$ with $\Phi(y) = \gamma$ and define $\Pi'\left(\gamma b \gamma^{-1}\right) = y\Pi(b) y^{-1}$ for each $b \in B$. $\square$

Let M be a manifold with boundary such that its interior $M - \partial M$ is homeomorphic to a locally symmetric space $\Gamma\backslash G/K$ of finite volume. (We will actually throughout section 4.2. only require that M and each connected component of ∂M are aspherical.) Let $\rho : \pi_1 M \to SL(\mathbb{C})$ be a representation. (e.g. coming from a representation $G \to GL(N, \mathbb{C})$, which by perfectness of G has necessarily image in $SL(N;\mathbb{C})$). Let $Q_* : H_*\left(BSL(\mathbb{C})^+; \mathbb{Q}\right) \to H_*(BSL(\mathbb{C}); \mathbb{Q})$ be Quillen's isomorphism. To push forward the fundamental class $[M_+]$ one would like to have a map $R : M_+ \to BSL(\mathbb{C})^+$ such that the following diagram commutes up to homotopy:



If this is the case, then one can define $Q_* R_* [M_+] \in H_*(BSL(\mathbb{C}); \mathbb{Q})$ and thus obtain an element in $K_*(\mathbb{C}) \otimes \mathbb{Q}$.

Assuming Corollary 5, the construction of R will now be completely analogous to the construction in [7], Section 8.1.

Definition 4 *: A homomorphism $\rho : \pi_1 M \to SL(N;\mathbb{C})$ is said to preserve parabolics if, for each connected component $\partial_0 M$ of ∂M, $\pi_1 \partial_0 M$ is mapped into some maximal parabolic subgroup of $SL(N;\mathbb{C})$.*

Lemma 6 *: Let M be a manifold with boundary such that M and each connected component of ∂M are aspherical. Let $\rho : \pi_1 M \to SL(N, \mathbb{C})$ be a homomorphism preserving parabolics. Then there exists a continuous map $R : M_+ \to BSL(\mathbb{C})^+$ such that $R \circ q = incl \circ B\rho$.*

Proof: Let F be the homotopy fiber of $BSL(\mathbb{C}) \to BSL(\mathbb{C})^+$. It is well-known (e.g. [7], p.336) that $\pi_1 F$ is isomorphic to the Steinberg group $St(\mathbb{C})$.

Let $\partial_0 M$ be some connected component of ∂M. By assumption, ρ maps $\pi_1\partial_0 M$ to a maximal parabolic subgroup B'. It follows then from Corollary 5 that there is a homomorphism $\tau : \pi_1\partial_0 M \to St\,(N;\mathbb{C})$ such that $\Phi\tau = \rho\mid_{\pi_1\partial_0 M}$.

By assumption, $\partial_0 M$ is aspherical. Hence τ is induced by some continuous mapping $g_0 : \partial_0 M \to F$, and the diagram

$$\begin{array}{ccc} \partial_0 M & \xrightarrow{i_0} & M \\ {\scriptstyle g_0}\downarrow & & \downarrow{\scriptstyle B\rho} \\ F & \xrightarrow{j} & BSL\,(\mathbb{C}) \end{array}$$

commutes up to some homotopy H_t.

This construction can be repeated for all connected components $\partial_0 M, \partial_1 M, \ldots, \partial_s M$ of ∂M. For each $r = 0, 1, \ldots, s$ we get a continuous map $g_r : \partial_r M \to F$ such that $jg_r \sim (B\rho)\, i_r$. Altogether, we get a continuous map $g : \partial M \to F$ such that jg is homotopic to $(B\rho)\, i$.

By [7], Lemma 8.1. this implies the existence of R. $\square$

4.3 Nontriviality

We show nontriviality of the constructed element by applying the Borel regulator. In this section we will not consider arbitrary (parabolics-preserving) representations $\pi_1 M \to SL\,(N,\mathbb{C})$, as we did in the previous section, but only those coming from a representation $G \to SL\,(N;\mathbb{C})$ for an inclusion $\pi_1 M = \Gamma \subset G$.

We remark that, for a subring $A \subset \mathbb{C}$, to each element $\gamma_A\,(M) \in K_n\,(A) \otimes \mathbb{Q}$ we also get an element $\gamma\,(M) \in K_n\,(\mathbb{C}) \otimes \mathbb{Q}$. Of course it suffices to prove $r_n\,(\gamma\,(M)) = vol\,(M)$ for the latter, as this implies $r_n\,(\gamma_A\,(M)) = vol\,(M)$. Thus we will confine to the case $A = \mathbb{C}$ in this section.

Theorem 3 *: Let M be a compact, orientable, n-dimensional manifold with boundary ∂M, whose interior is a locally symmetric space $\Gamma\backslash G/K$ of finite volume.*

Let $B \subset G$ be a maximal parabolic subgroup and $\Phi : S \to G$ the universal central extension. Assume that there exists a homomorphism $\Pi : B \to S$ with $\Phi\Pi = id_B$.

Let $\rho : \pi_1 M \to GL\,(N,\mathbb{C})$ be a representation, preserving parabolics. Then there exists a continuous mapping $R_\rho : M_+ \to BG^+$ such that: $\gamma\,(M) := I_n^{-1} pr_n\,(B\rho)_* \, Q_n R_n\,[M_+] \in K_n\,(\mathbb{C}) \otimes \mathbb{Q}$ satisfies

$$r_n\,(\gamma\,(M)) = c_\rho vol\,(M)\,.$$

Proof:

In principle, the existence of a continuous map $R : M_+ \to BG^+$ can be proved by literally the same argument as in the proof of Lemma 6 (which handled the case $G = SL\,(N,\mathbb{C})$). However, it turns out that (to compute the volume) we will need to be more explicit in the choice of a homotopy $H : B\Gamma \times I \to BG$.

We note that it suffices to prove $< v_n, Q_n R_n\,[M_+] >= vol\,(M)$ because this implies $r_n\,(\gamma\,(M)) = c_\rho vol\,(M)$ by the same argument as in the proof of Theorem 2.

Let $p : M \to M_+$ be the canonical projection, mapping each component of ∂M into one point. Let $\sum_{i=1}^r \sigma_i \in C_n(M, \partial M)$ be a (relative) triangulation, which then represents the fundamental class $[M, \partial M] \in H_n(M, \partial M; \mathbb{R})$. We may assume that all vertices of all σ_i are in ∂M and there is only one vertex in each connected component of ∂M. $\sum_{i=1}^r p(\sigma_i) \in C_n(M_+)$ represents the fundamental class $[M_+] \in H_n(M_+; \mathbb{R})$. (This follows from the well-known isomorphism $H_n(M, \partial M; \mathbb{R}) \cong H_n(M_+; \mathbb{R})$.) M_+ can be considered as a compactification of $int(M)$ and inherits a volume form $dvol$ on $int(M) \subset M_+$. Defining $w_n(\sigma) := \int_{\Delta^n} \sigma^* dvol$ (where $M_+ - int(M)$ is to be considered as a null set) we get a cocycle w_n with $vol(M) = < w_n, \sum_{i=1}^r p(\sigma_i) > = < w_n, \sum_{i=1}^r str(p(\sigma_i)) >$. On the other hand, the homotopy equivalence $h : M \to B\Gamma$ can be defined as follows. Perform a homotopy equivalence on M such that all boundary components have exactly one point x_0 in common, which is the above common vertex of the σ_i. Then each simplex σ_i determines an n-tuple of elements $\left(1, \gamma_1^i, \ldots, \gamma_n^i\right) \in \left(\pi_1(M, x_0)\right)^{n+1}$, namely the edges from the 0-th to the j-th vertex for $j = 1, \ldots, n$. $< v_n, \left(1, \gamma_1^i, \ldots, \gamma_n^i\right) >$ is the volume of the straight simplex (with all vertices in one point) whose edges from the 0-th to the j-th vertex are represented by γ_j^i, for $j = 1, \ldots, n$. Hence, $< v_n, \left(1, \gamma_1^i, \ldots, \gamma_n^i\right) > = vol(str(p(\sigma_i)))$. This implies $< v_n, \sum_{i=1}^r \left(1, \gamma_1^i, \ldots, \gamma_n^i\right) > = vol(M)$.

R is constructed with the help of a map $g : \partial M \to F$ and a homotopy $H : B\Gamma \times I \to BG$. Let $H_1 = H(., 1)$ and $H_0 = H(., 0)$. We have just shown that $< v_n, (H_0)_n \left(\sum_{i=1}^r \left(\gamma_0^i, \ldots, \gamma_n^i\right)\right) > = vol(M)$. A representative of $Q_n R_n [M_+]$ is given by $H_1\left(\sum_{i=1}^r \left(\gamma_0^i, \ldots, \gamma_n^i\right)\right)$. Thus, to prove $< v_n, Q_n R_n [M_+] > = vol(M)$, it suffices to prove that $< v_n, (H_1)_n \left(\sum_{i=1}^r \left(\gamma_0^i, \ldots, \gamma_n^i\right)\right) = vol(M)$.

The problem is now that v_n is a cocycle, but $\sum_{i=1}^r \left(\gamma_0^i, \ldots, \gamma_n^i\right)$ is only a relative cycle. Therefore a homotopy may change the value of $< v_n, \sum_{i=1}^r a_i \left(\gamma_0^i, \ldots, \gamma_n^i\right) >$. To guarantee that, for a suitably chosen homotopy H, this is not the case we have to look at an explicit description of H_t.

To this behalf, we first need an explicit description of the homotopy fibration and of its volume cocycle. Recall from Quillen's construction that BG^+ is constructed out of BG by attaching cells, in particular $BG \subset BG^+$. Let

$$PBG = \left\{w : [0,1] \to BG^+ \text{ continuous }, w(0) \in BG\right\}.$$

It is well-known that the inclusion $BG \to PBG$ is a homotopy equivalence, and that the endpoint map $p : PBG \to BG^+$, given by $w \to w(1)$, is a fibration with fiber

$$F_G = \left\{w : [0,1] \to BG^+ \text{ continuous }, w(0) \in BG, w(1) = *\right\}$$

for some point $* \in BG$.

Recall from section 2.2. that v_n is defined as a cellular cocycle on BG. Of course, there are isomorphisms

$$H^n_{cell}(BG) \cong H^n_{sing}(BG) \cong H^n_{sing}(PBG),$$

thus there must be a singular cocycle $\overline{v}_n \in C_n^{sing}(PBG)$ whose cohomology class corresponds to the cohomology class of v_n under this isomorphism. To evaluate the volume cocycle on the homotoped chain, we will need an explicit description of $\overline{v}_n$.

Fix some $\tilde{x}_0 \in G/K$. There is a continuous mapping $s : BG \to G/K$, which maps each simplex $(g_0, \ldots, g_n)$ to the straight simplex with vertices $g_0\tilde{x}_0, \ldots, g_n\tilde{x}_0$. Let $dvol$

be the volume form of G/K, then $\sigma \to \int_{\Delta^n} \sigma^* dvol$ is a singular cocycle v_n^{sing} extending v_n. Define $\overline{v}_n := p^* v_n^{sing}$ for the projection $p : PBG \to BG$ given by $p(w) = w(0)$. Since p is a homotopy equivalence, $\overline{v}_n$ indeed corresponds to v_n under the above isomorphism of cohomology groups.

$$
\begin{array}{ccccc}
\partial_0 M & \longrightarrow & M & \longrightarrow & M_+ \\
\downarrow & & \downarrow & & = \\
BT & \longrightarrow & B\Gamma & \longrightarrow & M_+ \\
Bi \downarrow & & Bi \downarrow & & \\
BB & \longrightarrow & BG & & \\
g \downarrow & & H_1 \downarrow & & \\
F_G & \longrightarrow & PBG & \longrightarrow & BG^+
\end{array}
$$

We come to the explicit construction of the homotopy. It suffices to construct it for each connected component of ∂M, to be denoted $\partial_0 M$. $T := \pi_1 \partial_0 M \subset \Gamma$ is a subgroup of some maximal parabolic subgroup $B' \subset G$. Let $\Phi : S \to G$ be the universal central extension. By assumption we have a homomorphism $\Pi : B \to S$. Conjugating B' into B we get $\Pi' : B' \to S$.

Let

$$F_{B'} := \{ w \in F : w(0) \in BB' \subset BG \} .$$

$F_{B'} \to PBB' \to BG^+$ is a fibration. It follows from the long exact homotopy sequence that

$$\pi_1 F_{B'} = S.$$

Since $\partial_0 M \simeq BT$ is aspherical, we can thus choose a homotopy such that BT is mapped to $F_{B'} \subset F_G$.

Let $(B\Gamma)_0$ be the 0-skeleton of $B\Gamma$ and $(B\Gamma)_0 - (BT)_0$ the complement of $(BT)_0$ in $(B\Gamma)_0$. We note that $(BT \cup ((B\Gamma)_0 - (BT)_0)) \times I \cup B\Gamma \times \{0\}$ is a sub-CW-complex of $B\Gamma \times I$. It is well-known that the inclusion of a sub-CW-complex is a cofibration. By assumption we have a map $H_0 : B\Gamma \to PBG$. and a homotopy

$$\overline{H} : BT \times I \to PBG$$

extending H_0 on $BT \times \{0\}$ (and such that $\overline{H}(BT \times \{1\}) \subset F_G$). Since BT is aspherical and the image of its fundamental group is contained in B' we can actually choose H_0 to have image in PBB'.

Moreover, we define $\overline{H}:\left((B\Gamma)_0-(BT)_0\right)\times I\to PBG$ by $\overline{H}(g,t)=H_0(g)$. Thus we have constructed a continuous mapping $\overline{H}:\left(BT\cup(B\Gamma)_0\right)\times I\cup B\Gamma\times\{0\}\to PBG$. By the cofibration property we may extend $\overline{H}$ to a continuous mapping

$$H:B\Gamma\times I\to PBG.$$

We will now use the special properties of H to show that
$<\overline{v}_n,H_1\left(\sum_{i=1}^r\left(\gamma_0^i,\ldots,\gamma_n^i\right)\right)>=<p^*s^*dvol,H_1\left(\sum_{i=1}^r\left(\gamma_0^i,\ldots,\gamma_n^i\right)\right)>=vol\left(M\right)$.

If $\gamma_0^i,\ldots,\gamma_n^i\not\in T$, then, by construction, $H_1\left(\gamma_0^i,\ldots,\gamma_n^i\right)$ is a singular simplex in PBG which has vertices in (the constant paths) $\gamma_0^i,\ldots,\gamma_n^i$. Thus $pH_1\left(\gamma_0^i,\ldots,\gamma_n^i\right)$ has vertices $\gamma_0^i,\ldots,\gamma_n^i$. This implies that $spH_1\left(\gamma_0^i,\ldots,\gamma_n^i\right)$ is the straight simplex σ_i in G/K with vertices $\gamma_0x_0,\ldots,\gamma_nx_0$. In particular $vol\left(spH_1\left(\gamma_0^i,\ldots,\gamma_n^i\right)\right)=vol\left(\sigma_i\right)$.

If some of the γ_k^i is an element of T, then $H_1\left(\gamma_k^i\right)\in F_{B'}$, hence $pH_1\left(\gamma_k^i\right)\in BB'$. Thus $\gamma_k^i\tilde{x}_0$ and $H_1\left(\gamma_k^i\right)\tilde{x}_0$ are mapped to the same point in ∂M_+. Thus, $vol\left(spH_1\left(\gamma_0^i,\ldots,\gamma_n^i\right)\right)=vol\left(\sigma_i\right)$ also in this case.

Hence

$$<\overline{v}_n,H_1\left(\gamma_0^i,\ldots,\gamma_n^i\right)>=vol\left(\sigma_i\right).$$

This implies $<\overline{v}_n,H_1\sum_{i=1}^r\left(\gamma_0^i,\ldots,\gamma_n^i\right)>=vol\left(M\right)$. □

5 Examples

5.1 K-theory of $\mathbb{Z}$

Let $N\in\mathbb{N}$. It follows from the Selberg Lemma that $SL\left(N,\mathbb{Z}\right)$ contains a torsion-free subgroup Γ of finite index.

Then $\widehat{M}:=\Gamma\backslash SL\left(N,\mathbb{R}\right)/SO\left(N\right)$ is a (noncompact) odd-dimensional locally symmetric space of finite volume.

Even though $M=SL\left(N,\mathbb{Z}\right)\backslash SL\left(N,\mathbb{R}\right)/SO\left(N\right)$ is not a manifold, we may of course apply our construction to the finite covering $\widehat{M}$ and then divide the so-obtained element of $K_*\left(\mathbb{Z}\right)\otimes\mathbb{Q}$ by the degree of the covering to obtain an invariant of M.

Let $N\equiv 3\ mod\ 4$. $A=\mathbb{Z}$ does not contain an N-th root of unity, thus, in view of Lemma 5, the assumptions of Theorem 3 are satisfied. Hence, for each irreducible representation ρ of $SL\left(N,\mathbb{R}\right)$ we get a nontrivial element $\gamma\left(M\right)\in K_{\frac{1}{2}(N-1)(N+2)}\left(\mathbb{Z}\right)\otimes\mathbb{Q}$ with $r_n\left(\gamma\left(M\right)\right)=c_\rho vol\left(M\right)$.

For example, for $M=SL\left(3,\mathbb{Z}\right)\backslash SL\left(3,\mathbb{R}\right)/SO\left(3\right)$ and $\rho:SL\left(3,\mathbb{R}\right)\to GL\left(3,\mathbb{C}\right)$ the inclusion, we get $\gamma\left(M\right)\in K_5\left(\mathbb{Z}\right)\otimes\mathbb{Q}$ with $r_n\left(\gamma\left(M\right)\right)=\frac{1}{144\sqrt{2}\pi^3}vol\left(M\right)$ by the computation at the end of section 3.4.

Similarly one can construct elements in the K-theory of other number rings A which satisfy the assumption i) of Lemma 5.

5.2 Examples using hyperbolic manifolds

The examples in section 5.1. have been noncompact manifolds of finite volume. Compact examples can e.g. be obtained by Borel's construction of arithmetic hyperbolic n-manifolds using quadratic forms.

Let n be odd. Let u be an algebraic integer such that all roots of its minimal polynomial have multiplicity 1 and are real and negative (except u). Assume moreover that $(0,\ldots,0)$ is the only integer solution of $x_1^2+\ldots+x_n^2-ux_{n+1}^2=0$. Let $\widehat{\Gamma}\subset GL(n+1)$ be the group of maps preserving $x_1^2+\ldots+x_n^2-ux_{n+1}^2$. It is isomorphic to a discrete cocompact subgroup of $SO(n,1)$. By Selberg's lemma, it contains a torsionfree cocompact subgroup Γ. The compact manifold $M:=\Gamma\backslash\mathbb{H}^n$ gives us a nontrivial element $\gamma(M)\in K_n(\mathbb{Z}[u])\otimes\mathbb{Q}$.

The case $n=3$ has been discussed to some extent in [22]. If M is a hyperbolic 3-manifold, then $\pi_1 M$ can be conjugated to be contained in $SL(2,F)$, where F is an at most quadratic extension of the trace field ([20]), thus one gets an element in $K_3(F)\otimes\mathbb{Q}$. In [22], section 9, some examples of this construction are given. (The discussion in [22] is about elements in $B(F)\otimes\mathbb{Q}$ for the Bloch group $B(F)$, but of course analogously one is getting elements in $K_3(F)\otimes\mathbb{Q}$ associated to the respective manifolds.) For example (cf. [22], section 9.4.) for a number field F with just one complex place there exists a hyperbolic 3-manifold with this field as invariant trace field and thus giving an element in $K_3(F)\otimes\mathbb{Q}$.

5.3 Representation varieties

If $\psi:\pi_1 M\to G$ is a homomorphism, one can construct a ψ-equivariant map $f:\widetilde{M}\to G/K$ which is unique up to homotopy. In particular, $vol(\psi):=\int_F f^*dvol$, for a fundamental domain $F\subset\widetilde{M}$, is well-defined. Literally the same proof as for Theorem 1 shows

$$<v_n,(B\psi)_*[M]>=vol(\psi).$$

Thus, if $\rho:G\to GL(N,\mathbb{C})$ is a representation with $\rho^*b_n\neq 0$, and M possesses a fundamental class $[M]\in H_n(M;\mathbb{Z})$, one does again get a nontrivial element $\gamma(\psi):=I_n^{-1}pr_n(B\rho)_n(B\psi)_n[M]\in K_n(\mathbb{C})\otimes\mathbb{Q}$. Of course, continuous families of representations give us constant images in K-theory, because already $(B\psi)_n[M]\in H_n(G;\mathbb{Z})$ is constant. Thus we actually get a map from the set of connected components of the representation variety, $\pi_0 Rep(\pi_1 M,G)$, to $K_n(\mathbb{C})\otimes\mathbb{Q}$. We note that this map is *not* constant. This follows, for example, from the volume rigidity theorem (which for hyperbolic manifolds has been proved by Thurston and Dunfield and in the higher rank case is a consequence of Margulis superrigidity theorem) which states that elements of the component of $Rep(\pi_1 M,G)$ that contains the discrete representation are the only representations of maximal volume.

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