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THE COEFFICIENTS OF MULTIVALENT CLOSE-TO-CONVEX FUNCTIONS

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1. **Introduction.** Let $S(p)$ denote the class of functions which are regular and star-like of order p in the unit disk E, $|z|<1$, [2], [4]. A function

$$f(z) = a_1 z + a_2 z^2 + \cdots \qquad (|z| < 1)$$

is a member of $S(p)$ if and only if there exists a positive number ρ such that for $\rho<|z|<1$

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > 0, \qquad \int_0^{2\pi} \operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) d\theta = 2p\pi.$$

We shall say that a function $F(z)$ is in $\mathcal{K}(p)$, if it is regular in E and there exists a function $f(z)$ in $S(p)$ and a positive number ρ such that for $\rho<|z|<1$

$$\operatorname{Re}\left(\frac{zF'(z)}{f(z)}\right) > 0.$$

A function in $\mathcal{K}(p)$ is at most p-valent in E [5].

Goodman [1] has conjectured that if $f(z) = a_1z+a_2z^2+ \cdots$ $(|z|<1)$ is p-valent in E, then

$$(1.1) \quad |a_n| \leqq \sum_{t=1}^{p} \frac{2t(n+p)!}{(p+t)!(p-t)!(n-p-1)!(n^2-t^2)} |a_t|.$$

Inequality (1.1) reduces to the well-known Bieberbach conjecture when $p=1$. The conjecture was proven by Goodman and Robertson [3] for a function in $S(p)$, in case all its coefficients are real and by Robertson [7], in case $a_1=a_2= \cdots =a_{p-2}=0$, the remaining coefficients being complex. The author [5] proved (1.1) for $n=p+1$ for functions in $\mathcal{K}(p)$, no restrictions being made on the coefficients. In this paper, we will prove (1.1) for functions of the class $\mathcal{K}(p)$ for the case $a_1=a_2= \cdots =a_{p-2}=0$, the remaining coefficients being complex. The case $p=2$ of our proof gives (1.1) for the entire class $\mathcal{K}(2)$. Inequality (1.1) is known to be true for the class $\mathcal{K}(1)$ [6].

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2. **Preliminary lemmas.** The following lemma is stated in greater generality than needed. We will make use only of the special case $s=1$.

LEMMA 1. *Let*

$$P(z) = c_0 + c_1 z + c_2 z^2 + \cdots \qquad (|z| < 1)$$

be regular and satisfy the condition $\operatorname{Re}(P(z)) > 0$ *in* E, *then for* $n \geqq 2$ *and* $s \geqq 1$

$$\left| \frac{c_n}{c_0} - \frac{c_s c_{n-s}}{c_0^2} \right| \leqq 2 \left| \frac{\operatorname{Re} c_0}{c_0} \right| \leqq 2.$$

These inequalities are sharp for all n and for all s, equality being attained for each n and for each s by the function

$$P(z) = (\operatorname{Re} c_0) \left(\frac{1+z}{1-z} \right) + i \operatorname{Im} c_0 \qquad (\operatorname{Re} c_0 > 0).$$

PROOF. Since any function $P(z)$, regular in E and satisfying $\operatorname{Re} P(z) > 0$ and $P(0) = c_0$ in E, is the limit of a sequence of functions of the form

$$P(z) = (\operatorname{Re} c_0) \sum_{k=1}^{m} \lambda_k \frac{1 + e^{it_k} z}{1 - e^{it_k} z} + i \operatorname{Im} c_0 \tag{2.1}$$

where $\lambda_k \geqq 0$, $1 \leqq k \leqq m$, and $\sum_{k=1}^{m} \lambda_k = 1$, we need only prove the lemma for functions of the form (2.1). For such functions,

$$c_n = (\operatorname{Re} c_0) \sum_{k=1}^{m} 2\lambda_k e^{int_k} \qquad (n \geqq 1).$$

Therefore,

$$\begin{aligned} \left| \frac{c_n}{c_0} - \frac{c_s c_{n-s}}{c_0^2} \right| &= \left| \frac{\operatorname{Re} c_0}{c_0} \sum_{k=1}^{m} 2\lambda_k e^{int_k} - \left(\frac{\operatorname{Re} c_0}{c_0} \right)^2 \sum_{k=1}^{m} 2\lambda_k e^{ist_k} \sum_{k=1}^{m} 2\lambda_k e^{i(n-s)t_k} \right| \\ &= 2 \left| \frac{\operatorname{Re} c_0}{c_0} \right| \left| \sum_{k=1}^{m} \lambda_k (e^{int_k} - 2e^{ist_k} Q) \right| \qquad (2.2) \\ &= 2 \left| \frac{\operatorname{Re} c_0}{c_0} \right| \left| \sum_{k=1}^{m} \lambda_k B_k \right| \\ &\leqq 2 \left| \frac{\operatorname{Re} c_0}{c_0} \right| \sum_{k=1}^{m} \lambda_k |B_k| \end{aligned}$$

where

$$Q = \frac{\operatorname{Re} c_0}{c_0} \sum_{k=1}^{m} \lambda_k e^{i(n-s)t_k}$$

and

$$B_k = e^{int_k} - 2e^{ist_k}Q, \qquad 1 \leqq k \leqq m.$$

Thus we need only prove that

$$\sum_{k=1}^{m} \lambda_k \left| B_k \right| \leqq 1.$$

By the Schwarz inequality

$$\left(\sum_{k=1}^{m} \lambda_k \left| B_k \right| \right)^2 \leqq \sum_{k=1}^{m} \lambda_k \sum_{k=1}^{m} \lambda_k \left| B_k \right|^2 = \sum_{k=1}^{m} \lambda_k \left| B_k \right|^2.$$

Moreover, we have

$$\sum_{k=1}^{m} \lambda_k \left| B_k \right|^2 = 1 + 4\left| Q \right|^2 - 4 \operatorname{Re} \sum_{k=1}^{m} \lambda_k e^{-i(n-s)t_k} Q \tag{2.3}$$

$$\begin{aligned} \operatorname{Re} \sum_{k=1}^{m} \lambda_k e^{-i(n-s)t_k} Q &= \operatorname{Re} \sum_{k=1}^{m} \lambda_k e^{-i(n-s)t_k} \left(\frac{\operatorname{Re} c_0}{c_0} \sum_{q=1}^{m} \lambda_q e^{i(n-s)t_q} \right) \\ &= \operatorname{Re} \left(\frac{\operatorname{Re} c_0}{c_0} \sum_{q=1}^{m} \lambda_q e^{i(n-s)t_q} \sum_{k=1}^{m} \lambda_k e^{-i(n-s)t_k} \right) \\ &= \operatorname{Re} \left(\frac{\operatorname{Re} c_0}{c_0} \left| \sum_{q=1}^{m} \lambda_q e^{i(n-s)t_q} \right|^2 \right) \qquad (2.4) \\ &= \left(\frac{\operatorname{Re} c_0}{\left| c_0 \right|} \right)^2 \left| \sum_{q=1}^{m} \lambda_q e^{i(n-s)t_q} \right|^2 \\ &= \left| Q \right|^2. \end{aligned}$$

From (2.3) and (2.4) we obtain

$$\sum_{k=1}^{m} \lambda_k \left| B_k \right|^2 = 1.$$

This completes the proof of the lemma.

LEMMA 2. *For* $n \geqq p+2$

$$\frac{2(p-1)(n+p)!}{(2p-1)!(n-p-1)!(n^2-(p-1)^2)} + \frac{2(n+p)!}{(2p-1)!(n-p-1)!(n^2-p^2)} + \sum_{k=p+1}^{n-1}\left(\frac{4(p-1)(k+p)!}{(2p-1)!(k-p-1)!(k^2-(p-1)^2)} + \frac{4(k+p)!}{(2p-1)!(k-p-1)!(k^2-p^2)}\right) + 2 = \frac{2n(n+p)!}{(2p-1)!(n-p-1)!(n^2-(p-1)^2)} \tag{2.5}$$

and

$$\frac{2p(n+p)!}{(2p)!(n-p-1)!(n^2-p^2)} + \sum_{k=p+1}^{n-1}\left(\frac{4p(k+p)!}{(2p)!(k-p-1)!(k^2-p^2)}\right) + 2 = \frac{2n(n+p)!}{(2p)!(n-p-1)!(n^2-p^2)}. \tag{2.6}$$

PROOF. Denote the left side of (2.5) by A_n. Both sides of (2.5) are equal to $(8/3)p(p+1)(p+2)$ when $n=p+2$. Assume the equality is true for a particular value of $n \geqq p+2$. Replacing n by $n+1$, the left side of (2.5) can be written in the form

$$\frac{2(p-1)(n+p+1)!}{(2p-1)!(n-p)!((n+1)^2-(p-1)^2)} + \frac{2(n+p+1)!}{(2p-1)!(n-p)!((n+1)^2-p^2)} + \frac{2(p-1)(n+p)!}{(2p-1)!(n-p-1)!(n^2-(p-1)^2)} + \frac{2(n+p)!}{(2p-1)!(n-p-1)!(n^2-p^2)} + A_n. \tag{2.7}$$

Replacing A_n in (2.7) by the right side of (2.5), combining the third and fifth terms and simplifying, (2.7) can be written in the form

$$\frac{2(p-1)(n+p+1)!}{(2p-1)!(n-p)!((n+1)^2-(p-1)^2)}$$
$$+\frac{2(n+p)!}{(2p-1)!(n-p-1)!(n-p+1)}$$
$$+\frac{2(n+p)!}{(2p-1)!(n-p)!(n-p+1)}+\frac{2(n+p-1)!}{(2p-1)!(n-p)!}$$
$$=\frac{2(p-1)(n+p+1)!}{(2p-1)!(n-p)!((n+1)^2-(p-1)^2)}$$
$$+\frac{2(n+p)!}{(2p-1)!(n-p-1)!(n-p+1)}\left[1+\frac{1}{n-p}\right]$$
$$+\frac{2(n+p-1)!}{(2p-1)!(n-p)!}$$

(2.8)
$$=\frac{2(p-1)(n+p+1)!}{(2p-1)!(n-p)!((n+1)^2-(p-1)^2)}$$
$$+\frac{2(n+p)!}{(2p-1)!(n-p)!}+\frac{2(n+p-1)!}{(2p-1)!(n-p)!}$$
$$=\frac{2(p-1)(n+p+1)!}{(2p-1)!(n-p)!((n+1)^2-(p-1)^2)}$$
$$+\frac{2(n+p+1)(n+p-1)!}{(2p-1)!(n-p)!}$$
$$=\frac{2(n+p+1)!}{(2p-1)!(n-p)!((n+1)^2-(p-1)^2)}$$
$$\cdot\left[(p-1)+\frac{(n+1)^2-(p-1)^2}{n+p}\right]$$
$$=\frac{2(n+1)(n+p+1)!}{(2p-1)!(n-p)!((n+1)^2-(p-1)^2)}.$$

The last expression is the right side of equality (2.5) with n replaced by $n+1$. This then completes the proof of equality (2.5) by induction.

Denote the left side of (2.6) by B_n. Both sides of (2.6) are equal to $(p+2)(2p+1)$ when $n=p+2$. Assume (2.6) is true for some $n \geqq p+2$. Replacing n by $n+1$, the left side of (2.6) can be written in the form

$$\frac{2p(n+p+1)!}{(2p)!(n-p)!((n+1)^2-p^2)}+\frac{2p(n+p)!}{(2p)!(n-p-1)!(n^2-p^2)}+B_n. \tag{2.9}$$

Replacing B_n by the right side of (2.6), (2.9) can be written in the form

$$\begin{aligned} &\frac{2p(n+p+1)!}{(2p)!(n-p)!((n+1)^2-p^2)}+\frac{2(n+p)(n+p)!}{(2p)!(n-p-1)!(n-p)(n+p)} \\ &=\frac{2p(n+p+1)!}{(2p)!(n-p)!((n+1)^2-p^2)}+\frac{2(n+p)!((n+1)^2-p^2)}{(2p)!(n-p)!((n+1)^2-p^2)} \\ &=\frac{2p(n+p+1)!}{(2p)!(n-p)!((n+1)^2-p^2)}\left[1+\frac{n+1-p}{p}\right] \\ &=\frac{2(n+1)(n+p+1)!}{(2p)!(n-p)!((n+1)^2-p^2)}\,. \end{aligned} \tag{2.10}$$

The last expression is the right side of (2.6) with n replaced by $(n+1)$. This then completes the proof of (2.6) by induction.

3. Coefficient bounds.

THEOREM. *Let*

$$F(z)=\sum_{n=p-1}^{\infty}a_nz^n \qquad (|z|<1)$$

be a member of $\mathcal{K}(p)$, *then for* $n\geqq p+1$

$$|a_n| \leqq \sum_{t=p-1}^{p}\frac{2t(n+p)!}{(p+t)!(n-p-1)!(n^2-t^2)}\,|a_t| \tag{3.1}$$

and these inequalities are sharp in both the variables $|a_{p-1}|$ *and* $|a_p|$.

PROOF. Since the author [**5**] has already proven (3.1) for $n=p+1$, we will assume in what follows that $n\geqq p+2$. We may assume without loss of generality that $F(z)$ is regular on $|z|=1$. Then [**5**, pp. 176–177] there exists $f(z)=b_{p-1}z^{p-1}+b_pz^p+\cdots$ regular and star-like for $|z|\leqq 1$ such that $\mathrm{Re}(zF'(z)/f(z))>0$ ($|z|\leqq 1$). Let

$$\frac{zF'(z)}{f(z)}=P(z)=\sum_{n=0}^{\infty}c_nz^n.$$

Comparing coefficients we obtain for $n\geqq(p-1)$

$$na_n=\sum_{k=p-1}^{n}b_kc_{n-k}. \tag{3.2}$$

Considering (3.2) for the cases $n=p-1$ and $n=p$ and solving for b_{p-1} and b_p in terms of a_{p-1} and a_p, we obtain

$$b_{p-1} = \frac{(p-1)a_{p-1}}{c_0}, \qquad b_p = \frac{pa_p}{c_0} - \frac{(p-1)a_{p-1}c_1}{c_0^2}. \tag{3.3}$$

Combining (3.2) and (3.3), we obtain

$$\begin{aligned} na_n &= c_0 b_n + \sum_{k=p+1}^{n-1} b_k c_{n-k} \\ &\quad + \left(\frac{c_{n-p+1}}{c_0} - \frac{c_1 c_{n-p}}{c_0^2}\right)(p-1)a_{p-1} + \frac{pa_p c_{n-p}}{c_0}. \end{aligned} \tag{3.4}$$

Using Lemma 1 with $s=1$ and the well-known inequality $|c_k| \leqq 2|c_0|$, we obtain

$$\begin{aligned} n|a_n| &\leqq |c_0||b_n| + \sum_{k=p+1}^{n-1} 2|b_k||c_0| \\ &\quad + 2(p-1)|a_{p-1} + 2p|a_p|. \end{aligned} \tag{3.5}$$

Since $f(z)$ is star-like of order p in E with $b_1 = b_2 = \cdots = b_{p-2} = 0$, it follows from [7] that for $k \geqq p+1$

$$\begin{aligned} |b_k| &\leqq \frac{2(p-1)(k+p)!}{(2p-1)!(k-p-1)!(k^2-(p-1)^2)} |b_{p-1}| \\ &\quad + \frac{2p(k+p)!}{(2p)!(k-p-1)!(k^2-p^2)} |b_p| \\ &\leqq \frac{2(p-1)(k+p)!}{(2p-1)!(k-p-1)!(k^2-(p-1)^2)} \frac{(p-1)|a_{p-1}|}{|c_0|} \\ &\quad + \frac{2p(k+p)!}{(2p)!(k-p-1)!(k^2-p^2)} \\ &\quad \cdot \left(\frac{p|a_p|}{|c_0|} + \frac{2(p-1)|a_{p-1}|}{|c_0|}\right) \\ &= \left(\frac{2(p-1)(k+p)!}{(2p-1)!(k-p-1)!(k^2-(p-1)^2)} \right. \\ &\quad \left. + \frac{2(k+p)!}{(2p-1)!(k-p-1)!(k^2-p^2)}\right) \frac{(p-1)|a_{p-1}|}{|c_0|} \\ &\quad + \left(\frac{2p(k+p)!}{(2p)!(k-p-1)!(k^2-p^2)}\right) \frac{p|a_p|}{|c_0|}. \end{aligned} \tag{3.6}$$

Combining (3.5), (3.6) and Lemma 2

$$n|a_n| \leqq \frac{2n(p-1)(n+p)!}{(2p-1)!(n-p-1)!(n^2-(p-1)^2)}|a_{p-1}| + \frac{2np(n+p)!}{(2p)!(n-p-1)!(n^2-p^2)}|a_p|.$$

Dividing the last inequality by n, we obtain (3.1). Inequality (3.1) is known to be sharp [3] for the class of functions in $S(p)$ whose power series have real coefficients, which is a subclass of $\mathcal{K}(p)$.

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