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# MEROMORPHIC MULTIVALENT CLOSE-TO-CONVEX FUNCTIONS

BY  
A. E. LIVINGSTON<sup>(1)</sup>

**1. Introduction.** Recently the author [5] has discussed some properties of the class of  $p$ -valent regular close-to-convex functions, called  $\mathcal{K}(p)$ . It is the purpose of this paper to generalize some of these results to the meromorphic case.

Let  $f(z)$  be meromorphic for  $|z| < 1$  with  $q$  ( $1 \leq q \leq p$ ) poles at the origin and  $f(z) \neq 0$  for  $|z| < 1$ . We shall say that  $f(z)$  is in  $S_1^*(p)$  if there exists a  $\rho$  ( $0 < \rho < 1$ ) such that for  $z = re^{i\theta}$  ( $\rho < r < 1$ ),

$$(1.1) \quad \operatorname{Re} \left[ \frac{zf'(z)}{f(z)} \right] < 0$$

and

$$(1.2) \quad \int_0^{2\pi} d \arg f(z) = \int_0^{2\pi} \operatorname{Re} \left[ \frac{zf'(z)}{f(z)} \right] d\theta = -2p\pi.$$

We shall say that  $f(z)$  is in  $S_2^*(p)$  if it is regular on  $|z| = 1$  and if (1.1) and (1.2) hold for  $|z| = 1$ . If  $f(z)$  is in  $S_1^*(p)$ , there exists a  $\delta$  ( $0 < \delta < 1$ ) such that  $f(rz)$  is in  $S_2^*(p)$  if  $\delta < r < 1$ .

We set  $S^*(p) = S_1^*(p) \cup S_2^*(p)$  and say that a function in  $S^*(p)$  is starlike of order  $p$ .

Condition (1.2) along with the argument principle implies that a function in  $S^*(p)$  has exactly  $p$  poles in  $|z| < 1$ . It is easily seen that a function  $f(z)$ , meromorphic in  $|z| < 1$ , is in  $S^*(p)$  if and only if the function  $[f(z)]^{-1}$  is regular and  $p$ -valently starlike in  $|z| < 1$ . Since the reciprocal of a  $p$ -valent function is  $p$ -valent, a function in  $S^*(p)$  is  $p$ -valent in  $|z| < 1$ . Also, using the fact that a regular  $p$ -valent starlike function can be written as the  $p$ th power of a regular univalent starlike function, it is easily seen that a function in  $S^*(p)$  with  $p$  poles at the origin can be written as the  $p$ th power of a meromorphic univalent starlike function.

Let  $F(z)$  be meromorphic in  $|z| < 1$  with  $q$  ( $1 \leq q \leq p$ ) poles at the origin and with at most  $p$  poles in  $|z| < 1$ . We shall say that  $F(z)$  is in  $\mathcal{K}_1^*(p)$  if there exists a function in  $S^*(p)$  and a  $\rho$  ( $0 < \rho < 1$ ) such that for  $z = re^{i\theta}$  ( $\rho < r < 1$ )

$$(1.3) \quad \operatorname{Re} \left[ \frac{zF'(z)}{f(z)} \right] > 0.$$

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We shall say that  $F(z)$  is in  $\mathcal{K}_2^*(p)$  if  $F(z)$  is regular on  $|z| = 1$  and if there exists a function  $f(z)$  in  $S_2^*(p)$  such that (1.3) is satisfied for  $|z| = 1$ . If  $F(z)$  is in  $\mathcal{K}_1^*(p)$ , there exists a  $\delta$  ( $0 < \delta < 1$ ) such that  $F(rz)$  is in  $\mathcal{K}_2^*(p)$  if  $\delta < r < 1$ .

We set  $\mathcal{K}^*(p) = \mathcal{K}_1^*(p) \cup \mathcal{K}_2^*(p)$  and say that a function in  $\mathcal{K}^*(p)$  is close-to-convex of order  $p$ .

The class  $\mathcal{K}^*(1)$  was defined by Libera and Robertson [4] and Pommerenke [7]. It was shown in both papers that a function in  $\mathcal{K}^*(1)$  need not be univalent. To show that a function in  $\mathcal{K}^*(p)$  need not be  $p$ -valent, let  $F(z)$  be such that

$$\frac{zF'(z)}{z^{-p}} = \frac{1 + z^{2p}}{1 - z^{2p}} \quad (|z| < 1).$$

Then

$$F(z) = -\frac{1}{pz^p} + \frac{2}{p}z^{2p} + \frac{2}{3p}z^{3p} + \dots \quad (0 < |z| < 1).$$

If  $F(z)$  was  $p$ -valent, then

$$F(z^{1/p}) = -\frac{1}{pz} + \frac{2}{p}z + \dots \quad (0 < |z| < 1)$$

would be univalent, and so would

$$-pF(z^{1/p}) = \frac{1}{z} - 2z + \dots \quad (0 < |z| < 1).$$

But this is impossible, since the coefficient of  $z$  has modulus greater than 1. Thus  $F(z)$  is at least  $2p$ -valent.

Necessary and sufficient conditions for a function to be in  $\mathcal{K}^*(1)$  have been given in [4] and [7]. In §2 we obtain necessary conditions for a function  $F(z)$  to be in  $\mathcal{K}^*(p)$  and show that these conditions with the added assumptions of regularity on  $|z| = 1$  and  $F'(z) \neq 0$  in  $|z| \leq 1$  are sufficient.

Recently, Royster [8] has shown that if

$$f(z) = \sum_{n=-p}^{\infty} a_n z^n \quad (0 < |z| < 1)$$

is in  $S^*(p)$  then  $|a_n| = O(1/n)$ . In §3 we will extend this result to functions in  $\mathcal{K}^*(p)$  with  $p$  poles at the origin. This result was obtained for  $\mathcal{K}^*(1)$  by Libera and Robertson [4] and Pommerenke [7].

## 2. The class $\mathcal{K}^*(p)$ .

**THEOREM 1.** *If  $F(z)$  is in  $\mathcal{K}^*(p)$ , then there exists  $\rho$  ( $0 < \rho < 1$ ) such that for  $z = re^{i\theta}$  ( $\rho < r < 1$ )*

$$(2.1) \quad \int_0^{2\pi} d \arg d F(z) = \int_0^{2\pi} \frac{d}{d\theta} \arg \left[ r e^{i\theta} F'(r e^{i\theta}) \right] d\theta = -2p\pi$$

and for any  $\theta_1$  and  $\theta_2$  with  $0 \leq \theta_1 < \theta_2 \leq 2\pi$

$$(2.2) \quad \int_{\theta_1}^{\theta_2} d \arg d F(z) < \pi.$$

**Proof.** Suppose  $F(z)$  is in  $\mathcal{K}_1^*(p)$ . Then there exists  $f(z)$  in  $S^*(p)$  and  $\rho$  ( $0 < \rho < 1$ ) such that (1.1), (1.2) and (1.3) hold for  $\rho < |z| < 1$ .

Since  $\operatorname{Re} [zF'(z)/f(z)] > 0$  for  $|z| = r$  ( $\rho < r < 1$ ), we may define

$$\arg [zF'(z)/f(z)]$$

to be single valued and continuous for  $|z| = r$  and such that

$$(2.3) \quad \left| \arg \frac{zF'(z)}{f(z)} \right| < \frac{\pi}{2} \quad (|z| = r).$$

Furthermore, since  $zF'(z) \neq 0$  for  $|z| = r$ , we may define  $\arg [zF'(z)]$  to be single valued and continuous for  $|z| = r$ . Since  $f(z) = [f(z)/zF'(z)] [zF'(z)]$ , we may define  $\arg [f(z)] = \arg [zF'(z)] - \arg [zF'(z)/f(z)]$  to be a single valued and continuous determination of  $\arg [f(z)]$  for  $|z| = r$ . Then

$$(2.4) \quad \left| \arg zF'(z) - \arg f(z) \right| = \left| \arg \frac{zF'(z)}{f(z)} \right| < \frac{\pi}{2} \quad (|z| = r).$$

It is easily seen that (2.4) implies

$$(2.5) \quad -\pi + \int_{\theta_1}^{\theta_2} d \arg f(z) < \int_{\theta_1}^{\theta_2} d \arg d F(z) < \pi + \int_{\theta_1}^{\theta_2} d \arg f(z)$$

for  $\theta_1 < \theta_2$  and  $|z| = r$ . Since  $f(z)$  is in  $S^*(p)$ ,

$$\int_{\theta_1}^{\theta_2} d \arg f(z) < 0 \quad (|z| = r).$$

Thus we obtain (2.2) for  $|z| = r$  from the right side of (2.5). Letting  $\theta_1 = 0$  and  $\theta_2 = 2\pi$  in (2.5) and noting that

$$\int_0^{2\pi} d \arg f(z) = -2p\pi$$

we obtain

$$(2.6) \quad -(2p + 1)\pi < \int_0^{2\pi} d \arg d F(z) < -(2p - 1)\pi.$$

However, the integral appearing in (2.6) is an integral multiple of  $2\pi$ . Thus (2.1) holds for  $|z| = r$ . Since  $r$  was arbitrary ( $\rho < r < 1$ ) (2.1) and (2.2) hold for  $\rho < |z| < 1$ .

If  $F(z)$  is in  $\mathcal{K}_2^*(p)$ , then the preceding argument with  $r = 1$  shows that (2.1) and (2.2) hold for  $|z| = 1$ . But since  $F(z)$  is regular near  $|z| = 1$ , we can show the existence of a  $\rho$  ( $0 < \rho < 1$ ) such that (2.1) and (2.2) hold for  $\rho < |z| \leq 1$ .

Using (2.1) and the argument principle we immediately obtain the following corollary.

**COROLLARY 1.** *If  $F(z)$  is in  $\mathcal{K}^*(p)$ , then  $F'(z)$  has at least  $(p + 1)$  poles in  $|z| < 1$  and if  $F'(z) \neq 0$  for  $|z| < 1$ , then  $F'(z)$  has exactly  $(p + 1)$  poles in  $|z| < 1$ .*

**THEOREM 2.** *Let  $F(z)$  be meromorphic in  $|z| < 1$  with  $q$  ( $1 \leq q \leq p$ ) poles at the origin. If  $F'(z) \neq 0$  for  $0 < |z| \leq 1$  and  $F(z)$  is regular on  $|z| = 1$  and if (2.1) and (2.2) hold for  $|z| = 1$ , then  $F(z)$  is in  $\mathcal{K}_2^*(p)$ .*

**Proof.** Consider the function  $G(z)$ , regular for  $|z| = 1$ , given by

$$G(z) = \int_0^z \frac{dz}{z^2 F'(z)} = b_q z^q + \dots$$

Since  $zF'(z) \neq 0$  for  $|z| = 1$  we may define  $\arg [zF'(z)]$  to be single valued and continuous for  $|z| = 1$ . Since  $zG'(z) = [zF'(z)]^{-1}$ , we may define  $\arg zG'(z) = -\arg zF'(z)$ . Thus, for  $|z| = 1$

$$\int_0^{2\pi} d \arg d G(z) = 2p\pi$$

and

$$\int_{\theta_1}^{\theta_2} d \arg d G(z) > -\pi \quad (\theta_1 < \theta_2).$$

The author has shown (Theorem 3 [5]) that under these conditions  $G(z)$  is in  $\mathcal{K}(p)$ . That is, there exists  $g(z)$ , regular for  $|z| \leq 1$  such that

$$\operatorname{Re} \left[ \frac{zg'(z)}{g(z)} \right] > 0 \quad (|z| = 1)$$

and

$$\operatorname{Re} \left[ \frac{zG'(z)}{g(z)} \right] > 0 \quad (|z| = 1).$$

The function  $f(z) = [g(z)]^{-1}$  is in  $S^*(p)$  and

$$\frac{zG'(z)}{g(z)} = zG'(z)f(z) = \frac{f(z)}{zF'(z)}.$$

Thus

$$\operatorname{Re} \left[ \frac{zF'(z)}{f(z)} \right] = \operatorname{Re} \left[ \frac{g(z)}{zG'(z)} \right] > 0 \quad (|z| = 1).$$

Therefore  $F(z)$  is in  $\mathcal{K}_2^*(p)$ .

Using the same procedure as above and by appealing to Theorem 2 [5], we may remove the condition of regularity on  $|z| = 1$ , if  $q = p$ . We thus have the following theorem.

**THEOREM 3.** *Let*

$$F(z) = \sum_{n=-p}^{\infty} a_n z^n \quad (0 < |z| < 1)$$

be meromorphic for  $|z| < 1$  and  $F'(z) \neq 0$ . If there exists a  $\rho$  ( $0 < \rho < 1$ ) such that (2.1) and (2.2) hold for  $\rho < |z| < 1$ , then  $F(z)$  is in  $\mathcal{K}_2^*(p)$ .

We will have need of the next lemma in what follows.

**LEMMA 1.** *Let  $F(z)$  be in  $\mathcal{K}_2^*(p)$ . Then, there exists a function*

$$f(z) = \sum_{n=-p}^{\infty} b_n z^n \quad (0 < |z| < 1) \quad (|b_{-p}| = 1),$$

in  $S_2^*(p)$ , such that

$$\operatorname{Re} \left[ \frac{zF'(z)}{f(z)} \right] > 0 \quad (|z| = 1).$$

**Proof.** There exists a function  $g(z)$  in  $S_2^*(p)$  with  $s$  poles ( $1 \leq s \leq p$ ) at the origin such that,

$$\operatorname{Re} \left[ \frac{zF'(z)}{g(z)} \right] > 0 \quad (|z| = 1).$$

The function  $g(z)$  has  $(p - s)$  nonzero poles in  $|z| < 1$ . Let  $\alpha_1, \alpha_2, \dots, \alpha_{p-s}$  be these poles and let

$$h(z) = z^{s-p} \prod_{i=1}^{p-s} (z - \alpha_i) (1 - \bar{\alpha}_i z)$$

and

$$f(z) = h(z)g(z) = \sum_{n=-p}^{\infty} c_n z^n \quad (0 < |z| < 1).$$

Since  $[zh'(z)/h(z)]$  is purely imaginary on  $|z| = 1$  and  $\operatorname{Re} [zg'(z)/g(z)] < 0$  for  $|z| = 1$ , then  $\operatorname{Re} [zf'(z)/f(z)] < 0$  for  $|z| = 1$ . Furthermore, since  $f(z)$  has  $p$  poles in  $|z| \leq 1$ , all of them at the origin, and since  $f(z) \neq 0$  in  $|z| \leq 1$ ,

$$\int_0^{2\pi} \operatorname{Re} \left[ \frac{e^{i\theta} f'(e^{i\theta})}{f(e^{i\theta})} \right] d\theta = -2p\pi.$$

Thus,  $f(z)$  in  $S_2^*(p)$ . Furthermore,

$$\frac{zF'(z)}{f(z)} = \frac{z^{p-s}zF'(z)}{\prod_{i=1}^{p-s}(z - \alpha_i)(1 - \bar{\alpha}_iz)g(z)}.$$

But  $z^{p-s}[\prod_{i=1}^{p-s}(z - \alpha_i)(1 - \bar{\alpha}_iz)]^{-1}$  is real and positive on  $|z| = 1$ . Therefore

$$\operatorname{Re} \left[ \frac{zF'(z)}{f(z)} \right] > 0 \quad (|z| = 1).$$

Replacing  $f(z)$  by  $1/|c_{-p}|f(z)$ , the proof of the lemma is completed.

**THEOREM 4.** *If  $F(z)$  is in  $\mathcal{H}^*(p)$  and has all its poles at the origin, then necessarily it has  $p$  poles there and  $F'(z) \neq 0$  for  $|z| < 1$ .*

**Proof.** Suppose

$$F(z) = \sum_{n=-q}^{\infty} a_n z^n \quad (0 < |z| < 1) \quad (1 \leq q \leq p).$$

There exists a  $\rho$  ( $0 < \rho < 1$ ) such that  $F(rz)$  is in  $\mathcal{H}_2^*(p)$  if  $\rho < r < 1$ . By Lemma 1, there exists

$$f(z) = \sum_{n=-p}^{\infty} b_n z^n \quad (0 < |z| < 1)$$

in  $S_2^*(p)$  such that

$$\operatorname{Re} \left[ \frac{rzF'(rz)}{f(z)} \right] > 0 \quad (|z| = 1).$$

Since  $f(z) \neq 0$  for  $|z| \leq 1$ ,

$$\frac{rzF'(rz)}{f(z)} = \sum_{n=p-q}^{\infty} c_n z^n$$

is regular for  $|z| \leq 1$ . Thus,

$$\operatorname{Re} \left[ \frac{rzF'(rz)}{f(z)} \right] > 0 \quad (|z| \leq 1).$$

Therefore, we must necessarily have  $q = p$  and  $F'(rz) \neq 0$  for  $|z| \leq 1$ . Thus,  $F'(z) \neq 0$  for  $|z| \leq r$ . Since  $r$  was arbitrary ( $\rho < r < 1$ ),  $F'(z) \neq 0$  for  $|z| < 1$ .

If  $F(z)$  has all its poles at the origin we may improve Lemma 1 by removing the condition of regularity on  $|z| = 1$ .

**LEMMA 2.** *Let*

$$F(z) = \sum_{n=-p}^{\infty} a_n z^n \quad (0 < |z| < 1)$$

be in  $\mathcal{H}^*(p)$ . Then there exists

$$f(z) = \sum_{n=-p}^{\infty} b_n z^n \quad (0 < |z| < 1) \quad (|b_{-p}| = 1)$$

in  $S^*(p)$  such that

$$\operatorname{Re} \frac{zF'(z)}{f(z)} > 0 \quad (|z| < 1).$$

**Proof.** There exists a  $\rho$  ( $0 < \rho < 1$ ) such that the function  $F_r(z) = F(rz)$  is in  $\mathcal{H}_2^*(p)$  if  $\rho < r < 1$ . Then by Lemma 1 there exists

$$f_r(z) = \sum_{n=-p}^{\infty} c_n z^n \quad (0 < |z| < 1) \quad (|c_{-p}| = 1)$$

in  $S_2^*(p)$ , such that

$$\operatorname{Re} \left[ \frac{zF'_r(z)}{f_r(z)} \right] > 0 \quad (|z| \leq 1).$$

Let  $r_i$  ( $\rho < r_i < 1$ ) be an increasing sequence tending to 1. The functions  $[f_{r_i}(z)]^{-1}$  are regular and  $p$ -valently starlike and have the moduli of their first  $p$  coefficients fixed. The class of regular and  $p$ -valently starlike functions with the moduli of their first  $p$  coefficients fixed forms a normal family of functions [1]. Thus, we can obtain a subsequence  $[f_{r_{i_k}}(z)]^{-1}$  tending uniformly in every closed subset of  $|z| < 1$  to a function  $f(z)$  regular and  $p$ -valently starlike and such that

$$f(z) = \sum_{n=p}^{\infty} d_n z^n \quad (|z| < 1) \quad (|d_p| = 1).$$

Since  $F_{r_{i_k}}(z)$  tends to  $F(z)$  as  $r_{i_k}$  tends to 1 and since

$$\operatorname{Re}[zF_{r_{i_k}}(z)[f_{r_{i_k}}(z)]^{-1}] > 0 \quad \text{for } |z| < 1$$

we have

$$\operatorname{Re} [zF'(z)f(z)] > 0 \quad \text{for } |z| < 1.$$

But

$$g(z) = [f(z)]^{-1} = \sum_{n=-p}^{\infty} b_n z^n \quad (0 < |z| < 1) \quad (|b_{-p}| = 1)$$

is in  $S^*(p)$  and

$$\operatorname{Re} \left[ \frac{zF'(z)}{g(z)} \right] = \operatorname{Re} [zF'(z)f(z)] > 0 \quad \text{for } |z| < 1.$$

**3. The coefficients of a function in  $\mathcal{H}^*(p)$ .** We will make use of the following lemma, proven by Royster [8] and the author [6].



LEMMA 3. *Let*

$$f(z) = \sum_{n=-p}^{\infty} b_n z^n \quad (0 < |z| < 1) \quad (|b_{-p}| = 1)$$

be in  $S^*(p)$ , then for  $n \geq 1$

$$|b_n| \leq \frac{2p}{(n+p)\sqrt{p}} \left( \sum_{k=-p}^{-1} |k| |b_k|^2 \right)^{1/2}.$$

The following lemma was proven for  $p = 1$  by Pommerenke [7].

LEMMA 4. *Let*

$$F(z) = \frac{1}{z^p} + \sum_{n=-(p-1)}^{\infty} a_n z^n \quad \text{and} \quad f(z) = \frac{e^{i\beta}}{z^p} + \sum_{n=-(p-1)}^{\infty} b_n z^n, \quad (0 < |z| < 1)$$

and let  $U(z) = \text{Re} [zF'(z)/f(z)]$ , then for  $r < 1$

$$(3.1) \quad na_n = -pe^{-i\beta}b_n + \frac{1}{\pi} \int_0^{2\pi} \left[ U(re^{i\theta}) \frac{e^{-in\theta}}{r^n} \right. \\ \left. \times \left[ f(re^{i\theta}) - \sum_{k=n}^{\infty} b_k (re^{i\theta})^k \right] d\theta. \right.$$

**Proof.** Let

$$\frac{zF'(z)}{f(z)} = -pe^{-i\beta} + \sum_{k=1}^{\infty} C_k z^k \quad (|z| < 1).$$

Then

$$\begin{aligned} \frac{-p}{z^p} + \sum_{n=-(p-1)}^{\infty} na_n z^n &= \left[ -pe^{-i\beta} + \sum_{k=1}^{\infty} C_k z^k \right] \left[ \frac{e^{i\beta}}{z^p} + \sum_{n=-(p-1)}^{\infty} b_n z^n \right] \\ &= \frac{-p}{z^p} - pe^{-i\beta} \left[ \sum_{n=-(p-1)}^{\infty} b_n z^n \right] + e^{i\beta} \left[ \sum_{k=1}^{\infty} C_k z^{k-p} \right] \\ &\quad + \sum_{n=-(p-2)}^{\infty} \left[ \sum_{k=1}^{n+p-1} C_k b_{n-k} \right] z^n. \end{aligned}$$

Thus, for  $n \geq 1$

$$(3.2) \quad na_n = -pe^{-i\beta}b_n + e^{i\beta}C_{p+n} + \sum_{k=1}^{n+p-1} C_k b_{n-k}.$$

Now

$$C_k = \frac{1}{r^k \pi} \int_0^{2\pi} U(re^{i\theta}) e^{-ik\theta} d\theta.$$

Substituting into (3.2), we obtain

$$\begin{aligned} na_n &= -pe^{-i\beta}b_n + \frac{1}{\pi} \int_0^{2\pi} \left[ U(re^{i\theta}) \frac{e^{-in\theta}}{r^n} \right] \\ &\quad \times \left[ \frac{e^{i\beta}}{r^p e^{ip\theta}} + \sum_{k=1}^{n+p-1} r^{n-k} e^{i(n-k)\theta} b_{n-k} \right] d\theta \\ &= -pe^{-i\beta}b_n + \frac{1}{\pi} \int_0^{2\pi} \left[ U(re^{i\theta}) \frac{e^{-in\theta}}{r^n} \right] \\ &\quad \times \left[ f(re^{i\theta}) - \sum_{k=n}^{\infty} b_k (re^{i\theta})^k \right] d\theta. \end{aligned}$$

THEOREM 5. *Let*

$$F(z) = \sum_{n=-p}^{\infty} a_n z^n \quad (0 < |z| < 1) \quad (a_{-p} \neq 0)$$

be in  $\mathcal{X}^*(p)$ , then  $|a_n| = O(n^{-1})$ .

**Proof.** We may assume without loss of generality that  $a_{-p} = 1$ . There exists, by Lemma 2,

$$f(z) = \frac{e^{i\beta}}{z^p} + \sum_{n=-(p-1)}^{\infty} b_n z^n \quad (0 < |z| < 1)$$

in  $S^*(p)$  such that

$$\left[ \operatorname{Re} \frac{zF'(z)}{f(z)} \right] > 0 \quad (|z| < 1).$$

Let  $U(z) = \operatorname{Re} [zF'(z)/f(z)]$ , then by a well-known result on harmonic functions,

$$\frac{1}{\pi} \int_0^{2\pi} U(re^{i\theta}) d\theta = 2U(0) = -2p \cos \beta \leq 2p.$$

By Lemma 4, we have for  $n \geq 1$

$$\begin{aligned} (3.3) \quad n|a_n| &\leq p|b_n| + \frac{1}{\pi r^n} \left| \int_0^{2\pi} U(re^{i\theta}) e^{-in\theta} \sum_{k=n}^{\infty} b_k (re^{i\theta})^k d\theta \right| \\ &\quad + \frac{1}{\pi r^n} \left| \int_0^{2\pi} U(re^{i\theta}) f(re^{i\theta}) e^{-in\theta} d\theta \right| \\ &\leq p|b_n| + \frac{2p}{r^n} \sum_{k=n}^{\infty} |b_k| r^k + \frac{1}{\pi r^n} \int_0^{2\pi} U(re^{i\theta}) |f(re^{i\theta})| d\theta. \end{aligned}$$

The Area Theorems of Golusin [2] and Kobori [3] give for  $n \geq 1$

$$\sum_{k=n}^{\infty} k |b_k|^2 \leq \sum_{k=1}^{\infty} k |b_k|^2 \leq \sum_{k=-p}^{-1} |k| |b_k|^2.$$

We thus have,

$$\begin{aligned} \frac{2p}{r^n} \sum_{k=n}^{\infty} |b_k| r^k &\leq \frac{2p}{r^n} \left[ \sum_{k=n}^{\infty} k |b_k|^2 \right]^{1/2} \left[ \sum_{k=n}^{\infty} \frac{r^{2k}}{k} \right]^{1/2} \\ (3.4) \qquad &\leq \frac{2p}{r^n} \left[ \sum_{k=-p}^{-1} |k| |b_k|^2 \right]^{1/2} \left[ \frac{1}{n} \sum_{k=n}^{\infty} r^{2k} \right]^{1/2} \\ &= 2p \left[ \sum_{k=-p}^{-1} |k| |b_k|^2 \right]^{1/2} [n(1-r^2)]^{-1/2}. \end{aligned}$$

Also for  $n \geq p$ , by Lemma 3

$$\begin{aligned} (3.5) \qquad |b_n| &\leq \frac{2p}{(p+n)\sqrt{p}} \left[ \sum_{k=-p}^{-1} |k| |b_k|^2 \right]^{1/2} \\ &\leq \frac{1}{\sqrt{p}} \left[ \sum_{k=-p}^{-1} |k| |b_k|^2 \right]^{1/2}. \end{aligned}$$

Since  $[f(z)]^{-1}$  is  $p$ -valently star like we have

$$|f(re^{i\theta})|^{-1} \geq \frac{r^p}{1+r^{2p}}$$

or

$$|f(re^{i\theta})| \leq \frac{(1+r)^{2p}}{r^p}.$$

Therefore, for  $n \geq p$

$$\begin{aligned} (3.6) \qquad \frac{1}{\pi r^n} \int_0^{2\pi} U(re^{i\theta}) |f(re^{i\theta})| d\theta &\leq \frac{(1+r)^{2p}}{r^{p+n}} \frac{1}{\pi} \int_0^{2\pi} U(re^{i\theta}) d\theta \\ &\leq \frac{2p(1+r)^{2p}}{r^{p+n}} \leq \frac{2p 4^p}{r^{2n}}. \end{aligned}$$

From (3.3), (3.4), (3.5) and (3.6) we have for  $n \geq p$  and any  $r < 1$

$$n |a_n| \leq \left[ \sqrt{p + 2p [n(1-r^2)]^{-1/2}} \right] \left[ \sum_{k=-p}^{-1} |k| |b_k|^2 \right]^{1/2} + 2p 4^p r^{-2n}.$$

Let  $r^2 = (1 - 1/n)$ , then for  $n \geq p + 1$

$$\begin{aligned} n|a_n| &\leq (\sqrt{p+2p}) \left[ \sum_{k=-p}^{-1} |k| |b_k|^2 \right]^{1/2} + 2p 4^p (1 + 1/(n-1))^n \\ &\leq (\sqrt{p+2p}) \left[ \sum_{k=-p}^{-1} |k| |b_k|^2 \right]^{1/2} + 2p 4^p \frac{(p+1)}{p} e. \end{aligned}$$

Thus,  $|a_n| = O(n^{-1})$ .

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