Computers and Mathematics with Applications 57 (2009) 1142–1146

Contents lists available at ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

Landau's theorem for planar harmonic mappings

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ARTICLE INFO

Article history:
Received 11 December 2007
Received in revised form 20 November 2008
Accepted 12 January 2009

Keywords:
Landau theorem
Bloch theorem
Planar harmonic mapping
Schwarz Lemma
Coefficient estimate

ABSTRACT

Chen, Gauthier and Hengartner obtained two versions of Landau's theorem for bounded planar harmonic mappings. Later, Dorff and Nowak improved their estimates. Furthermore, Grigoryan and Huang independently improved their results. In this note, we improve these last results by obtaining better coefficient estimates for bounded and normalized planar harmonic mappings. In particular, our theorems are sharp when $M = 1$, which are consistent with Landau's theorem when $M = 1$.

1. Introduction

Suppose that $f(z) = u(z) + iv(z)$, $z = x + iy$ is a twice continuously differentiable function on the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. Then f is a harmonic mapping on $\mathbb{U}$ if and only if f satisfies $\Delta f = 4f_{z\bar{z}} = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$ for $z \in \mathbb{U}$, where we use the common notations for its formal derivatives:

$$f_z = \frac{1}{2}(f_x - if_y), \qquad f_{\bar{z}} = \frac{1}{2}(f_x + if_y).$$

For such function f, we define

$$\Lambda_f = |f_z| + |f_{\bar{z}}| \quad \text{and} \quad \lambda_f = ||f_z| - |f_{\bar{z}}||.$$

It is known [1] that a harmonic mapping is locally univalent if and only if its Jacobian $J_f(z) = |f_z|^2 - |f_{\bar{z}}|^2 \neq 0$ for $z \in \mathbb{U}$. Since $\mathbb{U}$ is simply connected, $f(z)$ can be written as $f = h + \overline{g}$ with $f(0) = h(0)$, g and h are analytic on $\mathbb{U}$. Thus, we have

$$J_f(z) = |f_z|^2 - |f_{\bar{z}}|^2 = |h'(z)|^2 - |g'(z)|^2.$$

The classical Landau theorem states that if f is an analytic function on the unit disk $\mathbb{U}$ with $f(0) = f'(0) - 1 = 0$ and $|f(z)| < M$ for $z \in \mathbb{U}$, then f is univalent in the disk $|z| < r_0$ with

$$r_0 = \frac{1}{M + \sqrt{M^2 - 1}},$$

and $f(|z| < r_0)$ contains a disk $|w| < R_0$ with $R_0 = Mr_0^2$. This result is sharp, with the extremal function $f(z) = Mz\frac{1-Mz}{M-z}$ (see [2,3]). The Bloch theorem asserts the existence of a positive constant number b such that if f is an analytic function on the unit disk $\mathbb{U}$ with $f'(0) = 1$, then $f(\mathbb{U})$ contains a schlicht disk of radius b, that is, a disk of radius b which is the univalent image of some region in $\mathbb{U}$. The supremum of all such constants b is called the Bloch constant (see [4,5]). Chen, Gauthier and

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doi:10.1016/j.camwa.2009.01.009

Hengartner [4] obtained two versions of Landau's theorem for bounded harmonic mappings in the unit disk $\mathbb{U}$. Unfortunately their results are not sharp. Better estimates were given in [6] and later in [7–9].

Specifically, Grigoryan and Huang independently proved the following result.

Theorem A (*Grigoryan [7], Huang [8]*). *Let f be a harmonic mapping of the unit disk $\mathbb{U}$ with $f(0) = 0$, $J_f(0) = 1$, and $|f(z)| < M$ for $z \in \mathbb{U}$. Then, f is univalent in the disk $\mathbb{U}_{\rho_1} = \{z : |z| < \rho_1\}$ with*

$$\rho_1 = 1 - \frac{2\sqrt{2}M}{\sqrt{\pi + 8M^2}}, \tag{1.1}$$

and $f(\mathbb{U}_{\rho_1})$ contains the schlicht disk $\mathbb{U}_{R_1}$ with

$$R_1 = \frac{\pi}{4M} + 4M - \sqrt{2\pi + 16M^2}. \tag{1.2}$$

This result is the best known but not sharp. In [8], Huang also proved the following theorem, which improved Theorem 7 in [6], but it is also not sharp.

Theorem B (*Huang [8]*). *Let f be a harmonic mapping of the unit disk $\mathbb{U}$ with $f(0) = 0$, $\lambda_f(0) = 1$, and $|f(z)| < M$ for $z \in \mathbb{U}$. Then, f is univalent in the disk $\mathbb{U}_{\rho_2}$ with*

$$\rho_2 = 1 - \frac{1}{\sqrt{1 + \frac{1}{2M}}}, \tag{1.3}$$

and $f(\mathbb{U}_{\rho_2})$ contains a schlicht disk $\mathbb{U}_{R_2}$ with

$$R_2 = 1 + 4M - 4M\sqrt{1 + \frac{1}{2M}}. \tag{1.4}$$

In this paper, we first establish coefficient estimates for bounded and normalized harmonic mappings (see Lemma 2.1). Next, using these results, we give better results than those of Theorems A and B (see Theorems 2.4 and 2.5). In particular, Theorems 2.4 and 2.5 are sharp when $M = 1$, which are consistent with Landau's theorem when $M = 1$. In order to establish our main results, we recall the following lemma.

Lemma 1.1 (*Schwarz Lemma [4]*). *Let f be a harmonic mapping of the unit disk $\mathbb{U}$ with $f(0) = 0$ and $f(\mathbb{U}) \subset \mathbb{U}$. Then*

$$\Lambda_f(0) \le \frac{4}{\pi}, \tag{1.5}$$

$$|f(z)| \le \frac{4}{\pi}\arctan|z| \le \frac{4}{\pi}|z|, \quad \text{for } z \in \mathbb{U}. \tag{1.6}$$

2. Main results

We first establish the following coefficient estimates for normalized harmonic mappings.

Lemma 2.1. *Suppose that $f(z) = h(z) + \overline{g(z)}$ is a harmonic mapping of the unit disk $\mathbb{U}$ with $h(z) = \sum_{n=1}^{\infty} a_n z^n$ and $g(z) = \sum_{n=1}^{\infty} b_n z^n$ for $z \in \mathbb{U}$.*

(1) If $J_f(0) = 1$ and $|f(z)| < M$, then

$$|a_n|, |b_n| \le \sqrt{M^2 - 1}, \quad n = 2, 3, \ldots, \tag{2.1}$$

and

$$|a_n| + |b_n| \le \sqrt{2M^2 - 2}, \quad n = 2, 3, \ldots, \tag{2.2}$$

and

$$\lambda_f(0) \ge \begin{cases} \dfrac{\sqrt{2}}{\sqrt{M^2 - 1} + \sqrt{M^2 + 1}} & \text{if } 1 \le M \le \dfrac{\pi}{2\sqrt[4]{2\pi^2 - 16}}, \\ \dfrac{\pi}{4M} & \text{if } M > \dfrac{\pi}{2\sqrt[4]{2\pi^2 - 16}}. \end{cases} \tag{2.3}$$

(2) If $\lambda_f(0) = 1$ and $|f(z)| < M$, then the inequalities (2.1) and (2.2) also hold.

Proof. (1) Fix $r \in (0, 1)$. Then

$$f(re^{i\theta}) = \sum_{n=1}^{\infty} a_n r^n e^{in\theta} + \sum_{n=1}^{\infty} \overline{b_n} r^n e^{-in\theta} \quad \text{for } \theta \in [0, 2\pi).$$

By Parseval's identity and the hypothesis of $|f(z)| < M$, we obtain

$$\sum_{n=1}^{\infty} (|a_n|^2 + |b_n|^2) r^{2n} = \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^2 d\theta \leq M^2. \tag{2.4}$$

Notice that since $J_f(0) = |a_1|^2 - |b_1|^2 = 1$, we have $|a_1|^2 + |b_1|^2 \geq J_f(0) = 1$. Hence for any $n \geq 2$, we have

$$(|a_n|^2 + |b_n|^2) r^{2n} \leq M^2 - r^2.$$

Letting $r \to 1^-$, we obtain

$$|a_n|^2 + |b_n|^2 \leq M^2 - 1, \tag{2.5}$$

from which we conclude $M \geq 1$. This implies the inequalities (2.1), and the inequality (2.2) follows from

$$|a_n| + |b_n| \leq \sqrt{2(|a_n|^2 + |b_n|^2)} \leq \sqrt{2M^2 - 2}.$$

Now we prove (2.3). Since $|a_1|^2 - |b_1|^2 = J_f(0) = 1$, we have

$$|a_1| = \sqrt{|b_1|^2 + 1}. \tag{2.6}$$

From (2.4), we get that

$$|a_1|^2 + |b_1|^2 \leq M^2. \tag{2.7}$$

Hence it follows from (2.6) and (2.7) that

$$|b_1| \leq \sqrt{\frac{M^2 - 1}{2}}. \tag{2.8}$$

Thus we have

$$\begin{aligned} \lambda_f(0) &= \big||a_1| - |b_1|\big| = \sqrt{|b_1|^2 + 1} - |b_1| \\ &= \frac{1}{|b_1| + \sqrt{|b_1|^2 + 1}} \\ &\geq \frac{\sqrt{2}}{\sqrt{M^2 - 1} + \sqrt{M^2 + 1}}. \end{aligned} \tag{2.9}$$

On the other hand, by Lemma 1.1, it follows from the fact $J_f(0) = 1$ that

$$\lambda_f(0) = \frac{1}{\Lambda_f(0)} \geq \frac{\pi}{4M}. \tag{2.10}$$

Note that

$$\frac{\sqrt{2}}{\sqrt{M^2 - 1} + \sqrt{M^2 + 1}} \geq \frac{\pi}{4M} \Leftrightarrow 1 \leq M \leq \frac{\pi}{2\sqrt[4]{2\pi^2 - 16}}.$$

Hence the inequalities (2.3) follow from (2.9) and (2.10).
(2) Since $\lambda_f(0) = \big||a_1| - |b_1|\big| = 1$, we get that

$$|a_1|^2 + |b_1|^2 \geq \big||a_1|^2 - |b_1|^2\big| = |a_1| + |b_1| \geq \big||a_1| - |b_1|\big| = 1.$$

Thus the inequalities (2.1) and (2.2) follow as in part (1), and the proof is complete. □

Remark 2.2. Setting $M = 1$ in Lemma 2.1, according to the proof of Lemma 2.1, we get the following corollary.

Corollary 2.3. *Suppose that $f(z) = h(z) + \overline{g(z)}$ is a harmonic mapping of the unit disk $\mathbb{U}$ with $f(0) = 0$ and $|f(z)| < 1$ on $\mathbb{U}$.*

(1) *If $J_f(0) = 1$, then $f(z) = \alpha z$, where $|\alpha| = 1$.*
(2) *If $\lambda_f(0) = 1$, then $f(z) = \alpha z$ or $f(z) = \alpha \bar{z}$, where $|\alpha| = 1$.*

Next we are ready to improve Theorem A as follows.

Theorem 2.4. *Let f be a harmonic mapping of the unit disk $\mathbb{U}$ with $f(0) = J_f(0) - 1 = 0$, and $|f(z)| < M$ for $z \in \mathbb{U}$. Then f is univalent in the disk $\mathbb{U}_{r_1}$, and $f(\mathbb{U}_{r_1})$ contain the schlicht disk $\mathbb{U}_{\sigma_1}$, with*

$$\lambda_0(M) = \begin{cases} \dfrac{\sqrt{2}}{\sqrt{M^2-1}+\sqrt{M^2+1}} & \text{if } 1 \le M \le M_0 = \dfrac{\pi}{2\sqrt[4]{2\pi^2-16}} \approx 1.1296, \\ \dfrac{\pi}{4M} & \text{if } M > M_0, \end{cases} \tag{2.11}$$

and

$$r_1 = 1 - \frac{\sqrt{\sqrt{2M^2-2}}}{\sqrt{\lambda_0(M)+\sqrt{2M^2-2}}} = \begin{cases} 1 - \dfrac{\sqrt{M^2-1+\sqrt{M^4-1}}}{\sqrt{M^2+\sqrt{M^4-1}}} & \text{if } 1 \le M \le M_0, \\ 1 - \dfrac{\sqrt{4M\sqrt{2M^2-2}}}{\sqrt{\pi+4M\sqrt{2M^2-2}}}, & \text{if } M > M_0, \end{cases} \tag{2.12}$$

and

$$\sigma_1 = \lambda_0(M) + 2\sqrt{2M^2-2} - 2\sqrt{2M^2-2+\lambda_0(M)\sqrt{2M^2-2}} = \begin{cases} \dfrac{\sqrt{2M^2+2}}{2} + \dfrac{3\sqrt{2M^2-2}}{2} - 2\sqrt{M^2-1+\sqrt{M^4-1}} & \text{if } 1 \le M \le M_0, \\ \dfrac{\pi}{4M} + 2\sqrt{2M^2-2} - \sqrt{8M^2-8+\dfrac{\pi}{M}\sqrt{2M^2-2}} & \text{if } M > M_0. \end{cases} \tag{2.13}$$

The above result is sharp when $M = 1$.

Proof. If $f(z) = h(z) + \overline{g}(z)$ satisfies the hypothesis of Theorem 2.4, where $h(z) = \sum_{n=1}^{\infty} a_n z^n$ and $g(z) = \sum_{n=1}^{\infty} b_n z^n$ are analytic in $z \in \mathbb{U}$, then by Lemma 2.1, we have

$$\lambda_f(0) \ge \lambda_0(M) = \begin{cases} \dfrac{\sqrt{2}}{\sqrt{M^2-1}+\sqrt{M^2+1}} & \text{if } 1 \le M \le M_0 = \dfrac{\pi}{2\sqrt[4]{2\pi^2-16}}, \\ \dfrac{\pi}{4M} & \text{if } M > M_0 = \dfrac{\pi}{2\sqrt[4]{2\pi^2-16}}. \end{cases} \tag{2.14}$$

To prove the univalence of $f(z)$ in $\mathbb{U}_{r_1}$, we adopt the method used in [6]. For $z_1 \ne z_2$ in $\mathbb{U}_r (0 < r < r_1)$, by Lemma 2.1(1), we have

$$\begin{aligned} |f(z_1) - f(z_2)| &= \left| \int_{[z_1,z_2]} f_z(z)\mathrm{d}z + f_{\overline{z}}(z)\mathrm{d}\overline{z} \right| = \left| \int_{[z_1,z_2]} h'(z)\mathrm{d}z + \overline{g'(z)}\mathrm{d}\overline{z} \right| \\ &\ge \left| \int_{[z_1,z_2]} h'(0)\mathrm{d}z + \overline{g'(0)}\mathrm{d}\overline{z} \right| - \left| \int_{[z_1,z_2]} (h'(z)-h'(0))\mathrm{d}z + (\overline{g'(z)} - \overline{g'(0)})\mathrm{d}\overline{z} \right| \\ &\ge \int_{[z_1,z_2]} |h'(0) + \overline{g'(0)}\mathrm{e}^{-2i\theta}|\mathrm{d}s - \left| \int_{[z_1,z_2]} (h'(z)-h'(0))\mathrm{d}z \right| - \left| \int_{[z_1,z_2]} (g'(z)-g'(0))\mathrm{d}z \right| \\ &\ge \lambda_f(0)|z_2 - z_1| - |z_1 - z_2| \sum_{n=2}^{\infty} (|a_n| + |b_n|) n r^{n-1} \\ &\ge |z_2 - z_1| \left(\lambda_0(M) - \sum_{n=2}^{\infty} \sqrt{2M^2-2} \cdot n r^{n-1} \right) \\ &= |z_2 - z_1| \left[\lambda_0(M) - \sqrt{2M^2-2} \cdot \frac{2r - r^2}{(1-r)^2} \right] \\ &= \frac{|z_2 - z_1|}{(1-r)^2} \left[(\lambda_0(M) + \sqrt{2M^2-2}) r^2 - 2(\lambda_0(M) + \sqrt{2M^2-2}) r + \lambda_0(M) \right] \end{aligned}$$

$$= \frac{|z_2 - z_1|(\lambda_0(M) + \sqrt{2M^2-2})}{(1-r)^2}(r - r_1)\left(r - 1 - \frac{\sqrt{\sqrt{2M^2-2}}}{\sqrt{\lambda_0(M) + \sqrt{2M^2-2}}}\right) > 0.$$

This implies $f(z_1) \neq f(z_2)$.

From Lemma 2.1, we know that $M \geq 1$. Noticing that $f(0) = 0$, for any $z' = r_1 e^{i\theta} \in \partial\mathbb{U}_{r_1}$, we have

$$\begin{aligned} |f(z')| &\geq |a_1 z' + \bar{b}_1 \overline{z'}| - \left|\sum_{n=2}^{\infty}\left(a_n z'^n + \bar{b}_n \overline{z'}^n\right)\right| \\ &\geq \lambda_f(0) r_1 - \sum_{n=2}^{\infty}(|a_n| + |b_n|) r_1^n \\ &\geq \lambda_0(M) r_1 - \sum_{n=2}^{\infty}\sqrt{2M^2-2}\cdot r_1^n = \lambda_0(M) r_1 - \sqrt{2M^2-2}\cdot\frac{r_1^2}{1-r_1} \\ &= \lambda_0(M) + 2\sqrt{2M^2-2} - 2\sqrt{2M^2 - 2 + \lambda_0(M)\sqrt{2M^2-2}} = \sigma_1. \end{aligned}$$

Hence $f(z)$ is univalent on $\mathbb{U}_{r_1}$ and $f(\mathbb{U}_{r_1})$ contains the disk $\mathbb{U}_{\sigma_1}$, where r_1 defined by (2.12) and σ_1 defined by (2.13).

Finally, it is evident that $r_1 = \sigma_1 = 1$ for $M = 1$ is the best possible. This completes the proof of Theorem 2.4. □

With the aid of Lemma 2.1(2), using the same method as in our proof of Theorem 2.4, or let 1 instead of $\lambda_0(M)$ in (2.12) and (2.13), we can improve Theorem B as follows. We omit the details.

Theorem 2.5. *Let f be a harmonic mapping of the unit disk $\mathbb{U}$ with $f(0) = 0$, $\lambda_f(0) = 1$, and $|f(z)| < M$ for $z \in \mathbb{U}$. Then, f is univalent in the disk $\mathbb{U}_{r_2}$ with*

$$r_2 = 1 - \sqrt{\frac{\sqrt{2M^2-2}}{1 + \sqrt{2M^2-2}}}, \tag{2.15}$$

and $f(\mathbb{U}_{r_2})$ contains a schlicht disk $\mathbb{U}_{\sigma_2}$ with

$$\sigma_2 = 1 + 2\sqrt{2M^2-2} - 2\sqrt{2M^2 - 2 + \sqrt{2M^2-2}}, \tag{2.16}$$

and the result is sharp when $M = 1$.

Remark 2.6. (1) Setting $M = 1$ in Theorem 2.5, we get $r_2 = \sigma_2 = 1$. This and Theorem 2.4 are consistent with Landau's theorem for $M = 1$.

(2) Noting that $\sqrt{2M^2-2} < 2M$, and that r_1, σ_1, r_2, and σ_2 are monotone functions of $\sqrt{2M^2-2}$, we verify that

$$r_1 > \rho_1, \sigma_1 > R_1; \quad r_2 > \rho_2, \sigma_2 > R_2, \tag{2.17}$$

where ρ_1, R_1, ρ_2, R_2 and $r_1, \sigma_1, r_2, \sigma_2$ defined by (1.1)–(1.4), (2.12), (2.13), (2.15)–(2.16) respectively.

Acknowledgments

The author thanks the referees for their helpful comments and suggestions in improving the manuscript. He also thanks Professor Huaihui Chen for providing many useful materials and suggestions.

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