Southeast Asian Bulletin of Mathematics (2013) 37: 211–220

Southeast Asian
Bulletin of
Mathematics

On Bloch Constants for Certain Harmonic Mappings

Ming-Sheng Liu and Zhi-Wen Liu

School of Mathematical Sciences, South China Normal University Guangzhou 510631, Guangdong, China
Email: liumsh@scnu.edu.cn; liuziwen1985@163.com

Received 29 April 2009
Accepted 26 June 2009

Communicated by S. Parvathi

AMS Mathematics Subject Classification(2000): 30C99, 30C62

Abstract. Let $f(z) = h(z) + \overline{g(z)}$ be a harmonic mapping of the unit disk U, where $h(z)$ and $g(z)$ are analytic in U. In this paper, we give a lower estimate of the Bloch constant for harmonic mappings of the form $L(f)$, where L represents the linear complex operator $L = z\frac{\partial}{\partial z} - \overline{z}\frac{\partial}{\partial \overline{z}}$ defined on the class of complex-valued C^1 functions in the plane. Also, we obtain three versions of Landau's theorems for harmonic mappings of the form $L(f)$, which are sharp when $M = 1$ or $\Lambda = 1$.

Keywords: Landau theorem; Bloch constant; Linear complex operator; Harmonic mapping; Univalent.

1. Introduction and Preliminaries

Harmonic mappings can be regarded as generalizations of holomorphic functions. A two times continuously differentiable complex-valued function $f(z) = u(z) + iv(z)$ is said to be a harmonic function in a domain $D \subseteq \mathbb{C}$ if and only if its real part and imaginary part are real-valued harmonic functions of a domain D. Note that f is harmonic in D if f satisfies the harmonic equation $\Delta f = 0$, where Δ represents the Laplacian operator

$$\Delta f = 4f_{z\overline{z}} = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}, \ \ z = x + iy \in D,$$

we use the common notations for its formal derivatives:

$$f_z = \frac{1}{2}(f_x - if_y), \;\; f_z = \frac{1}{2}(f_x + if_y).$$

Without loss of generality, we consider the class of harmonic mappings defined in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$. For such function f, we define

$$\Lambda_f(z) = \max_{0\le\theta\le2\pi} |f_z(z) + e^{-2i\theta} f_{\overline{z}}(z)| = |f_z(z)| + |f_{\overline{z}}(z)|,$$

and

$$\lambda_f(z) = \min_{0\le\theta\le2\pi} |f_z(z) + e^{-2i\theta} f_{\overline{z}}(z)| = ||f_z(z)| - |f_{\overline{z}}(z)||.$$

We denote the Jacobian of f by J_f, in [11], Lewy showed that a harmonic mapping $f(z)$ is locally univalent in a domain D if and only if $J_f \neq 0$ for any $z \in D$. Of course, local univalence of f does not imply global univalence in a domain D. Note that $|J_f| = \Lambda_f \lambda_f$. A harmonic mapping f is said to be K-quasiregular ($K \geq 1$) on a domain D if $\Lambda_f \leq K\lambda_f$ holds everywhere on D. Recall that a mapping of the unit disk U is said to be an open mapping if it maps any open subset of U to an open set in $\mathbb{C}$.

If D is simply connected, then $f(z)$ can be written as $f(z) = h(z) + \overline{g(z)}$ with $f(0) = h(0)$, where $h(z)$ and $g(z)$ are analytic in D (for details see [15, 16, 18]). Thus we have

$$J_f(z) = |f_z|^2 - |f_{\overline{z}}|^2 = |h'(z)|^2 - |g'(z)|^2.$$

For $r > 0$, we let U_r denote the disk with center at the origin and radius r. The classical Landau theorem concerns determining the possibly largest schlicht disk for the properly normalized bounded analytic functions. It states that if f is an analytic function in the unit disk U with $f(0) = f'(0) - 1 = 0$ and $|f(z)| < M$ for $z \in U$, then f is univalent in the disk $|z| < r_0$ with $r_0 = 1/(M + \sqrt{M^2 - 1})$, and $f(U_{r_0})$ contains a disk $|w| < R_0$ with $R_0 = Mr_0^2$. This result is sharp, with the extremal function $f(z) = Mz(1 - Mz)/(M - z)$ (see [4]). Furthermore, for holomorphic functions in U with the only restriction that $f'(0) = 1$, there is the Bloch theorem which asserts the existence of a positive constant b such that $f(U)$ contains a schlicht disk, that is, a disk of radius b which is the univalent image of some region in U. The Bloch constant is defined as the supremum of all such b.

Recently, many authors considered Landau and Bloch theorems for planar harmonic mappings (see [5, 7, 8, 9, 10, 13, 14]) and Landau's theorem for biharmonic mappings (see [3, 6, 12]). Some authors studied other problems of harmonic functions [17].

In [1], the authors considered the following differential operator L defined on the class of complex-valued C^1 functions:

$$L = z\frac{\partial}{\partial z} - \overline{z}\frac{\partial}{\partial \overline{z}}.$$

Clearly, L is a complex linear operator and satisfies the usual product rule:

$$L(af + bg) = aL(f) + bL(g),$$

and

$$L(fg) = fL(g) + gL(f),$$

where a, b are complex constants, f and g are complex functions in $\mathbb{C}$. In addition, the operator L possesses a number of interesting properties. For instance, it is easy to see that the operator L preserves both harmonicity and biharmonicity. Many other basic properties are stated in [2].

The main aim of this paper is to consider Landau and Bloch theorems for the harmonic mappings of the form $L(f)$, where f belongs to the class of harmonic mappings.

In order to establish our main results, we need the following lemmas.

Lemma 1.1. [12] *Suppose that $f(z) = h(z) + \overline{g(z)}$ is a harmonic mapping of the unit disk U with $h(z) = \sum_{n=1}^{\infty} a_n z^n$ and $g(z) = \sum_{n=1}^{\infty} b_n z^n$ are analytic on U.*

(i) *If $J_f(0) = 1$ and $|f(z)| \le M$ for $z \in U$, then*

$$|a_n| + |b_n| \le \sqrt{2M^2 - 2}, \quad n = 2, 3, \cdots . \tag{1}$$

and

$$\lambda_f(0) \ge \lambda_0(M) = \begin{cases} \frac{\sqrt{2}}{\sqrt{M^2-1}+\sqrt{M^2+1}} & 1 \le M \le M_0 = \frac{\pi}{2\sqrt[4]{2\pi^2-16}}, \\ \frac{\pi}{4M} & M > M_0 \approx 1.1296. \end{cases} \tag{2}$$

(ii) *If $\lambda_f(0) = 1$ and $|f(z)| \le M$, then the inequalities (1) also hold.*

Remark 1.2. 1.2 From (1), we know that $M \ge 1$. Setting $M = 1$ in Lemma 1.1, then $a_n = b_n = 0$ for $n = 2, 3, \cdots$, thus $f(z) = a_1 z + \overline{b_1 z}$ with $||a_1| - |b_1|| = 1$.

Lemma 1.3. [12, 13] *Suppose that $f(z) = h(z) + \overline{g(z)}$ is a harmonic mapping of the unit disk U with $h(z) = \sum_{n=1}^{\infty} a_n z^n$ and $g(z) = \sum_{n=1}^{\infty} b_n z^n$ and $\lambda_f(0) = 1$. If $\Lambda_f(z) \le \Lambda$ for $z \in U$, then*

$$|a_n| + |b_n| \le \frac{\Lambda^2 - 1}{n\Lambda}, \quad n = 2, 3, \cdots . \tag{3}$$

Above estimates are sharp for all $n = 2, 3, \cdots$, with the extremal functions $f_n(z)$ and $\overline{f_n(z)}$

$$f_n(z) = \Lambda^2 z - (\Lambda^3 - \Lambda) \int_0^z \frac{dz}{\Lambda + z^{n-1}}.$$

Remark 1.4. From (3), we know that $\Lambda \geq 1$. Setting $\Lambda = 1$ in Lemma 1.3, then $a_n = b_n = 0$ for $n = 2, 3, \cdots$, thus $f(z) = a_1 z + \overline{b_1 z}$ with $||a_1| - |b_1|| = 1$.

2. Main Results

We first establish a version of Landau's theorem for the harmonic mappings of the form $L(f)$, where f belongs to the class of harmonic mappings.

Theorem 2.1. *Let $f(z)$ be a harmonic mapping of the unit disk U, with $f(0) = J_f(0) - 1 = 0$, and $|f(z)| \leq M$ for $z \in U$. Then $L(f)$ is univalent in the disk U_{r_1}, and $L(f)(U_{r_1})$ contains a schlicht disk U_{σ_1}, where r_1 is the minimum positive root of the following equation:*

$$\lambda_0(M) - \sqrt{2M^2 - 2} \cdot \frac{4r - 3r^2 + r^3}{(1-r)^3} = 0, \tag{4}$$

for $M > 1$, and $r_1 = 1$ for $M = 1$,

$$\sigma_1 = \lambda_0(M) r_1 - \sqrt{2M^2 - 2} \cdot \frac{2r_1^2 - r_1^3}{(1-r_1)^2}, \tag{5}$$

where $\lambda_0(M)$ is defined by (2). This result is sharp for $M = 1$.

Proof. If $f(z) = h(z) + \overline{g(z)}$ satisfies the hypothesis of Theorem 2.1, where $h(z) = \sum_{n=1}^{\infty} a_n z^n$ and $g(z) = \sum_{n=1}^{\infty} b_n z^n$ are analytic on U. We defined

$$H := L(f) = z f_z - \overline{z} f_{\overline{z}},$$

then

$$H_z = f_z + z f_{zz},$$

and

$$H_{\overline{z}} = -f_{\overline{z}} - \overline{z} f_{\overline{z}\,\overline{z}}.$$

Also, we observe that $J_f(0) = J_H(0) = 1$, by Lemma 1.1, we have

$$|a_n| + |b_n| \leq \sqrt{2M^2 - 2}, \ \ n = 2, 3, \cdots. \tag{6}$$

and

$$\lambda_f(0) \geq \lambda_0(M). \tag{7}$$

Thus, for $z_1 \neq z_2$ in U_r $(0 < r < r_1)$, by (6) and (7), we have

$$\begin{aligned}
&|H(z_1) - H(z_2)| \\
= &\left| \int_{[z_1,z_2]} H_z(z)dz + H_{\overline{z}}(z)d\overline{z} \right| \\
\geq &\left| \int_{[z_1,z_2]} f_z(0)dz - f_{\overline{z}}(0)d\overline{z} \right| - \left| \int_{[z_1,z_2]} z f_{zz} dz - \overline{z} f_{\overline{z}\,\overline{z}} d\overline{z} \right| \\
&- \left| \int_{[z_1,z_2]} (f_z(z) - f_z(0))dz - (f_{\overline{z}}(z) - f_{\overline{z}}(0))d\overline{z} \right| \\
\geq &|z_1 - z_2| \left(\lambda_f(0) - \sum_{n=2}^{\infty} n(n-1)(|a_n| + |b_n|) r^{n-1} - \sum_{n=2}^{\infty} (|a_n| + |b_n|) n r^{n-1} \right) \\
\geq &|z_1 - z_2| \left(\lambda_0(M) - \sqrt{2M^2 - 2} \cdot \sum_{n=2}^{\infty} n(n-1) r^{n-1} - \sqrt{2M^2 - 2} \cdot \sum_{n=2}^{\infty} n r^{n-1} \right) \\
= &|z_1 - z_2| \left(\lambda_0(M) - \sqrt{2M^2 - 2} \cdot \frac{2r}{(1-r)^3} - \sqrt{2M^2 - 2} \cdot \frac{2r - r^2}{(1-r)^2} \right) \\
= &|z_1 - z_2| \left(\lambda_0(M) - \sqrt{2M^2 - 2} \cdot \frac{4r - 3r^2 + r^3}{(1-r)^3} \right) \\
> &\ 0,
\end{aligned}$$

this implies $F(z_1) \neq F(z_2)$, which proves the univalence of $L(f)$ in the disk U_{r_1}.

Finally, in the same way, we now consider any $z = r_1 e^{i\theta} \in \partial U_{r_1}$, by (6) and (7), we have

$$\begin{aligned}
|H(z)| &= |z f_z - \overline{z} f_{\overline{z}}| \\
&\geq |z f_z(0) - \overline{z} f_{\overline{z}}(0)| - |z(f_z(z) - f_z(0)) - \overline{z}(f_{\overline{z}}(z) - f_{\overline{z}}(0)| \\
&\geq \lambda_f(0) r_1 - r_1 \sum_{n=2}^{\infty} (|a_n| + |b_n|) n r_1^{n-1} \\
&\geq \lambda_0(M) r_1 - \sqrt{2M^2 - 2} \cdot \sum_{n=2}^{\infty} n r_1^n \\
&= \lambda_0(M) r_1 - \sqrt{2M^2 - 2} \cdot \frac{2r_1^2 - r_1^3}{(1 - r_1)^2} = \sigma_1.
\end{aligned}$$

Hence, $L(f)$ is univalent on U_{r_1} and $L(f)(U_{r_1})$ contains the disk U_{σ_1}, where r_1 is defined by (4) and σ_1 is defined by (5).

Finally, it is evident that $r_1 = \sigma_1 = 1$ for $M = 1$ is the best possible. This completes the proof of Theorem 2.1. ■

With the aid of Lemma 1.1, if we apply the same method as in our proof of Theorem 2.1, or let 1 instead of $\lambda_0(M)$, we get the following theorem.

Theorem 2.2. *Let $f(z)$ be a harmonic mapping of the unit disk U, with $f(0) = \lambda_f(0) - 1 = 0$, and $|f(z)| \le M$ for $z \in U$. Then $L(f)$ is univalent in the disk U_{r_2}, and $L(f)(U_{r_2})$ contains a schlicht disk U_{σ_2}, where r_2 is the minimum positive root of the following equation:*

$$1 - \sqrt{2M^2 - 2} \cdot \frac{4r - 3r^2 + r^3}{(1-r)^3} = 0,$$

for $M > 1$, and $r_2 = 1$ for $M = 1$,

$$\sigma_2 = r_2 - \sqrt{2M^2 - 2} \cdot \frac{2r_2^2 - r_2^3}{(1-r_2)^2}.$$

This result is sharp for $M = 1$.

Next, with the aid of Lemma 1.3, we can get another version of Landau's theorem for the harmonic mappings of the form $L(f)$ as follows.

Theorem 2.3. *Let $f(z)$ be a harmonic mapping of the unit disk U, with $f(0) = \lambda_f(0) - 1 = 0$, and $\Lambda_f(z) \le \Lambda$ for $z \in U$. Then $L(f)$ is univalent in the disk U_{r_3}, and $L(f)(U_{r_3})$ contains a schlicht disk U_{σ_3}, where*

$$r_3 = 1 - \sqrt{\frac{\Lambda^2 - 1}{\Lambda^2 + \Lambda - 1}}, \tag{8}$$

and

$$\sigma_3 = \begin{cases} 1 - \frac{2\sqrt{\Lambda^2-1}}{\sqrt{\Lambda^2+\Lambda-1}+\sqrt{\Lambda^2-1}} & \Lambda > 1, \\ 1 & \Lambda = 1, \end{cases} \tag{9}$$

above result is sharp when $\Lambda = 1$.

Proof. If $f(z) = h(z) + \overline{g(z)}$ satisfies the hypothesis of Theorem 2.3, where $h(z) = \sum_{n=1}^{\infty} a_n z^n$ and $g(z) = \sum_{n=1}^{\infty} b_n z^n$ are analytic on U, then

$$\lambda_f(0) = ||a_1| - |b_1|| = 1. \tag{10}$$

Let $H := L(f) = zf_z - \overline{z}f_{\overline{z}}$, by the proof of Theorem 2.1 and Lemma 1.3, we obtain that $\lambda_f(0) = \lambda_H(0) = 1$, and

$$|a_n| + |b_n| \le \frac{\Lambda^2 - 1}{n\Lambda}, \ \ n = 2, 3, \cdots. \tag{11}$$

To prove the univalence of $H(z)$ in U_{r_3}, we adopt the method as in our proof of Theorem 2.1. For $z_1 \neq z_2$ in $U_r (0 < r < r_3)$, by (10) and (11) we have

$$\begin{aligned}
& |H(z_1) - H(z_2)| \\
= & \left| \int_{[z_1, z_2]} H_z(z)dz + H_{\overline{z}}(z)d\overline{z} \right| \\
\geq & |z_1 - z_2| \Bigg(\lambda_f(0) - \sum_{n=2}^{\infty} n(n-1)(|a_n| + |b_n|)r^{n-1} \\
& - \sum_{n=2}^{\infty} (|a_n| + |b_n|)nr^{n-1} \Bigg) \\
\geq & |z_1 - z_2| \left(1 - \frac{\Lambda^2 - 1}{\Lambda} \cdot \sum_{n=2}^{\infty} (n-1)r^{n-1} - \frac{\Lambda^2 - 1}{\Lambda} \cdot \sum_{n=2}^{\infty} r^{n-1} \right) \\
= & |z_1 - z_2| \left(1 - \frac{\Lambda^2 - 1}{\Lambda} \cdot \frac{r}{(1-r)^2} - \frac{\Lambda^2 - 1}{\Lambda} \cdot \frac{r}{1-r} \right) \\
= & |z_1 - z_2| \left(1 - \frac{\Lambda^2 - 1}{\Lambda} \cdot \frac{2r - r^2}{(1-r)^2} \right).
\end{aligned}$$

Since

$$r_3 = 1 - \sqrt{\frac{\Lambda^2 - 1}{\Lambda^2 + \Lambda - 1}}$$

is the minimum positive root of the following equation:

$$1 - \frac{\Lambda^2 - 1}{\Lambda} \cdot \frac{2r - r^2}{(1-r)^2} = 0.$$

This implies $H(z_1) \neq H(z_2)$, which proves the univalence of $L(f)$ in the disk U_{r_3}.

From Lemma 1.3, we get that $\Lambda \geq 1$. When $\Lambda > 1$, by means of (10), for $|z| = r_3$, we have

$$\begin{aligned}
|H(z)| &= |zf_z - \overline{z}f_{\overline{z}}| \\
&\geq |zf_z(0) - \overline{z}f_{\overline{z}}(0)| - |z(f_z(z) - f_z(0)) - \overline{z}(f_{\overline{z}}(z) - f_{\overline{z}}(0)| \\
&\geq \lambda_f(0)r_3 - r_3 \sum_{n=2}^{\infty} (|a_n| + |b_n|)nr_3^{n-1} \\
&\geq r_3 - \frac{\Lambda^2 - 1}{\Lambda} \cdot \sum_{n=2}^{\infty} r_3^n \\
&= r_3 - \frac{\Lambda^2 - 1}{\Lambda} \cdot \frac{r_3^2}{1 - r_3} \\
&= 1 - \frac{2\sqrt{\Lambda^2 - 1}}{\sqrt{\Lambda^2 + \Lambda - 1} + \sqrt{\Lambda^2 - 1}} = \sigma_3.
\end{aligned}$$

When $\Lambda = 1$, from Remark 1.4 and (8), we get that $f(z) = a_1 z + \overline{b_1 z}$ with $||a_1| - |b_1|| = 1$, and $r_3 = 1$, thus for $|z| = r_3 = 1$,

$$|H(z)| = |a_1 z - \overline{b_1 z}| \geq ||a_1| - |b_1|| = 1.$$

Hence, $L(f)(z)$ is univalent on U_{r_3} and $L(f)(U_{r_3})$ contains the disk U_{σ_3}, where r_3 is defined by (8) and σ_3 is defined by (9).

Finally, it is evident that $r_1 = \sigma_1 = 1$ for $M = 1$ is the best possible. This completes the proof of Theorem 2.3. ∎

Using Theorem 2.3, now we consider the case f is a K-quasiregular harmonic mapping of the unit disk U.

Theorem 2.4. *Let $f(z)$ be a K-quasiregular harmonic mapping of the unit disk U, with $f(0) = \lambda_f(0) - 1 = 0$. Then $L(f)(U)$ contains a schlicht disc of radius at least*

$$\sigma_4 = \frac{1}{\sqrt{2}} - \frac{\sqrt{8K^2 - 2}}{\sqrt{4K^2 + 2K - 1} + \sqrt{4K^2 - 1}}.$$

Proof. As in the proof of Theorem 5 in [5], let $g(\omega) = \sqrt{2} f(\frac{\omega}{\sqrt{2}})$ for $\omega \in U$. Since f is a K-quasiregular, we have g is also K-quasiregular, thus $\lambda_g(0) = \lambda_f(0) = 1$, and

$$\Lambda_g(\omega) \leq K\lambda_g(\omega) = K\lambda_f(\frac{\omega}{\sqrt{2}}) < 2K, \ \ \omega \in U.$$

By Theorem 2.3, we see that $L(g)$ is univalent on U_{r_4} with

$$r_4 = 1 - \frac{\sqrt{4K^2 - 1}}{\sqrt{4K^2 + 2K - 1}},$$

and $L(g)(U_{r_4})$ contains a schlicht disk U_{R_4} with

$$R_4 = 1 - \frac{2\sqrt{4K^2 - 1}}{\sqrt{4K^2 + 2K - 1} + \sqrt{4K^2 - 1}}.$$

Notice that $L(g) = \sqrt{2} L(f(\frac{\omega}{\sqrt{2}}))$, hence, $L(f)(U)$ contains a schlicht disc of radius at least σ_4, and the proof is complete. ∎

If the function is normalized by $J_f(0) = 1$, the radius of the schlicht disk will be a little smaller.

Theorem 2.5. *Let $f(z)$ be a K-quasiregular harmonic mapping of the unit disk U, with $f(0) = J_f(0) - 1 = 0$. Then $L(f)(U)$ contains a schlicht disc of radius at least*

$$\sigma_5 = \frac{1}{\sqrt{2K}} - \frac{\sqrt{8K^2 - 2}}{\sqrt{K}(\sqrt{4K^2 + 2K - 1} + \sqrt{4K^2 - 1})}.$$

Proof. Since f is a K-quasiregular and $J_f(0) = 1$, we have

$$\lambda_f(0) \geq \frac{\sqrt{J_f(0)}}{\sqrt{K}} = \frac{1}{\sqrt{K}}.$$

Applying Theorem 2.4 to the function $f/\lambda_f(0)$ gives the conclusion of the theorem. ∎

Finally, we consider the case f is an open harmonic mapping of the unit disk U.

Theorem 2.6. *Let $f(z)$ be an open harmonic mapping of the unit disk U normalized by $f_z(0) - 1 = f_{\bar{z}}(0) = 0$. Then $L(f)(U)$ contains a schlicht disc of radius at least*

$$\sigma_6 \approx 0.014333.$$

Proof. As in the proof of Theorem 7 in [5], we get f is K_r-quasiregular on U_r with $K_r = \frac{1+r}{1-r}$ and $0 < r < 1$, by using Theorem 2.4 to the function $F(\omega) = \frac{f(r\omega)}{r}$, $\omega \in U$, we have $L(f)(U_r)$ contains a schlicht disc of radius at least

$$\begin{aligned}
\sigma_6(r) &= \frac{r}{\sqrt{2}} - \frac{r\sqrt{8K_r^2 - 2}}{\sqrt{4K_r^2 + 2K_r - 1} + \sqrt{4K_r^2 - 1}} \\
&= \frac{r}{\sqrt{2}}\left(1 - \frac{2\sqrt{4K_r^2 - 1}}{\sqrt{4K_r^2 + 2K_r - 1} + \sqrt{4K_r^2 - 1}}\right) \\
&= \frac{r}{\sqrt{2}}\left(\frac{\sqrt{4K_r^2 + 2K_r - 1} - \sqrt{4K_r^2 - 1}}{\sqrt{4K_r^2 + 2K_r - 1} + \sqrt{4K_r^2 - 1}}\right) \\
&= \frac{r}{\sqrt{2}}\left(\frac{\sqrt{4(\frac{1+r}{1-r})^2 + 2(\frac{1+r}{1-r}) - 1} - \sqrt{4(\frac{1+r}{1-r})^2 - 1}}{\sqrt{4(\frac{1+r}{1-r})^2 + 2(\frac{1+r}{1-r}) - 1} + \sqrt{4(\frac{1+r}{1-r})^2 - 1}}\right) \\
&= \frac{r}{\sqrt{2}}\left(\frac{\sqrt{r^2 + 10r + 5} - \sqrt{3r^2 + 10r + 3}}{\sqrt{r^2 + 10r + 5} + \sqrt{3r^2 + 10r + 3}}\right).
\end{aligned}$$

Note that the function $\frac{r}{\sqrt{2}}\left(\frac{\sqrt{r^2+10r+5} - \sqrt{3r^2+10r+3}}{\sqrt{r^2+10r+5} + \sqrt{3r^2+10r+3}}\right)$ attains its maximum value ≈ 0.014333 when $r \approx 0.41796$, we get

$$\sigma_6(0.41796) \approx 0.014333.$$

This completes the proof of Theorem 2.6. ∎

References

[1] Z. Abdulhadi, Y. Muhanna, S. Khuri, On univalent solutions of the biharmonic equations, *J. Inequal. Appl.* **5** (2005) 469–478.

[2] Z. Abdulhadi, Y. Muhanna, S. Khuri, On some properties of solutions of the biharmonic equation, *Appl. Math. Comput.* **177** (2006) 346–351.

[3] Z. Abdulhadi, Y. Muhanna, Landau's theorems for biharmonic mappings, *J. Math. Anal. Appl.* **338** (2008) 705–709.

[4] H.H. Chen, On the Bloch constant. In: *Approximation Complex Analysis, and Potential Theory*, KLuwer Acad Publ, Dordrecht, 2001.

[5] H.H. Chen, P.M. Gauthier, W. Hengartner, Bloch constants for planar harmonic mappings, *Proc. Amer. Math. Soc.* **128** (11) (2000) 3231–3240.

[6] S. Chen, S. Ponnusamy, X. Wang, Landau's theorems for certain biharmonic mappings, *Appl. Math. Comput.* **208** (2) (2009) 427–433.

[7] F. Colonna, The Bloch constants of bounded harmonic mappings, *Indiana Univ. Math. J.* **38** (1989) 829–840.

[8] M. Dorff, M. Nowak, Landau's theorem for planar harmonic mappings, *Comput. Meth. Funct. Theory* **4** (1) (2000) 151–158.

[9] X.Z. Huang, Estimates on Bloch constants for planar harmonic mappings, *J. Math. Anal. Appl.* **337** (2008) 880–887.

[10] A. Grigoryan, Landau and Bloch theorems for harmonic mappings, *Complex Variable Theory Appl.* **51** (1) (2006) 81–87.

[11] H. Lewy, On the non-vanishing of the Jacobian in certain one-to-one mappings, *Bull. Amer. Math. Soc.* **42** (1936) 689–692.

[12] M.S. Liu, Landau's theorems for biharmonic mappings, *Complex Variables and Elliptic Equations* **53** (9) (2008) 843–855.

[13] M.S. Liu, Estimates on Bloch constants for planar harmonic mappings, *Sci. China Ser. A-Math.* **52** (1) (2009) 87–93.

[14] M.S. Liu, Landau's theorem for planar harmonic mappings, *Computers and Mathematics with Applications* **57** (7) (2009) 1142–1146.

[15] Y. Muhanna, G. Schober, Harmonic mappings onto convex mapping domains, *Canad. J. Math.* **XXXIX** (6) (1987) 1489–1530.

[16] W. Szapiel, Bounded harmonic mappings, *J. Anal. Math.* **111** (2010) 47–76.

[17] H.S. Zhan, The harmonic functions on a complete Riemannian manifold with negative sectional curvature, *Southeast Asian Bull. Math.* **32** (2) (2008) 399–404.

[18] Y.C. Zhu, Y. C. Zhu, Criteria for some subclasses of biholmorphic starlike mappings in Banach spaces, *Southeast Asian Bull. Math.* **34** (3) (2010) 579–593.