groove. This fold is well shown in Keibel's[4] figures of embryos of the pig, and must represent the supramaxillary fold of the Teleostomi, the lachrymal groove representing a part of the supramaxillary furrow of those fishes. The supramaxillary fold is apparently not continued onward anterior to this point, as it is in Chimaera and Ceratodus, and the Schnauzenfalte of His's[5] descriptions of human embryos, notwithstanding that it strikingly resembles the median portion of the supramaxillary fold of Chimaera and Ceratodus, is probably not a part of that fold. The lips and nasal apertures of the Mammalia could, accordingly, not be derived from those in Ceratodus without marked reversions, but they could readily be derived from those in Amia or Polypterus by the simple shifting of the secondary upper lip from a position oral to the nasal apertures to one between those apertures.

In the Amphibia the formation of the nasal appertures, as described by authors, is markedly different from that above set forth, but this is certainly due simply to condensations and abbreviations of the normal developmental processes, for the posterior nasal apertures of the adults of these vertebrates lie, as they do in the Amniota, between the primary and secondary dental arcades, and the nasal apertures of either side are, in embryos of certain of these vertebrates, connected by an epithelial cord (Gymnophiona) or line (Urodela) derived from the external epidermis; this cord or line certainly indicating the line where nasal processes have fused with each other above the nasal groove to form a normal nasal bridge.

[1] Müller, J., und Henle, J., *Systematische beschreibung der Plagiostomen*, 1841, Berlin, xxii + 200 pp., 60 Taf.

[2] Allis, E. P., Jr., *Q. J. Microsc. Sci., London*, N. S. **45**, 1901, (87–236), pl. 10–12.

[3] Peter, K., *Handbuch vergl. exper. Entwickelungslehre d. Wirbeltiere* von O. Hertwig, Bd. 2, Teil 2, 1906, (1–82).

[4] Keibel, Fr., *Anat. Anz., Jena*, **8**, 1893, (473–487).

[5] His, W., *Arch. Anat. Physiol., Anat. Abth., Leipzig* Jahrg. 1892, (384–424).

NATURAL AND ISOGONAL FAMILIES OF CURVES ON A SURFACE

By Joseph Lipka

DEPARTMENT OF MATHEMATICS, MASSACHUSETTS INSTITUTE OF TECHNOLOGY

Communicated by E. H. Moore, December 13, 1916

1. If F is a function of the coördinates of a point and ds is the element of arc length in any space, the curves along which $\int Fds$ *is a minimum* are said to form a *natural family of curves*. Such families include many interesting special cases. Thus if W is the negative potential function and h is a given constant of energy in a conservative

field of force, our natural family may be: (1) a system of *trajectories* arising from the principle of least action, where $F = \sqrt{W + h}$; (2) a system of *brachistochrones* or curves of quickest descent, where $F = \sqrt{(W + h)^{-1}}$; (3) a system of *general catenaries* or positions of equilibrium of homogeneous, flexible, inextensible strings, where $F = (W+h)$. Again, if F is the variable index of refraction in an isotropic medium, the *paths of light* in such a medium form a natural family. The *conformal representation of the geodesics* of any surface upon certain other surfaces is also a natural family. Natural families of curves have been geometrically characterized by Kasner for a plane and for space of three dimensions,[1] and by me for any surface and any curved space of n dimensions.[2]

If we have any set of ∞^1 curves on a surface, the system of ∞^1 curves which cut every curve of this set at a constant angle, α, form a system of isogonal trajectories of the original set; there are ∞^1 such systems for varying values of the parameter α, and these form the complete family of ∞^2 *isogonal trajectories*. Isogonal families of curves have been geometrically characterized by Kasner for the plane,[1] and by me[3] for any surface.

In this paper, §2 gives a very general geometric transformation by which a family of isogonals may be transformed into a natural family; §3 gives the analytic representation of the geometric transformation of §2, and exhibits the interchange of the two families through repeated application of this transformation; §4 gives the relations existing between the point functions which characterize dual (natural-isogonal) families.

2. If we take an isothermal system of curves as parameter curves on the surface, we can write the element of arc length in the form

$$ds^2 = \lambda\,(u, v)\,[du^2 + dv^2].$$

The variation problem

$$\int F\,(u,v)\,ds = minimum \qquad (1)$$

then leads to a family of ∞^2 curves whose differential equation is given by[3,4]

$$v'' = [\,(\log F\sqrt{\lambda})_v - (\log F\sqrt{\lambda})_u v']\,[1 + v'^2] \qquad (type\ N) \qquad (2)$$

and the problem of finding the isogonals of a simple system of curves

$$v' = \tan\omega\,(u,\ v) \qquad (3)$$

leads to a family of ∞^2 curves whose differential equation is given by[3]

$$v'' = (\omega_u + \omega_v v')\ (1 + v'^2) \qquad (type\ I_\omega) \qquad (4)$$

If we have any other simple system of curves

$$v' = \tan \alpha\,(u, v), \tag{5}$$

the differential equation of its isogonals is similarly given by

$$v'' = (\alpha_u + \alpha_v v')\,(1 + v'^2) \qquad (\textit{type } I_\alpha) \tag{6}$$

For these families, ∞^1 curves pass through each point on the surface, one in each direction.

The geodesic curvature for any curve on the surface is

$$\frac{1}{\rho} = \frac{v'' - [\,(\log \sqrt{\lambda})_v - (\log \sqrt{\lambda})_u\, v'\,]\,[\,1 + v'^2]}{\sqrt{\lambda}\,(1 + v'^2)^{\frac{3}{2}}} \tag{7}$$

If we apply (7) to the unrelated I_ω and I_α curves, we have, for a curve in any direction v',

$$\left[\frac{1}{\rho}\right]_{I_\omega} = \frac{[\,\omega_u - (\log \sqrt{\lambda})_v\,] + [\,\omega_v + (\log \sqrt{\lambda})_u\,]\, v'}{\sqrt{\lambda}\,(1 + v'^2)}, \tag{8}$$

$$\left[\frac{1}{\rho}\right]_{I_\alpha} = \frac{[\,\alpha_u - (\log \sqrt{\lambda})_v\,] + [\,\alpha_v + (\log \sqrt{\lambda})_u\,]\, v'}{\sqrt{\lambda}\,(1 + v'^2)}, \tag{9}$$

and hence

$$\left[\frac{1}{\rho}\right]_{I_\omega} - \left[\frac{1}{\rho}\right]_{I_\alpha} = \frac{(\omega_u - \alpha_u) + (\omega_v - \alpha_v)\, v'}{\sqrt{\lambda}\,(+ v'^2)}. \tag{10}$$

On the other hand, if we apply (7) to the N curves, we have, for a curve in any direction v',

$$\left[\frac{1}{\rho}\right]_N = \frac{(\log F)_v - (\log F)_u\, v'}{\sqrt{\lambda}\,(1 + v'^2)} \tag{11}$$

and for a curve in a direction $\left(-\frac{1}{v'}\right)$, i.e., in a direction perpendicular to the direction v',

$$\left[\frac{1}{\rho}\right]_N = \frac{(\log F)_u + (\log F)_v\, v'}{\sqrt{\lambda}\,(1 + v'^2)}. \tag{12}$$

Comparing (12) and (10) we see that the right members of these equations will coincide if

$$\log F = \omega - \alpha, \quad \text{or} \quad F = e^{\omega - \alpha}. \tag{13}$$

Thus if we have two distinct I families and we choose one curve of each family passing through the same point in the same direction, the difference of their geodesic curvatures is equal to the geodesic curvature of a curve through that point but in a perpendicular direction of a related N family. This is evidently true for every point and in every direction.

Considering our ∞^2 curves as composed of ∞^3 geodesic curvature elements (u, v, v', v''), and defining corresponding geodesic curvature elements on a surface as two elements which have the same initial point and the same direction, we may state the following result:

Given any two isogonal families. If in each direction through each point we construct a geodesic curvature element whose geodesic curvature is equal to the difference of the geodesic curvatures of corresponding elements of the two isogonal families, and then rotate each new element in the same direction through a right angle (keeping its geodesic curvature unchanged), the ∞^3 *new elements will form a natural family.*

If ω *and* α *are the functions determining the two isogonal families, then the above transformation leads to the natural family whose characteristic function,* F, *is the exponential of* $\omega - \alpha$.

According as we subtract the geodesic curvatures of the I_α curves from those of the I_ω curves or vice versa, we get the N family, $F = e^{\omega-\alpha}$ or $F_1 = e^{\alpha-\omega}$, so that $F = 1/F_1$; hence

Two isogonal families give rise, by the above mentioned transformation, to two natural families whose characteristic point functions are reciprocals, and such that corresponding geodesic curvature elements have their geodesic curvatures numerically equal but opposite in sign.

3. The analytic curvature transformation which changes an I family into an N family, is

$$(T)\ u_1 = u,\ v_1 = v,\ v_1' = -\frac{1}{v'},\ v_1'' = -\frac{v'' - [(\log\sqrt{\lambda} - \alpha)_v - (\log\sqrt{\lambda} - \alpha)_u v'][1 + v'^2]}{v'^3}$$

where α is an *arbitrary* point function.

This changes

$$v'' = (\omega_u + \omega_v v')\,(1 + v'^2) \qquad (type\ I) \qquad (4)$$

into

$$v'' = \{[(\omega - \alpha)_v + (\log\sqrt{\lambda})_v] - [(\omega - \alpha)_u + (\log\sqrt{\lambda})_u]v'\}\ \{1 + v'^2\}\ (type\ N) \qquad (14)$$

and by comparison with (2), we have

$$\log F = \omega - \alpha, \quad \text{or} \quad F = e^{\omega - \alpha} \qquad (15)$$

Hence (T) *is the analytic statement of the geometric transformation described in* §2.

It is interesting to note the results of repeated applications of the transformation (T) on the I family, (4). (T^2), (T^3), and (T^4) lead respectively to

$$v'' = \{[(\alpha - \omega - \log\sqrt{\lambda})_u - (\alpha - \log\sqrt{\lambda})_v] + [(\alpha - \omega - \log\sqrt{\lambda})_v + (\alpha - \log\sqrt{\lambda})_u]\, v'\}\ \{1 + v'^2\}, \tag{16}$$

$$v'' = \{[(\alpha - \log\sqrt{\lambda})_u - \omega_v] + [(\alpha - \log\sqrt{\lambda})_v + \omega_u]v'\}\ \{1 + v'^2\} \tag{17}$$

$$v'' = (\omega_u + \omega_v\, v')\,(1 + v'^2). \tag{18}$$

Now equations (2) and (4) are special forms of a more general equation

$$v'' = (\psi - \phi\, v')\,(1 + v'^2), \tag{19}$$

which reduces to type I or type N according to the restriction

$$\psi_v + \phi_u = 0 \quad \text{or} \quad \psi_u - \phi_v = 0 \tag{20}$$

respectively. Applying the criteria (20) to equations (16), (17), and (18), we may draw the following conclusions:

Given any I family, (T) always transforms this into an N family, and (T^4) always gives the orginal I family. In general (T^2) and (T^3) give neither an I nor an N family; but if the auxiliary arbitrary function, α, is so chosen that the system $v' = \tan(\alpha - \log\sqrt{\lambda})$ is an isothermal system,[5] or if our surface is developable[6] and the system $v' = \tan\alpha$ is isothermal, then (T^2) gives an I family and (T^3) gives an N family.

Given any $N-I$ family, i.e., the isogonals of an isothermal system (cf. §4), (T) always transforms this into an N family, (T^2) gives neither an I family nor an N family, in general, (T^3) always gives an I family, and (T^4) always gives the original $N-I$ family. If the auxiliary arbitrary function is so chosen that the system $v' = \tan(\alpha - \log\sqrt{\lambda})$ is isothermal, or if our surface is developable and the system $v' = \tan\alpha$ is isothermal, then (T), (T^2), and (T^3) give $N-I$ families (not the original family).

4. If the N family

$$v'' = [(\log F\sqrt{\lambda})_v - (\log F\sqrt{\lambda})_u\, v']\ [1 + v'^2]$$

and the I family

$$v'' = (\omega_u + \omega_v\, v')\,(1 + v'^2)$$

coincide, then we must have

$$\left.\begin{array}{l} (\log F\sqrt{\lambda})_v = \omega_u \quad \text{and} \quad (\log F\sqrt{\lambda})_u = -\,\omega_v \\ \text{or} \\ \omega_{uu} + \omega_{vv} = 0 \quad \text{and} \quad (\log F\sqrt{\lambda})_{uu} + (\log F\sqrt{\lambda})_{vv} = 0 \end{array}\right\} \tag{21}$$

Therefore the curves $v' = \tan\omega$ and $v' = \tan(\log F\sqrt{\lambda})$ are isothermal systems, and the functions ω and $\log F\sqrt{\lambda}$ are conjugate harmonic. Thus the base system of our isogonals is isothermal, and if H is conjugate harmonic to ω, i.e., if $\omega + i H$ is a function of $u + i v$, then $F = \lambda^{-\frac{1}{2}} e^H$. Now for our parameter system, which is *any* isothermal system, $\omega = 0$ and therefore $H = 0$ and $F = \lambda^{-\frac{1}{2}} = F_0$, and we may write $F = F_0 e^H$. Hence

If F_0 is the characteristic function corresponding to the isogonals of an isothermal system, an N family can be identified with an I family when and only when its characteristic function is the product of F_0 and the exponential of a harmonic function.[7]

If we have two isothermal systems, $v' = \tan \omega$ and $v' = \tan \alpha$, and ω and α are conjugate harmonic, they form the base systems of two $N-I$ families whose characteristic functions are

$$F_\omega = \lambda^{-\frac{1}{2}} e^{\alpha} \quad \text{and} \quad F_\alpha = \lambda^{-\frac{1}{2}} e^{\omega}$$

respectively; hence

$$\frac{F_\alpha}{F_\omega} = e^{\omega - \alpha} \tag{21}$$

On the other hand, the isogonals of the system $v' = \tan \omega$ are transformed by (T) into the N family whose characteristic function is $F = e^{\omega - \alpha}$. Comparing this with (21) we may write

$$F = \frac{F_\alpha}{F_\omega} \tag{22}$$

If $\omega + i \alpha$ is a function of the complex variable $u + i v$, it determines two isothermal systems, $v' = \tan \omega$ and $v' = \tan \alpha$, which are the base systems of two $N - I$ families. Either of these families may be transformed by means of (T) and the remaining family into an N family whose characteristic function is the ratio of the characteristic functions of the two given families.

[1] Kasner, E., *Trans. Amer. Math. Soc., New York*, **10**, 1909, (201–219).

[2] Lipka, J., *Ibid.*, **13**, 1912, (77–95).

[3] Lipka, J., *Ann. Math., Princeton, N. J.*, **15**, 1913, (71–77).

[4] Throughout this paper, primes refer to total derivatives with respect to u, and literal subscripts to partial derivatives.

[5] The condition that the system $v' = \tan\beta$ be isothermal is $\beta_{uu} + \beta_{vv} = 0$.

[6] The condition for a developable surface is $(\log \lambda)_{uu} + (\log \lambda)_{vv} = 0$; cf. Note[3].

[7] Compare with Kasner, E., *New York, Bull. Amer. Math. Soc.*, **14**, 1908, (169–172).