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ON MEROMORPHIC SOLUTIONS OF A TYPE OF SYSTEM OF COMPOSITE FUNCTIONAL EQUATIONS*

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Abstract In this article, we mainly investigate the growth and existence of meromorphic solutions of a type of systems of composite functional equations, and obtain some interesting results. It extends some results concerning functional equations to the systems of functional equations.

Key words meromorphic solution; composite function; functional equations

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1 Introduction

Many authors investigated the problem of existence and growth of meromorphic solution of algebraic differential equations in the complex plane and obtained many results [12–25]. However, functional equations attracted considerable attention during the last few decades only in the complex plane. In the 1970's, R.Goldstein wrote a series of articles concerning the value distribution of solutions of certain functional equations containing composite functions. A substantial part of his research was concerned with functional equations of the form $f(g(z)) = q(f(z))$ and $f(g(z)) = q(f(z))+h(z)$, where f and h are meromorphic functions, g is entire and q is a rational function of order $k \geq 1$. Goldstein applied the Nevanlinna theory to prove several results on the growth of the functions involved [3–6]. During the last decade, meromorphic solutions of functional equations were studied by W.Bergweiler [7–9], J.Heittokangas [10], et al.

Recently, H.Silvennoinen considered the growth and existence of meromorphic solutions of functional equations of the form $f(p(z)) = R(z, f(z))$, he obtained:

Theorem A [11] Let f be a non-constant meromorphic solution of the equation

$$f(g(z)) = \frac{\sum\limits_{i=0}^{m_1} a_i(z) f^i(z)}{\sum\limits_{j=0}^{n_1} b_j(z) f^j(z)},$$

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where g is an entire function, a_i, b_j are small meromorphic functions with respect to f. Then, g is a polynomial.

A question is, whether the assertion of Theorem A remains valid, if we replace the equation

$$f(g(z)) = \frac{\sum\limits_{i=0}^{m_1} a_i(z) f^i(z)}{\sum\limits_{j=0}^{n_1} b_j(z) f^j(z)}$$

by the following form

$$\begin{cases} f_1(p(z)) = R_1(z, f_2(z)), \\ f_2(p(z)) = R_2(z, f_1(z)). \end{cases}$$

In this article, we would study the problem of existence of meromorphic solutions for a system of functional equations

$$\begin{cases} f_1(p(z)) = R_1(z, f_2(z)), \\ f_2(p(z)) = R_2(z, f_1(z)), \end{cases} \tag{1}$$

where $p(z)$ is an entire function, $R_1(z, f_2(z)) = \frac{\sum\limits_{i=0}^{m_1} a_i(z) f_2^i(z)}{\sum\limits_{j=0}^{n_1} b_j(z) f_2^j(z)}, R_2(z, f_1(z)) = \frac{\sum\limits_{i=0}^{m_2} c_i(z) f_1^i(z)}{\sum\limits_{j=0}^{n_2} d_j(z) f_1^j(z)}$ are irreducible rational function, $a_i(z), b_j(z), c_i(z), d_j(z)$ are small functions.

Our results are:

Theorem 1 Let (f_1, f_2) be a non-constant meromorphic solution of the system (1). Then, $p(z)$ is a polynomial.

Theorem 2 Let the polynomial $p(z) = e_k z^k + e_{k-1} z^{k-1} + \cdots + e_1 z + e_0$ be of degree $k \geq 2$, (f_1, f_2) be a transcendental meromorphic solution of the system (1), $a_{m_1} b_{n_1} \not\equiv 0, c_{m_2} d_{n_2} \not\equiv 0$, a_i, b_j, c_i, d_j be small functions. If

$$\overline{N}(r, f_i(p(z))) + \overline{N}(r, \frac{1}{f_i(p(z))}) = S(r, f_i),$$

then, system (1) is of the from

$$\begin{cases} f_1(p(z)) = a(z) f_2^s(z), \\ f_2(p(z)) = b(z) f_1^l(z), \end{cases}$$

where $a(z), b(z)$ are meromorphic, $T(r, a(z)) = S(r, f_i), T(r, b(z)) = S(r, f_i)$ and $s, l \in \mathbf{Z}, s \neq 0, l \neq 0$.

Theorem 3 Let $p(z) = az + b$, (f_1, f_2) be a meromorphic solution of system (1), and $\mu(f_1), \mu(f_2)$ be the lower orders of f_1, f_2, respectively. If

$$\mu(f_1) + \mu(f_2) < \frac{\log d_1 d_2}{\log |a|^2},$$

then, the components f_1, f_2 in (f_1, f_2) have at least one rational function, where $d_i = \max\{m_i, n_i\}$, $i = 1, 2$.

We use the standard notation of the Nevanlinna theory of meromorphic functions (see [1, 2]).

2 Some Lemmas

To obtain some results, we first need the following lemmas.

Lemma 1 [2] Let $R(z,w)=\frac{\sum_{i=0}^{p}a_i(z)w^i}{\sum_{j=0}^{q}b_j(z)w^j}$ be an irreducible rational function in $w(z)$ with meromorphic coefficients $\{a_i(z)\}$ and $\{b_j(z)\}$. If $w(z)$ is an meromorphic function, then,

$$T(r,R(z,w))=\max\{p,q\}T(r,w)+O\{\sum T(r,a_i)+\sum T(r,b_j)\}.$$

Lemma 2 [2] Let $g{:}(0,+\infty)\to\mathbf{R}$, $h{:}(0,+\infty)\to\mathbf{R}$ be monotone increasing functions such that $g(r)\le h(r)$ outside of an exceptional set E with finite linear measure. Then, for any $\alpha>1$, there exists r_0 such that $g(r)\le h(\alpha r)$ for all $r>r_0$.

Lemma 3 [6] Let f be a transcendental meromorphic function and $p(z)=a_k+a_{k-1}z^{k-1}+\cdots+a_1(z)+a_0, a_k\neq 0, k\ge 1$, be a polynomial of degree k. Given $0<\delta<|a_k|$, let $\lambda=|a_k|+\delta, \mu=|a_k|-\delta$. Then, given $\varepsilon>0$, for any $a\in\mathbf{C}\cup\{\infty\}$ and for r large enough, we have

$$kn(\mu r^k,\frac{1}{f-a})\le n(r,\frac{1}{f(p)-a})\le kn(\lambda r^k,\frac{1}{f-a}),$$

$$N(\mu r^k,\frac{1}{f-a})+O(\log r)\le N(r,\frac{1}{f(p)-a})\le N(\lambda r^k,\frac{1}{f-a})+O(\log r),$$

$$(1-\varepsilon)T(\mu r,f)\le T(r,f(p))\le(1+\varepsilon)T(\lambda r^k,f).$$

Lemma 4 [8] Let f be a non-constant meromorphic function and let g be a transcendental entire function. Then, there exists an increasing sequence $r_n\to\infty$, such that

$$T(r,f(g(z)))\ge T((M(\frac{r}{4},g))^{\frac{1}{30}},f)$$

holds for $r=r_n$.

Lemma 5 [8] Suppose that a meromorphic function f has finite lower order λ. Then, for every constant $c>1$ and a given ε, there exists a sequence $r_n=r_n(c,\varepsilon)\to\infty$, such that

$$T(cr_n,f)\le c^{\lambda+\varepsilon}T(r_n,f).$$

3 Proof of Theorem 1

Suppose that $p(z)$ is transcendental entire function, we have

$$\liminf_{r\to\infty}\frac{\log M(r,p(z))}{\log r}=\infty.$$

Hence for any given $K>30$ and for r large enough,

$$M(r,p)>r^K.$$

There exists an increasing sequence $r_n\to\infty$, as in Lemma 4 such that, for any n,

$$M(\frac{r_n}{4},p)>(\frac{r_n}{4})^K.$$

By Lemma 1 and the systems (1), we have

$$\begin{cases} T(r,f_1(p(z)))=T(r,R_1(z,f_2(z)))=\max\{m_1,n_1\}(1+o(1))T(r,f_2),\\ T(r,f_2(p(z)))=T(r,R_2(z,f_1(z)))=\max\{m_2,n_2\}(1+o(1))T(r,f_1),\end{cases}\tag{2}$$

outside a possible exceptional set of finite linear measure. According to Lemma 2, for $\forall\alpha>1, r\geq r_\alpha$, we obtain

$$T(r,f_1(p(z)))\leq\max\{m_1,n_1\}(1+o(1))T(\alpha r,f_2),\tag{3}$$

$$T(r,f_2(p(z)))\leq\max\{m_2,n_2\}(1+o(1))T(\alpha r,f_1).\tag{4}$$

It follows from Lemma 4 that

$$T(r_n,f_1(p(z)))\geq T((\frac{r_n}{4})^{\frac{K}{30}},f_1),\tag{5}$$

$$T(r_n,f_2(p(z)))\geq T((\frac{r_n}{4})^{\frac{K}{30}},f_2).\tag{6}$$

Note that $\frac{T(r,f_i)}{\log r}$ is an increasing function of r. As

$$(\frac{r_n}{4})^{\frac{K}{30}}>\alpha r_n,$$

we have, for sufficiently large n,

$$T((\frac{r_n}{4})^{\frac{K}{30}},f_1)>\frac{K/30(\log r_n-\log 4)}{\log r_n+\log\alpha}>\frac{K}{40}T(\alpha r_n,f_1),\tag{7}$$

$$T((\frac{r_n}{4})^{\frac{K}{30}},f_2)>\frac{K/30(\log r_n-\log 4)}{\log r_n+\log\alpha}>\frac{K}{40}T(\alpha r_n,f_2),\tag{8}$$

as $n\to\infty$. By (3), (5), (7) and (4), (6), (8), we get, respectively,

$$\max\{m_1,n_1\}(1+o(1))T(\alpha r_n,f_2)\geq T((\frac{r_n}{4})^{\frac{K}{30}},f_1)>\frac{K}{40}T(\alpha r_n,f_1),\tag{9}$$

$$\max\{m_2,n_2\}(1+o(1))T(\alpha r_n,f_1)\geq T((\frac{r_n}{4})^{\frac{K}{30}},f_1)>\frac{K}{40}T(\alpha r_n,f_2),\tag{10}$$

as $n\to\infty$.

Combining (9) and (10), we have

$$\begin{aligned}\frac{K^2}{1600}T(\alpha r_n,f_2)&<\max\{m_2,n_2\}(1+o(1))T(\alpha r_n,f_1)\\&<\max\{m_1,n_1\}(1+o(1))\max\{m_2,n_2\}T(\alpha r_n,f_2).\end{aligned}$$

Because K can be arbitrarily large, this is a contradiction. This shows that $p(z)$ is a polynomial.

4 Proof of Theorem 2

Let z_0 be a zero of f_1. By f_1 is transcendental, there exist at most t points, $\xi_1,\xi_2,\cdots,\xi_{t_1}$, $t_1\leq k$, such that $p(\xi_j)=z_0$ and $f_1(p(\xi_j))=0$. Hence,

$$t_1\overline{N}(r,\frac{1}{f_1})\leq\overline{N}(r,\frac{1}{f_1(p(z))}).$$

Similarly, we have

$$t_2\overline{N}(r,\frac{1}{f_2}) \le \overline{N}(r,\frac{1}{f_2(p(z))}).$$

Under the assumptions of Theorem 2 and Lemma of the logarithmic derivative, we obtain

$$\begin{aligned}T(r,\frac{f_i'(p)}{f_i(p)}) &= N(r,\frac{f_i'(p)}{f_i(p)}) + m(r,\frac{f_i'(p)}{f_i(p)})\\ &\le \overline{N}(r,f_i(p(z))) + \overline{N}(r,\frac{1}{f_i(p(z))}) + S(r,f_i) = S(r,f_i).\end{aligned}$$

$$T(r,\frac{f_i'}{f_i}) = \overline{N}(r,f_i) + \overline{N}(r,\frac{1}{f_i}) = S(r,f_i).$$

Now, we further rewrite (1) in the form

$$\begin{cases}\dfrac{b_{n_1}(z)}{a_{m_1}(z)}f_1(p(z)) = \dfrac{P_1(z,f_2)}{Q_1(z,f_2)} = u_1(z,f_2),\\ \dfrac{d_{n_2}(z)}{c_{m_2}(z)}f_2(p(z)) = \dfrac{P_2(z,f_1)}{Q_2(z,f_1)} = u_2(z,f_1),\end{cases} \tag{11}$$

where P_1, Q_1 and P_2, Q_2 are monic polynomials in f_2 and f_1 with coefficients of growth $S(r,f_i), i=1,2$, respectively. Denote $F_i = \frac{f_i'}{f_i}, U_i = \frac{u_i'}{u_i}, i=1,2$. We have $T(r,U_i) = S(r,f_i)$. Because

$$\frac{P_1'Q_1 - Q_1'P_1}{Q_1^2} = u_1' = U_1u_1 = \frac{U_1P_1}{Q_1},$$
$$\frac{P_2'Q_2 - Q_2'P_2}{Q_2^2} = u_2' = U_2u_2 = \frac{U_2P_2}{Q_2},$$

we have

$$P_1'Q_1 - Q_1'P_1 = U_1P_1Q_1, \tag{12}$$
$$P_2'Q_2 - Q_2'P_2 = U_2P_2Q_2. \tag{13}$$

Writing $f_i' = F_if_i$, from (12) and (13) we obtain

$$n_1f_2^{n_1-1}f_2'Q_1 - m_1f_2^{m_1-1}f_2'P_1 = U_1P_1Q_1,$$
$$n_2f_1^{n_2-1}f_1'Q_2 - m_2f_1^{m_2-1}f_1'P_2 = U_2P_2Q_2,$$

that is,

$$(n_1f_2^{n_1}Q_1 - m_1f_2^{m_1}P_1)F_2 = U_1P_1Q_1, \tag{14}$$
$$(n_2f_1^{n_2}Q_2 - m_2f_1^{m_2}P_2)F_1 = U_2P_2Q_2. \tag{15}$$

From (14), comparing the leading coefficients, we have

$$(n_1 - m_1)F_2 = U_1.$$

From (15), comparing the leading coefficients, we have

$$(n_2 - m_2)F_1 = U_2.$$

Hence, we obtain

$$u_1(z,f_2) = \alpha f_2^{n_1-m_1}, u_2(z,f_1) = \beta f_1^{n_2-m_2}$$

for $\alpha, \beta, \in \mathbf{C}$, and so

$$f_1(p(z)) = \alpha\frac{a_{m_1}(z)}{b_{n_1}(z)}f_2^{n_1-m_1}, f_2(p(z)) = \beta\frac{c_{m_2}(z)}{d_{n_2}(z)}f_1^{n_2-m_2}.$$

This proves Theorem 2.

5 Proof of Theorem 3

We assume conversely that f_1, f_2 are transcendental meromorphic functions.

By Lemma 1 and $T(r, f(z+c)) \leq (1+o(1))T(r+|c|, f)+M$ [26], where M is a constant, we have

$$\begin{cases} d_1T(r,f_2)=T(r,f_1(a(z+\dfrac{b}{a}))) \leq (1+o(1))T(|a|r+|\dfrac{b}{a}|,f_1)+S(r,f_2), \\ d_2T(r,f_1)=T(r,f_2(a(z+\dfrac{b}{a}))) \leq (1+o(1))T(|a|r+|\dfrac{b}{a}|,f_2)+S(r,f_1), \end{cases} \tag{16}$$

where $d_i=\max\{m_i,n_i\}, i=1,2$. There are a constants $c_i=|a|+\varepsilon_i, \varepsilon_i>0, i=1,2$, such that

$$T(|a|r+|\frac{b}{a}|,f_1) \leq T(c_1r,f_1), T(|a|r+|\frac{b}{a}|,f_2) \leq T(c_2r,f_2).$$

When r is large enough,

$$\begin{cases} d_1T(r,f_2) \leq (1+o(1))T(c_1r,f_1)+S(r,f_2), \\ d_2T(r,f_1) \leq (1+o(1))T(c_2r,f_2)+S(r,f_1), \end{cases}$$

outside a possible exceptional set with finite linear measure. According to Lemma 2, for given $\sigma>1$,

$$\begin{cases} d_1T(r,f_2) \leq (1+o(1))T(\sigma c_1r,f_1)+S(r,f_2), \\ d_2T(r,f_1) \leq (1+o(1))T(\sigma c_2r,f_2)+S(r,f_1). \end{cases} \tag{17}$$

Let $\mu(f_1), \mu(f_2)$ be the finite lower order in f_1, f_2, respectively.By Lemma 5, for any given $\varepsilon>0$, there exists a sequence $r_n \to \infty$ such that, for $r_n>r_0$,

$$T(c_1r_n,f_1) \leq c_1^{\mu(f_1)+\varepsilon}T(r_n,f_1), T(c_2r_n,f_2) \leq c_2^{\mu(f_2)+\varepsilon}T(r_n,f_2).$$

Hence,

$$\begin{cases} d_1T(r_n,f_2) \leq (1+o(1))(\sigma c_1)^{\mu(f_1)+\varepsilon}T(r_n,f_1)+S(r_n,f_2), \\ d_2T(r_n,f_1) \leq (1+o(1))(\sigma c_2)^{\mu(f_2)+\varepsilon}T(r_n,f_2)+S(r_n,f_1). \end{cases} \tag{18}$$

From (18), we get

$$\begin{cases} d_1 \leq (1+o(1))(\sigma c_1)^{\mu(f_1)+\varepsilon}\dfrac{T(r_n,f_1)}{T(r_n,f_2)}+\dfrac{S(r_n,f_2)}{T(r_n,f_2)}, \\ d_2 \leq (1+o(1))(\sigma c_2)^{\mu(f_2)+\varepsilon}\dfrac{T(r_n,f_2)}{T(r_n,f_1)}+\dfrac{S(r_n,f_1)}{T(r_n,f_1)}. \end{cases} \tag{19}$$

Take the lower limit as $n \to \infty$, and $\lim\limits_{n\to\infty} \inf \frac{S(r_n,f_i)}{T(r_n,f_i)}=0, i=1,2$. Then, (19) becomes

$$d_1d_2 \leq (\sigma c_1)^{\mu(f_1)+\varepsilon_3}(\sigma c_2)^{\mu(f_2)+\varepsilon_3},$$

where $\varepsilon_3=\max\{\varepsilon,\varepsilon_1,\varepsilon_2\}, \varepsilon_3 \to 0, \sigma \to 1$. Hence,

$$\mu(f_1)+\mu(f_2) \geq \frac{\log d_1d_2}{\log |a|^2},$$

thus, we have a contradiction to the assumptions of Theorem 3.

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