# Measure Theory and Lebesgue Integration

Joshua H. Lifton

Originally published 31 March 1999
Revised 5 September 2004

**Abstract**

This paper originally came out of my 1999 Swarthmore College Mathematics Senior Conference. I've made minor touch-ups to make it more presentable.

This paper begins where the Swarthmore College Mathematics and Statistics course Math 47: Introduction to Real Analysis left off. Namely, basic measure theory is covered with an eye toward exploring the Lebesgue integral and comparing it to the Riemann integral. Knowledge of the notation and techniques used in an introductory analysis course is assumed throughout.

# Contents

# Acknowledgments

Many thanks to all members of the Swarthmore College Mathematics and Statistics Department for their encouragement and tolerance of my mathematics education.

# Chapter 1

# How to Count Rectangles: A Review of Integration

## 1.1 Riemann Revisited

The development of the integral in most introductory analysis courses is centered almost exclusively on the Riemann integral. This relatively intuitive approach begins by taking a partition $P = \{x_0, \ldots, x_n\}$ of the domain of the real-valued function $f$ in question. Given $P$, the Riemann sum of $f$ is simply

$$\sum_{i=1}^{n} (x_i - x_{i-1}) \cdot f(c_i) \qquad \text{where } x_{i-1} < c_i < x_i.$$

The integral of $f$, if it exists, is the limit of the Riemann sum as $n \to \infty$.

## 1.2 Shortcomings of Riemann Integration

Although the Riemann integral suffices in most daily situations, it fails to meet our needs in several important ways. First, the class of Riemann integrable functions is relatively small.

Second and related to the first, the Riemann integral does not have satisfactory limit properties. That is, given a sequence of Riemann integrable functions $\{f_n\}$ with a limit function $f = \lim\limits_{n\to\infty} f_n$, it does not necessarily follow that the limit function $f$ is Riemann integrable.

Third, all $L_p$ spaces except for $L_\infty$ fail to be complete under the Riemann integral.

Examples 1.1 and 1.2 illustrate some of these problems.

**Example 1.1** *Consider the sequence of functions* $\{f_n\}$ *over the interval* $E = [0, 1]$.

$$f_n(x) = \begin{cases} 2^n & \text{if } \frac{1}{2^n} \leq x \leq \frac{1}{2^{n-1}} \\ 0 & \text{otherwise} \end{cases}$$

*The limit function of this sequence is simply* $f = 0$. *In this example, each function in the sequence is integrable as is the limit function. However, the limit of the sequence of integrals is not equal to the integral of the limit of the sequence. That is,*

$$\lim_{n\to\infty} \int_0^1 f_n(x)\,dx = 1 \neq 0 = \int_0^1 \lim_{n\to\infty} f_n(x)\,dx.$$

*We will return to this function as an example of how the Lebesgue integral fails as well.*

**Example 1.2** *Consider the sequence of functions* $\{d_n\}$ *over the interval* $E = [0, 1]$.

$$d_n(x) = \begin{cases} 1 & \text{if } x \in \{r_n\} \\ 0 & \text{otherwise} \end{cases}$$

*where* $\{r_n\}$ *is the set of the first* $n$ *elements of some decided upon enumeration of the rational numbers. Each function* $d_n$ *is Riemann integrable since it is discontinuous only at* $n$ *points. The limit function* $D = \lim_{n\to\infty} d_n$ *is given by*

$$D(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$$

*This function, known as the Dirichlet function, is discontinuous everywhere and therefore not Riemann integrable. Another way of showing that* $D(x)$ *is not Riemann integrable is to take upper and lower sums, which result in* 1 *and* 0, *respectively.*

## 1.3 A New Way to Count Rectangles: Lebesgue Integration

An equally intuitive, but long in coming method of integration, was presented by Lebesgue in 1902. Rather than partitioning the domain of the function, as in the Riemann integral, Lebesgue chose to partition the range. Thus, for each interval in the partition, rather than asking for the value of the function between the end points of the interval in the domain, he asked how much of the domain is mapped by the function to some value between two end points in the range. See Figure 1.1.

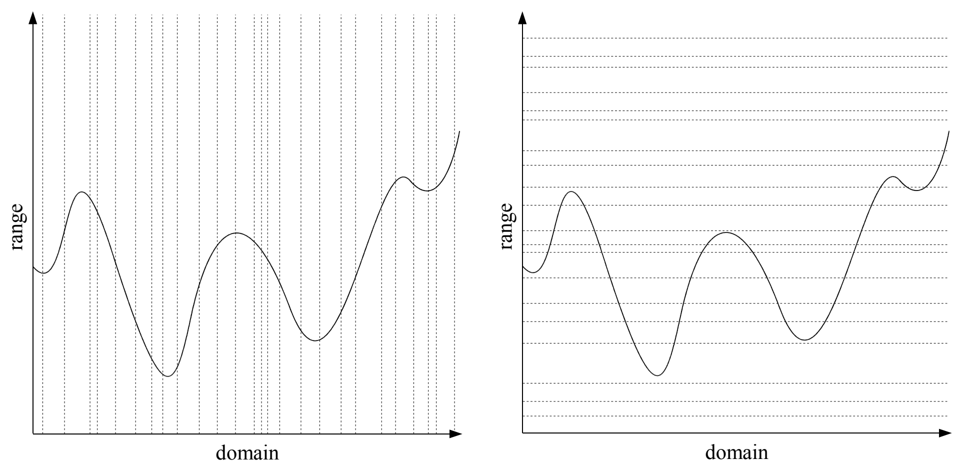


Figure 1.1: Two ways to count rectangles – partitioning the range as opposed to partitioning the domain of a function.

Partitioning the range of a function and counting the resultant rectangles becomes tricky since we must employ some way of determining (or measuring) how much of the domain is sent to a particular portion of a partition of the range. Measure theory addresses just this problem.

As it turns out, the Lebesgue integral solves many of the problems left by the Riemann integral. With this in mind, we now turn to measure theory.

# Chapter 2

# Measure Theory

Measure theory is a rich subject in and of itself. However, we present it here expressly for the purpose proposed at the end of §1.3; to define the "length" of an arbitrary set so as to formalize the idea of the Lebesgue integral. As such, only the very basics of measure theory are presented here and many of the rote proofs are left to the reader.

## 2.1 Measure

Given an interval $E = [a, b]$ and a set $\mathcal{S}$ of subsets of $E$ which is closed under countable unions, we define the following.

**Definition 2.1** *A* **set function** *on $\mathcal{S}$ is a function which assigns to each set $A \in \mathcal{S}$ a real number.*

**Definition 2.2** *A set function $\mu$ on $\mathcal{S}$ is called a* **measure** *if the following properties hold.*

- *Semi-Positive-Definite:* $0 \leq \mu(A) \leq b - a$ *for all* $A \in \mathcal{S}$.
- *Trivial Case:* $\mu(\emptyset) = 0$.
- *Monotonicity:* $\mu(A) \leq \mu(B)$ *for all* $A, B \in \mathcal{S}, \mathcal{A} \subset \mathcal{B}$
- *Countable Additivity: if* $A = \bigcup_{n=1}^{\infty} A_n$, *then* $\mu(A) = \sum_{n=1}^{\infty} \mu(A_n)$, *where* $A_n \in \mathcal{S}$ *for* $n = 1, 2, \ldots$ *and* $A_n \bigcap A_m = \emptyset$ *for* $n \neq m$.

As can be seen from the definition, the concept of measure is a general one indeed.

**Example 2.3** *Examples of measure abound. To begin at the beginning, consider the trivial measure:* $\mu(A) = 0$ *for all* $A \in \mathcal{S}$*.*

**Example 2.4** *Let* $\{x_1, \ldots, x_n\} \subset E$ *be a finite set of points and* $E = [a, b]$*. Let* $a < x_i < x_j < b$ *for all* $i < j$*. Consider the set* $A = E/\{x_1, \ldots, x_n\}$*, the interval* $E$ *without the points* $x_1, \ldots, x_n$*. Let our measure be such that the measure of any interval with endpoints* $x < y$ *is* $y - x$*. Then,*

$$\begin{aligned} m(A) &= m([a, x_1) \cup (x_1, x_2) \cup \ldots \cup (x_{n-1}, x_n) \cup (x_n, b]) \\ &= m([a, x_1)) + m((x_1, x_2)) + \ldots + m((x_{n-1}, x_n)) + m((x_n, b]) \\ &= (x_1 - a) + (x_2 - x_1) + \ldots + (x_n - x_{n-1}) + (b - x_n) \\ &= b - a \\ &= m(E). \end{aligned}$$

*Notice that the only complication which may arise in this example is how we define our set of subsets,* $\mathcal{S}$*. We will explore this further in §2.3.*

## 2.2 Outer and Inner Measure

To keep us grounded in reality, we would like to use a measure which puts our intuitive notion of length at ease. Thus we define the following.

**Definition 2.5** *The* **outer measure** *of any interval* $I$ *on the real number line with endpoints* $a < b$ *is* $b - a$ *and is denoted as* $m^*(I)$*.*

We would like to generalize this definition of outer measure to all sets of real numbers, not just intervals. To do this, we use the following theorem, upon which the remainder of our work depends heavily.

**Theorem 2.6** *Every non-empty open set* $G \subset \mathcal{R}$ *can be uniquely expressed as a finite or countably infinite union of pairwise disjoint open intervals.*
**Proof Idea:** *Let* $x \in G$ *and consider the open interval* $I_x = (a_x, b_x)$ *constructed from*

$$a_x = glb\{y \,|\, (y, x) \subset G\}$$

*and*

$$b_x = lub\{z \,|\, (x, z) \subset G\}.$$

$I_x$ *is called the component of* $x$ *in* $G$*.* $G = \cup_{x \in G} I_x$ *is exactly the decomposition of* $G$ *we are looking for. It is straight-forward to show that, given*

*$x_1, x_2 \in G$, either $I_{x_1}$ and $I_{x_2}$ are disjoint or they are equal. The countability of this collection of open intervals follows easily from their disjointness; simply pick one rational number from each interval and use it to label that interval. Since the intervals are disjoint, no two will have the same label and since the rationals are countable, so will be the intervals which they label. Uniqueness of the decomposition is, as usual, a straight-forward proof by contradiction.*

With this theorem, it is possible to extend the definition of outer measure to open sets of real numbers.

**Definition 2.7** *The* **outer measure** *$m^*(G)$ of an open set $G \subset E$ is given by $\sum_i m^*(I_i)$ where the $I_i$ form the unique decomposition of $G$ into a finite or a countably infinite union of pairwise disjoint open intervals. See Theorem 2.6.*

From this directly follows a definition of outer measure for any set.

**Definition 2.8** *The* **outer measure** *$m^*(A)$ of any set $A \subset \mathcal{R}$ is given by $glb\{m^*(G)|A \subset G$ and $G$ open in $E\}$.*

Now that outer measure is well-defined for arbitrary subsets of $E$, we turn to a closely related measure, inner measure.

**Definition 2.9** *The* **inner measure** *of any set $A \subset E$, denoted $m_*(A)$, is defined as $m^*(E) - m^*(E/A)$, where $E/A$ is the compliment of $A$ with respect to $E$.*

Intuitively, the inner measure is in some ways "measuring" the same thing as the outer measure, only in a more roundabout way. We cannot take it for granted, however, that the inner and outer measures of any given set are the same, although we are very interested in the cases when they are, as we shall soon see. For now, though, let us state without proof some simple observations regarding inner and outer measure.

**Lemma 2.10** *The measures $m_*$ and $m^*$ both exhibit monotonicity. That is, given $A \subset B \subset E$, it follows that,*

- $m^*(A) \leq m^*(B)$
- $m_*(A) \leq m_*(B)$

**Lemma 2.11** *Given a set $A \subset E$, it follows that $m_*(A) \leq m^*(A)$.*

**Lemma 2.12** *The outer measure $m^*$ exhibits subadditivity. That is, whenever $\{A_n \mid n = 1, 2, \ldots\}$ is a set of subsets of $E$, then*

$$m^*(\bigcup_{n=1}^{\infty} A_n) \leq \sum_{n=1}^{\infty} m^*(A_n).$$

## 2.3 Shortcomings of Outer and Inner Measure

Now that we have some concrete measures to work with, we must examine whether they satisfy our needs. The first shortcoming we discover is presented below.

**Theorem 2.13** *The outer measure, $m^*$, is not countably additive on the set of all subsets of $E$.*
**Proof Idea:** *By construction. It is possible to construct pairwise disjoint sets $A_n$ such that $E = \cup_{n=1}^{\infty} A_n$ and $m^*(E) \neq \sum_{n=1}^{\infty} m^*(A_n)$.*

This is rather disappointing. It would be nice to have a measure which is defined on all subsets of $E$ and still satisfies our intuition enough to use it as the basis for the development of the Lebesgue integral. However, it turns out that the set of subsets of $E$ on which the outer measure $m^*$ is countably additive is large enough to make it worth our while to continue using the outer measure. We discuss this point in Chapter 3.

# Chapter 3

# Measurable Sets & Measurable Functions

## 3.1 Measurable Sets

Let's look at a certain subset of all the sets on which inner and outer measure are well-defined – the set of *Lebesgue measurable* sets.

**Definition 3.1** *A set $A \subset E$ is* **Lebesgue measurable***, or* **measurable***, if $m^*(A) = m_*(A)$, in which case the* **measure of A** *is denoted simply by $m(A)$ and is given by $m(A) = m^*(A) = m_*(A)$.*

A straight-forward extension of this definition applies to unbounded sets,

**Definition 3.2** *The measure for an unbounded set $A$ is defined simply as,*

$$m(A) = \lim_{n\to\infty} m(A \bigcap [-n, n]).$$

**Lemma 3.3** *Measurable sets have the following properties.*

- *A set $A \subset E$ is measurable if and only if*

$$m^*(A) + m^*(E/A) = b - a,$$

  *where $a$ and $b$ are the endpoints of the interval $E$.*

- *A set $A \subset E$ is measurable if and only if its complement $E/A$ is measurable.*

**Proof:** *The lemma follows directly from the definition of measurable (3.1).*

**Example 3.4** *Referring back to Example 2.4, we see that a set consisting of an interval which is missing a finite number of points has the same outer measure as the interval itself. That is,*

$$m^*(E/\{x_i, \ldots, x_n\}) = m^*(E).$$

*Thus, by Lemma 3.3, $E/\{x_i, \ldots, x_n\}$ is measurable if and only if $m^*(\{x_i, \ldots, x_n\}) = 0$.*

The following theorem takes us back to where we left off at the end of §2.3.

**Theorem 3.5** *The outer measure, $m^*$, is countably additive on the set of all measurable subsets of $E$. That is, whenever $\{A_n \,|\, n = 1, 2, \ldots\}$ is a set of measurable subsets of $E$, then*

$$m^*(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} m^*(A_n)$$

**Proof Idea:** *Although the proof of this theorem is not difficult, it is somewhat lengthy in its technical detail and is therefore left for the enrichment of the reader.*

So, knowing that we intend to define the Lebesgue integral in terms of the outer measure, it seems that the size of the class of "Lebesgue integrable" functions may somehow be limited by the fact that, as far as we know, the outer measure is only defined on measurable sets. We will return to this idea in Chapter 4.

## 3.2 Measurable Functions

**Definition 3.6** *Let $A$ be a bounded measurable subset of $\mathcal{R}$ and $f : A \to \mathcal{R}$ a function. Then $f$ is said to be* **measurable on A if** $\{x \in A \,|\, f(x) > r\}$ **is measurable (as a set) for every real number** $r$**.**

There are many equivalent definitions of measurable which follow similar lines. In this definition we see the beginnings of being able to "count the rectangles" created by partitioning the range of a function rather than the domain. See Figure 3.1.

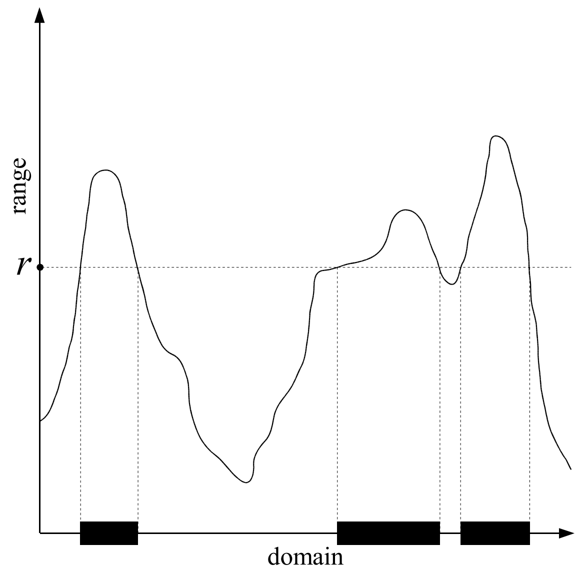


Figure 3.1: The function $f$ is *measurable* if the shaded region of the domain is measurable as a set for all choices of the real number $r$.

**Example 3.7** *Consider again the sequence of functions over* $E = [0, 1]$ *defined in Example 1.1,*

$$f_n(x) = \begin{cases} 2^n & \text{if } \frac{1}{2^n} \le x \le \frac{1}{2^{n-1}} \\ 0 & \text{otherwise} \end{cases}$$

*We want to show that each* $f_n$ *is a measurable function. There are three cases to consider, corresponding to three possible choices for the real number* $r$*. They are,*

- $r \ge 2^n$ : *The set* $\{x \in E \,|\, f_n(x) > r\}$ *is the null set and is therefore measurable.*
- $0 \le r < 2^n$ : *The set* $\{x \in E \,|\, f_n(x) > r\}$ *is the closed interval* $[\frac{1}{2^n}, \frac{1}{2^{n-1}}]$ *and is therefore measurable (with a measure of* $\frac{1}{2^n}$*).*
- $r < 0$ : *The set* $\{x \in E \,|\, f_n(x) > r\}$ *is the entire interval* $E$ *and is therefore measurable.*

*Thus, each* $f_n$ *is a measurable function.*

**Example 3.8** *Consider again the sequence of functions over* $E = [0, 1]$ *defined in Example 1.2,*

$$d_n(x) = \begin{cases} 1 & \text{if } x \in \{r_n\} \\ 0 & \text{otherwise} \end{cases}$$

*We want to show that each* $d_n$ *is a measurable function. There are three cases to consider, corresponding to three possible choices for the real number* $r$*. They are,*

- $r \geq 1$ : *The set* $\{x \in E \,|\, d_n(x) > r\}$ *is the null set and is therefore measurable.*
- $0 \leq r < 1$ : *The set* $\{x \in E \,|\, d_n(x) > r\}$ *is the set of the first* $n$ *rational numbers (in a decided upon enumeration) and is therefore measurable (with a measure of* $0$*).*
- $r < 0$ : *The set* $\{x \in E \,|\, d_n(x) > r\}$ *is the entire interval* $E$ *and is therefore measurable.*

*Thus, each* $d_n$ *is a measurable function.*

The above examples naturally bring up the question of whether or not the limit function of a sequence of measurable functions is itself a measurable function. Clearly, the limit function in Example 3.7 is measurable since it is the trivial function. But what of the Dirichlet function in Example 3.8? The following lemma answers this question.

**Lemma 3.9** *If each function in a sequence* $\{f_n\}$ *is measurable on a set* $A$ *and if* $f$ *is the pointwise limit function of* $\{f_n\}$*, then* $f$ *is measurable on* $A$ *as well.*

**Proof:** *The proof of the lemma requires a close examination of the definitions of limit and measurable, as follows. Let* $x \in A$ *and* $r \in \mathcal{R}$ *such that* $f(x) > r$*. Let* $p$ *be a natural number such that* $f(x) > r + 1/p$*. Then, by definition of limit, there exists a natural number* $N$ *such that for all* $n > N$*,* $f_n(x) > r + 1/p$*. Thus,*

$$f(x) = \lim_{n \to \infty} f_n(x) > r + 1/p > r.$$

*This implies that*

$$\{x \in A \,|\, f(x) > r\} = \bigcup_{p=1}^{\infty} \bigcup_{N=1}^{\infty} \bigcap_{n=N+1}^{\infty} \{x \in A \,|\, f_n(x) > r + 1/p\}.$$

*Since this set is measurable and* $r$ *was arbitrary, it follows that* $f$ *is measurable.*

## 3.3 Simple Functions

Similar to the way which step functions are put to use in the development the Riemann integral, our development of the Lebesgue integral will make use of a pedestrian class of measurable functions, aptly named *simple functions.*

**Definition 3.10** *A* **simple function** $f : A \to \mathcal{R}$ *is a measurable function which takes on finitely many values.*

We've already seen simple functions in Examples 1.1 and 1.2. The utility of simple functions becomes apparent in the following theorem.

**Theorem 3.11** *A function* $f : A \to \mathcal{R}$ *is measurable if and only if it is the pointwise limit of a sequence of simple functions.*
**Proof Idea:** *(⇒) The forward direction of the proof involves constructing a sequence of simple functions whose limit is the given measurable function* $f$*. This construction is omitted for brevity.*
*(⇐) The reverse direction is given by Lemma 3.9.*

**Example 3.12** *Example 3.8 combined with Theorem 3.11 (or even just Lemma 3.9) tells us that the Dirichlet function is indeed measurable.*

# Chapter 4

# The Lebesgue Integral

## 4.1 Integrating Bounded Measurable Functions

We introduce the Lebesgue integral by first restricting our attention to bounded measurable functions. The approach here is almost identical to that used in constructing the Riemann integral except that here we partition the range rather than the domain of the function.

Let $f : A \to \mathcal{R}$ be a bounded measurable function on a bounded measurable subset $A$ of R. Let $l = glb\{f(x) \,|\, x \in A\}$ and $u > lub\{f(x) \,|\, x \in A\}$ where $u$ is arbitrary insofar as it is greater than the least upper bound of $f$ on $A$.

As with the Riemann integral, we'll define the Lebesgue integral of $f$ over an interval $A$ as the limit of some "Lebesgue sum".

**Definition 4.1** *The* **Lebesgue sum** *of $f$ with respect to a partition $P = \{y_0, \ldots, y_n\}$ of the interval $[l, u]$ is given as*

$$L(f, P) = \sum_{i=1}^{n} y_i^* \, m(\{x \in A \,|\, y_{i-1} \leq f(x) < y_i\})$$

*where $y_i^* \in [y_{i-1}, y_i]$ for $i = 1, \ldots, n$ and $f$ is a bounded measurable function over a bounded measurable set $A \subset \mathcal{R}$.*

This is the new way to count rectangles; the $y_i^*$ is the height of the rectangle and the $m(\{x \in A \,|\, y_{i-1} \leq f(x) < y_i\})$ serves as the base of the rectangle. The definition of the actual Lebesgue integral is virtually identical to that of the Riemann integral.

**Definition 4.2** *A bounded measurable function* $f : A \to \mathcal{R}$ *is* **Lebesgue integrable on** $A$ *if there is a number* $L \in \mathcal{R}$ *such that, given* $\epsilon > 0$*, there exists a* $\delta > 0$ *such that* $|L(f, P) - L| < \epsilon$ *whenever* $||P|| < \delta$*.* $L$ *is known as the* **Lebesgue integral of** $f$ **on** $A$ *and is denoted by* $\int_A f\,dm$*.*

## 4.2 Criteria for Integrability

As with the definition of the Riemann integral, the definition of the Lebesgue integral does not illuminate us as to whether a given function is Lebesgue integrable and, if it is, what the Lebesgue integral is. The following theorem, presented without proof, fills this gap.

**Theorem 4.3** *A bounded measurable function* $f$ *is Lebesgue integrable on a bounded measurable set* $A$ *if and only if, given* $\epsilon > 0$*, there exist simple functions* $\underline{f}$ *and* $\overline{f}$ *such that*

$$\underline{f} \le f \le \overline{f}$$

*and*

$$\int_A \overline{f}\,dm - \int_A \underline{f}\,dm < \epsilon.$$

**Corollary 4.4** *If* $f$ *is a bounded measurable function on a bounded measurable set* $A$*, then* $f$ *is Lebesgue integrable on* $A$*. Furthermore,*

$$\begin{aligned}\int_A f\,dm =& lub\left\{\int_A \underline{f}\,dm \,|\, \underline{f} \text{ is simple and } \underline{f} \le f\right\}\\ =& glb\left\{\int_A \overline{f}\,dm \,|\, \overline{f} \text{ is simple and } \overline{f} \ge f\right\}.\end{aligned}$$

**Proof Idea:** *This corollary is a result of the proof of Theorem 4.3.*

**Example 4.5** *Let's find the Lebesgue integral of the Dirichlet function (see Example 1.2). We know by Example 3.12 that the Dirichlet function is measurable. Thus, by Corollary 4.4, it is Lebesgue integrable. Specifically, we claim that* $\int_{[0,1]} D\,dm = 0$*. Looking back to the definition of the Lebesgue integral (4.2), we must show that, given* $\epsilon > 0$*, there exists a* $\delta > 0$ *such that* $|L(D, P)| < \epsilon$ *whenever the width of the partition* $P$ *is less than* $\delta$*. Referring to Definition 4.1 for* $|L(D, P)|$*, this means that we must find* $\delta$ *such that*

$$|L(D, P)| = \sum_{i=1}^{n} y_i^*\, m(\{x \in [0, 1] \,|\, y_{i-1} \le D(x) < y_i\}) < \epsilon,$$

*where* $P = \{y_0 = 0, \ldots, y_n = 2\}$ *is a partition of the interval* $[0, 2]$ *in the range,* $y_i^* \in [y_{i-1}, y_i]$ *for* $i = 1, \ldots, n$*, and the absolute value signs have been omitted since all quantities here are inherently positive. We accomplish this by picking* $\delta = \epsilon/3$*. Then*

$$\begin{aligned}|L(D,P)| =& \sum_{i=1}^{n} y_i^* \, m(\{x \in [0,1] \,|\, y_{i-1} \le D(x) < y_i\}) \\ =& y_1^* \, m(\{x \in [0,1] \,|\, 0 \le D(x) < \epsilon/3\}) \\ &+ \quad y_j^* \, m(\{x \in [0,1] \,|\, y_{j-1} \le D(x) < y_j\}),\end{aligned}$$

*where* $y_{j-1} \le 1 < y_j$*. All other terms in the sum vanish because* $D(x)$ *only takes on two values,* $0$ *and* $1$*. The first term achieves its maximum when* $y_1^* = \epsilon/3$ *and when* $m(\{x \in [0,1] \,|\, 0 \le D(x) < \epsilon/3\} = 1$*. Thus,*

$$\begin{aligned}|L(D,P)| =& \sum_{i=1}^{n} y_i^* \, m(\{x \in [0,1] \,|\, y_{i-1} \le D(x) < y_i\}) \\ \le& [\epsilon/3] + [y_j^* \, m(\{x \in [0,1] \,|\, y_{j-1} \le D(x) < y_j\})].\end{aligned}$$

*Also, we know that* $y_j^* < 1 + \epsilon/3$*. Finding an upper bound on*

$$m(\{x \in [0,1] \,|\, y_{j-1} \le D(x) < y_j\})$$

*is equivalent to asking for an upper bound on the measure of the rational numbers in the interval* $[0, 1]$*. Since any non-empty open interval centered on a rational number contains other rational numbers, an upper bound on the measure of the rational numbers is simply the sum of the measures of all the intervals surrounding the rationals. That is, given an enumeration of the rationals, let the* $n^{th}$ *rational number be contained in an interval of measure* $\epsilon/3^{n+1}$*. In this way,*

$$m(\{x \in [0,1] \,|\, y_{j-1} \le D(x) < y_j\}) < \sum_{n=1}^{\infty} \frac{\epsilon}{3^{n+1}} = \frac{\epsilon}{6},$$

*and we finally get that*

$$\begin{aligned}|L(D,P)| \le& \epsilon/3 + y_j^* \, m(\{x \in [0,1] \,|\, y_{j-1} \le D(x) < y_j\}) \\ <& \epsilon/3 + (1 + \epsilon/3)(\epsilon/6) \\ =& \epsilon/3 + \epsilon/3 + \epsilon^2/18 \\ <& \epsilon,\end{aligned}$$

*where we have assumed without loss of generality that $\epsilon < 1$. This completes the proof that $\int_{[0,1]} D\,dm = 0$. Combined with Lemma 3.3, we also know that the irrationals must have measure 1 on the interval $[0, 1]$. This result makes intuitive sense since the rationals are countably infinite whereas the irrationals are uncountably infinite.*

## 4.3 Properties of the Lebesgue Integral

It is comforting to know that the Lebesgue integral shares many properties with the Riemann integral.

**Theorem 4.6** *Let $f$ and $g$ be bounded measurable functions on a bounded measurable set $A$. Then,*

- *Monotonicity: If $f \leq g$, then $\int_A f\,dm \leq \int_A g\,dm$.*
- *Linearity:*
$$\int_A (f + g)\,dm = \int_A f\,dm + \int_A g\,dm$$
*and*
$$\int_A cf\,dm = c\int_A f\,dm \text{ for } c \in \mathcal{R}.$$
- *For any numbers $l, u \in \mathcal{R}$ such that $l \leq f \leq u$, it follows that $l \cdot m(A) \leq \int_A f\,dm \leq u \cdot m(A)$.*
- *$|\int_A f\,dm| \leq \int_A |f|\,dm$.*
- *If $A$ and $B$ are disjoint bounded measurable sets and $f : A \cup B \to \mathcal{R}$ is a bounded measurable function, then*
$$\int_{A\cup B} f\,dm = \int_A f\,dm + \int_B f\,dm.$$
- *Countable Additivity: If $A = \cup_{i=1}^{\infty} A_i$ where the $A_i$ are pairwise disjoint bounded measurable sets, then*
$$\int_A f\,dm = \sum_{i=1}^{\infty} \int_{A_i} f\,dm.$$

Further comparison of the Riemann and the Lebesgue integral is made in Chapter 5.

## 4.4 Integrating Unbounded Measurable Functions

Thus far, we have restricted ourselves to integrating bounded measurable functions. The generalization of the Lebesgue integral to unbounded measurable functions is straight-forward but will not be given in full detail here. In general, though, it involves the introduction of $\pm\infty$ as possible values of the integral. As it turns out, all of the properties of the Lebesgue integral listed in §4.3 hold for unbounded measurable functions as well.

# Chapter 5

# Comparison of Lebesgue and Riemann Integrals

## 5.1 Summary of Events

Having achieved what we first set out to do – to define an integral based on a new way of counting rectangles, let us look back at the motivation behind this endeavor. As outlined in §1.2, we are striving to define an integral such that the class of integrable functions is large and well-behaved. Ideally, we would like the class of integrable functions to have nice limit properties; that the integral of the limit is the limit of the integral. Examples 1.1 and 1.2 show that this is not in general true of the Riemann integral.

Aside from examining the convergence properties of the Lebesgue integral, we are also interested in how it behaves relative to the Riemann integral. Is a Riemann integrable function Lebesgue integrable and, if so, what are the values of the respective integrals?

## 5.2 Convergence of the Lebesgue Integral

The following lemmas and theorems build upon each other and end with a broad statement which identifies when the limit and Lebesgue integral operations are interchangeable.

**Lemma 5.1** *Let $g$ be a non-negative measurable function on a bounded measurable set $A$. If $\{A_n\}$ are measurable subsets of $A$ such that*

$$A_1 \subset A_2 \subset A_3 \subset \cdots,$$

*and if $L \in \mathcal{R}$ is such that $L \geq \int_{A_n} g\,dm$ for all $n$, then $L \geq \int_{\cup A_n} g\,dm$.*
**Proof Idea:** *(Proof by belief). Given that the Lebesgue integral depends on the measure of subsets of the domain of the function, and that measure follows our intuition, this lemma is easy to believe without proof.*

**Theorem 5.2** *(Monotone Convergence Theorem) Let $A$ be a bounded measurable subset of $\mathcal{R}$ and $\{f_n\}$ be a sequence of measurable functions on $A$ such that $0 \leq f_1 \leq f_2 \leq \cdots$. Let $f$ be the pointwise limit of $\{f_n\}$. That is,*

$$\lim_{n\to\infty} f_n(x) = f(x)$$

*for all $x \in E$. Then $f$ is integrable and*

$$\lim_{n\to\infty} \int_A f_n\,dm = \int_A f\,dm.$$

**Proof:** *By Lemma 3.9, $f$ is measurable since each $f_n$ is measurable. Monotonicity of the Lebesgue integral (Theorem 4.6) implies that $\left\{\int_A f_n\,dm\right\}$ is an increasing sequence. So, not discounting $\infty$ as a possibility, this sequence has a limit $L$. Since $f \geq f_n$ implies that $\int_A f\,dm \geq \int_A f_n\,dm$ for all $n$, it follows that $\int_A f\,dm \geq L$.*

*On the other hand, we can also show that $\int_A f\,dm \leq L$. Let $c \in (0,1)$ and let $g : A \to \mathcal{R}$ be a simple function such that $0 \leq g \leq f$. Consider the sets $A_n = \{x \mid f_n(x) \geq cg(x)\}$, which happen to satisfy the hypotheses of Lemma 5.1. The reason for including $c$ is so that $cg < f$, which implies that $\bigcup_{n=1}^{\infty} A_n = A$. For any $n = 1, 2, \cdots$, we have*

$$L = \lim_{n\to\infty} \int_A f_n\,dm \geq \int_{A_n} f_n\,dm \geq c \int_{A_n} g\,dm.$$

*By Lemma 5.1, $L \geq c \int_{\cup A_n} g\,dm$. Since $\bigcup_{n=1}^{\infty} A_n = A$ and $c \in (0,1)$ was arbitrary, it follows that $L \geq \int_{\cup A_n} g\,dm$. By Corollary 4.4,*

$$\int_A f\,dm = lub\left\{\int_A \underline{f}\,dm \mid \underline{f} \text{ is simple and } \underline{f} \leq f\right\},$$

*we get $L \geq \int_A f\,dm$.*

*Combining the above two inequalities gives $L = \int_A f\,dm$, as desired.*

**Lemma 5.3** *(Fatou's Lemma) Let $A \subset \mathcal{R}$ be a bounded measurable set. If $\{f_n\}$ is a sequence of non-negative measurable functions on $A$ and*

$$f(x) = \underline{lim}_{n\to\infty} f_n(x) \equiv \lim_{n\to\infty} \{glb\, f_k(x) \mid k \geq n\}$$

*for every* $x \in A$, *then* $\int_A f\,dm \le \underline{\lim}_{n\to\infty} \int_A f_n\,dm$.

**Proof:** *Let*

$$g_n(x) = glb\{f_k(x) \,|\, k \ge n\}$$

*for each* $n = 1, 2, \cdots$ *and for each* $x \in A$. *Each* $g_n$ *is a measurable function since the set*

$$\{x \,|\, g_n(x) > r\} = \bigcup_{k=n}^{\infty} \{x \,|\, f_k(x) > r\}$$

*is measurable. Notice that* $g_n \le f_n$ *for all* $n$ *and that* $\{g_n\}$ *is a monotonically increasing sequence of non-negative functions.*

*By definition,*

$$\lim_{n\to\infty} g_n(x) = \lim_{n\to\infty} \{glb\, f_k(x) \,|\, k \ge n\} \equiv \underline{\lim}_{n\to\infty} f_n(x) = f(x).$$

*Therefore, by the Monotone Convergence Theorem (5.2),*

$$\lim_{n\to\infty} \int_A g_n\,dm = \int_A f\,dm.$$

*This, combined with*

$$\int_A g_n(x)\,dm \le \int_A f_n(x)\,dm \text{ for each } n \text{ and each } x \in A$$

*due to the monotonicity (Theorem 4.6) of the Lebesgue integral, leads to*

$$\lim_{n\to\infty} \int_A g_n\,dm = \int_A f\,dm \le \underline{\lim}_{n\to\infty} \int_A f_n\,dm,$$

*as desired.*

**Theorem 5.4** *(Lebesgue Dominated Convergence Theorem) Let* $A \in \mathcal{R}$ *be a bounded measurable set and let* $\{f_n\}$ *be a sequence of measurable functions on* $A$ *such that* $\lim_{n\to\infty} f_n(x) = f(x)$ *for all* $x \in A$. *If there exists a function* $g$ *whose Lebesgue integral is finite such that* $|f_n(x)| \le g(x)$ *for all* $n$ *and all* $x \in A$, *then*

$$\lim_{n\to\infty} \int_A f_n\,dm = \int_A f\,dm.$$

**Proof:** *To begin, the functions* $f_n$ *and* $f$ *have finite Lebesgue integrals since* $g$ *does (this fact does take some proof, but is omitted here). By our choice of* $g$ *we have* $f_n + g \ge 0$ *on the set* $A$. *So,*

$$\int_A (f+g)\,dm \le \underline{\lim}_{n\to\infty} \int_A (f_n+g)\,dm$$

*by Fatou's Lemma (5.3) and hence,*

$$\int_A f\,dm \le \underline{lim}_{n\to\infty} \int_A f_n\,dm$$

*by the linearity of the Lebesgue integral (Theorem 4.6).*

*The same argument can be used with the function $g - f_n$ to obtain*

$$-\int_A f\,dm \le \underline{lim}_{n\to\infty} \left(-\int_A f_n\,dm\right).$$

*But since*

$$\begin{aligned}\underline{lim}_{n\to\infty} \left(-\int_A f_n\,dm\right) &= \lim_{n\to\infty} \left\{glb \left(-\int_A f_k\,dm\right) \mid k \ge n\right\} \\ &= \lim_{n\to\infty} \left\{lub \int_A f_k\,dm \mid k \ge n\right\} \\ &\equiv \overline{lim}_{n\to\infty} \int_A f_n\,dm,\end{aligned}$$

*it follows that*

$$\int_A f\,dm \ge \overline{lim}_{n\to\infty} \int_A f_n\,dm.$$

*Finally, combining the above two inequalities,*

$$\int_A f\,dm \le \underline{lim}_{n\to\infty} \int_A f_n\,dm \le \overline{lim}_{n\to\infty} \int_A f_n\,dm \le \int_A f\,dm,$$

*shows that $\lim_{n\to\infty} \int_A f_n\,dm$ exists and is equal to $\int_A f\,dm$, as desired.*

## 5.3 Convergence of the Riemann Integral

Now that we have a grasp of the convergence properties of the Lebesgue integral, let's compare it with those of the Riemann integral. The Riemann equivalent to the Lebesgue Dominated Convergence Theorem (5.4), is stated below in Theorem 5.5.

**Theorem 5.5** *Let $a, b \in \mathcal{R}$, $a < b$, and let $\{f_n\}$ be a uniformly convergent sequence of Riemann integrable functions on $[a, b]$ such that $\lim_{n\to\infty} f_n(x) = f(x)$ for all $x \in [a, b]$. Then*

$$\lim_{n\to\infty} \int_a^b f_n(x)\,dx = \int_a^b f(x)\,dx.$$

Recall that $\{f_n\}$ is uniformly convergent if, given $\epsilon > 0$, there exists a natural number $N$ such that $|f(x) - f_n(x)| < \epsilon$ whenever $n > N$, for all $x \in [a, b]$. Clearly, the hypotheses placed on $\{f_n\}$ in order for the Lebesgue Dominated Convergence Theorem to hold are much less stringent than requiring $\{f_n\}$ to converge uniformly. Thus, we can at least expect that the class of Lebesgue integrable functions has somewhat better limit properties than those of the class of Riemann integrable functions. In fact, we have already witnessed this in Example 4.5 with the Dirichlet function.

## 5.4 A Final Comparison

The following theorem, stated without proof, explicitly relates the Riemann and Lebesgue integrals.

**Theorem 5.6** *If $f$ is Riemann integrable on [a,b], then $f$ is Lebesgue integrable on [a,b], and*

$$\int_a^b f(x)\,dx = \int_{[a,b]} f\,dm.$$

What's more, we've already seen that the converse of this theorem isn't true. Thus, not only does the class of Lebesgue integrable functions have better limit properties, but it is also larger than the class of Riemann integrable functions.

As promised in §1.2, we now turn to the Lebesgue integral to re-examine the function given in Example 1.1.

**Example 5.7** *Consider the sequence of functions $\{f_n\}$ over the interval $E = [0, 1]$.*

$$f_n(x) = \begin{cases} 2^n & \text{if } \frac{1}{2^n} \leq x \leq \frac{1}{2^{n-1}} \\ 0 & \text{otherwise} \end{cases}$$

*The limit function of this sequence is simply $f = 0$. In this example, each function in the sequence is Riemann integrable, as is the limit function. However, the limit of the sequence of Riemann integrals is not equal to the Riemann integral of the limit of the sequence. That is,*

$$\lim_{n\to\infty} \int_0^1 f_n(x)\,dx = 1 \neq 0 = \int_0^1 \lim_{n\to\infty} f_n(x)\,dx.$$

*By Theorem 5.6, it is also the case that*

$$\lim_{n\to\infty} \int_{[0,1]} f_n\,dm = 1 \neq 0 = \int_{[0,1]} \lim_{n\to\infty} f_n(x)\,dx.$$

This example goes to show that, although the Lebesgue integral is superior to the Riemann integral insofar as the size of the class of integrable functions and the limit properties of these functions, there are still functions which defy Lebesgue integration.

One point we have not yet touched on is the effect of the Lebesgue integral on the $L_p$ spaces. It turns out that $L_2$ is complete under the Lebesgue integral. This is of considerable importance, especially when dealing with the theory behind Fourier series. However interesting that topic may be, though, it will have to wait for another time.

# References

[1] HJ Wilcox, DL Myers, **An Introduction to Lebesgue Integration and Fourier Series**, Dover Publications, Inc., New York, 1978.

[2] M Rosenlicht, **Introduction to Analysis**, Dover Publications, Inc., New York, 1968.