A fixed point localization formula for the Fourier transform of regular semisimple coadjoint orbits

Matvei Libine

Department of Mathematics and Statistics, University of Massachusetts, 710 North Pleasant Street, Amherst, MA 01003, USA

Abstract

Let $G_{\mathbb{R}}$ be a Lie group acting on an oriented manifold M, and let ω be an equivariantly closed form on M. If both $G_{\mathbb{R}}$ and M are compact, then the integral $\int_M \omega$ is given by the fixed point integral localization formula (Theorem 7.11 in Berline et al. Heat Kernels and Dirac Operators, Springer, Berlin, 1992). Unfortunately, this formula fails when the acting Lie group $G_{\mathbb{R}}$ is not compact: there simply may not be enough fixed points present. A proposed remedy is to modify the action of $G_{\mathbb{R}}$ in such a way that all fixed points are accounted for.

Let $G_{\mathbb{R}}$ be a real semisimple Lie group, possibly *noncompact*. One of the most important examples of equivariantly closed forms is the symplectic volume form $d\beta$ of a coadjoint orbit Ω. Even if Ω is not compact, the integral $\int_{\Omega} d\beta$ exists as a distribution on the Lie algebra $\mathfrak{g}_{\mathbb{R}}$. This distribution is called *the Fourier transform of the coadjoint orbit*.

In this article, we will apply the localization results described in [L1,L2] to get a geometric derivation of Harish-Chandra's formula (9) for the Fourier transforms of regular semisimple coadjoint orbits. Then, we will make an explicit computation for the coadjoint orbits of elements of $\mathfrak{g}_{\mathbb{R}}^*$ which are dual to regular semisimple elements lying in a maximally split Cartan subalgebra of $\mathfrak{g}_{\mathbb{R}}$.

E-mail address: matvei@math.umass.edu.

1. Introduction

Equivariant forms were introduced in 1950 by Henri Cartan. There are many good texts on this subject including [BGV,GS]. Let $G_{\mathbb{R}}$ be a real Lie group acting on a compact oriented manifold M, let $\mathfrak{g}_{\mathbb{R}}$ be the Lie algebra of $G_{\mathbb{R}}$, and let $\omega : \mathfrak{g}_{\mathbb{R}} \to \mathscr{A}(M)$ be an equivariantly closed form on M. (Here $\mathscr{A}(M)$ denotes the algebra of smooth differential forms on M.) For $X \in \mathfrak{g}_{\mathbb{R}}$, we denote by M^X the set of zeroes of the vector field on M induced by the infinitesimal action of X. We assume that M^X is discrete. Then the integral localization formula [BGV, Theorem 7.11] says that when the Lie group $G_{\mathbb{R}}$ is *compact* the integral of $\omega(X)$ can be expressed as a sum over the set of zeroes M^X of certain *local* quantities of M and ω:

$$\int_M \omega(X) = \sum_{p \in M^X} \text{local invariant of } M \text{ and } \omega \text{ at } p.$$

This is the essence of the integral localization formula for equivariantly closed differential forms.

Unfortunately, this formula fails when the acting Lie group $G_{\mathbb{R}}$ is not compact: there simply may not be enough fixed points present. For example, let $G_{\mathbb{R}} = SL(2, \mathbb{R})$. Then there exists an essentially unique $G_{\mathbb{R}}$-invariant symmetric bilinear form B on $\mathfrak{g}_{\mathbb{R}} = \mathfrak{sl}(2, \mathbb{R})$, say $B(X, Y)$ is the Killing form $Tr(ad(X)ad(Y))$. This bilinear form B induces an $SL(2, \mathbb{R})$-equivariant isomorphism $I : \mathfrak{sl}(2, \mathbb{R}) \xrightarrow{\sim} \mathfrak{sl}(2, \mathbb{R})^*$. Let $\Omega \subset \mathfrak{sl}(2, \mathbb{R})^*$ denote the coadjoint orbit of

$$I \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Like all semisimple coadjoint orbits, Ω possesses a canonical symplectic form ω which is the top degree part of a certain equivariantly closed form. Although Ω is not compact, the symplectic volume $\int_\Omega \omega$ still exists as a distribution on $\mathfrak{sl}(2, \mathbb{R})$. Let $\mathfrak{sl}(2, \mathbb{R})'_{\text{split}}$ denote the set of $X \in \mathfrak{sl}(2, \mathbb{R})$ with distinct real eigenvalues. The set $\mathfrak{sl}(2, \mathbb{R})'_{\text{split}}$ is an open subset of $\mathfrak{sl}(2, \mathbb{R})$. Now, if we take any element $X \in \mathfrak{sl}(2, \mathbb{R})'_{\text{split}}$, then one can see that the vector field on Ω generated by X has no zeroes. Thus, if there were a fixed point integral localization formula like in the case of compact group $G_{\mathbb{R}}$, this formula would suggest that the distribution determined by $\int_\Omega \omega$ vanishes on the open set $\mathfrak{sl}(2, \mathbb{R})'_{\text{split}}$. But it is known that the restriction $\int_\Omega \omega$ to $\mathfrak{sl}(2, \mathbb{R})'_{\text{split}}$ is *not* zero.

On the other hand, the localization argument for character formulas described in [L1,L2] strongly suggests that there should be some kind of a localization formula when the group $G_{\mathbb{R}}$ is semisimple. One can regard the failure of the localization formula in the last example as an instance when some of the fixed points which should be counted are "hiding somewhere at infinity". A proposed remedy is to modify the action of $G_{\mathbb{R}}$ in such a way that all fixed points are accounted for.

We will show how it can be done in the case of the symplectic volume of a regular semisimple coadjoint orbit. Let $\lambda \in \mathfrak{g}_{\mathbb{R}}^*$ be a regular semisimple element and let $\Omega_{\lambda,\mathbb{R}}$ be its coadjoint orbit. Let $\mathscr{X}$ denote the *flag variety*, i.e. the variety of Borel subalgebras $\mathfrak{b} \subset \mathfrak{g} = \mathfrak{g}_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$. The space $\mathscr{X}$ is a smooth complex projective variety. The twisted moment map $\mu_{\lambda} : T^* \mathscr{X} \to \mathfrak{g}^*$ defined in [SchV1] is an embedding whose image contains $\Omega_{\lambda,\mathbb{R}}$. We will replace $\Omega_{\lambda,\mathbb{R}}$ with $\mu_{\lambda}^{-1}(\Omega_{\lambda,\mathbb{R}})$ and then it will be possible to apply the results of [SchV2] and the localization argument described in [L1,L2] to get a fixed point integral formula (9) for the Fourier transform of $\Omega_{\lambda,\mathbb{R}}$. This is not really a new formula, but rather a different way of looking at Harish-Chandra's formula for an invariant eigendistribution. Unless the group $G_{\mathbb{R}}$ is compact, the map $\mu_{\lambda} : T^* \mathscr{X} \to \mathfrak{g}^*$ is not $G_{\mathbb{R}}$-equivariant. Thus, the cycle $\mu_{\lambda}^{-1}(\Omega_{\lambda,\mathbb{R}})$ does not need to be $G_{\mathbb{R}}$-invariant, but we will show that it is homologous to a $G_{\mathbb{R}}$-invariant cycle which we will call C_{λ}. Replacing $\Omega_{\lambda,\mathbb{R}}$ with C_{λ} amounts to changing the action of $G_{\mathbb{R}}$ on $\Omega_{\lambda,\mathbb{R}}$.

As an example, we will compute the Fourier transforms of the coadjoint orbits of elements of $\mathfrak{g}_{\mathbb{R}}^*$ which are dual to regular semisimple elements lying in a maximally split Cartan subalgebra of $\mathfrak{g}_{\mathbb{R}}$.

In this setting, the main difficulty lies in identifying the cycle C_{λ}. It is most likely that this cycle C_{λ} can be identified as the characteristic cycle of some sheaf $\mathscr{F}_{\lambda}$ which has a simple description. Then, one will get an interesting perspective of looking at Harish-Chandra's formula for the Fourier transforms of regular semisimple coadjoint orbits.

This result strongly resembles the following result due to W. Rossmann which is described in [BGV,BV,Par,R1,V]. Let $G_{\mathbb{R}}$ be a connected real semisimple Lie group, and let $K_{\mathbb{R}} \subset G_{\mathbb{R}}$ be a maximal compact subgroup of $G_{\mathbb{R}}$ with Lie algebra $\mathfrak{k}_{\mathbb{R}}$. Let $\mathfrak{t}_{\mathbb{R}} \subset \mathfrak{k}_{\mathbb{R}}$ be a Cartan subalgebra of $\mathfrak{k}_{\mathbb{R}}$ and suppose that $\mathfrak{t}_{\mathbb{R}}$ is also a Cartan subalgebra of $\mathfrak{g}_{\mathbb{R}}$. Then W. Rossmann gives an explicit formula for the restriction of the Fourier transform of a semisimple coadjoint orbit to $\mathfrak{t}_{\mathbb{R}}$.

The difference between the result described in this article and that of W. Rossmann is that it seldom happens that a Cartan subalgebra $\mathfrak{t}_{\mathbb{R}} \subset \mathfrak{k}_{\mathbb{R}}$ is also a Cartan subalgebra of $\mathfrak{g}_{\mathbb{R}}$. Also, W. Rossmann does not give an explicit formula for the Fourier transform as a distribution on all of $\mathfrak{g}_{\mathbb{R}}$, but rather the restriction of the Fourier transform to $\mathfrak{t}_{\mathbb{R}}$.

I would like to thank W. Rossmann for his helpful suggestions and especially for pointing out that part (1) of Theorem 8 was stated and used in his proof of Theorem 1.4(b) in [R2]. (The proof of this statement was omitted because it was not the point of that paper.)

This article strongly suggests that some results which previously were known in the compact group setting only can be generalized to noncompact groups. It would be very interesting to have a uniform equivariant form theory which holds for both compact and noncompact groups and which generalizes presently known compact case.

2. The Fourier transform of a coadjoint orbit

We fix a *connected, complex algebraic, semisimple* Lie group G which is defined over $\mathbb{R}$. Let $G_{\mathbb{R}}$ be the identity component of the group of real points $G(\mathbb{R})$. We denote by $\mathfrak{g}$ and $\mathfrak{g}_{\mathbb{R}}$ the Lie algebras of G and $G_{\mathbb{R}}$ respectively. The group G (respectively $G_{\mathbb{R}}$) acts on $\mathfrak{g}$ (respectively $\mathfrak{g}_{\mathbb{R}}$) by the *adjoint* action, and the dual action of G (respectively $G_{\mathbb{R}}$) on $\mathfrak{g}^*$ (respectively $\mathfrak{g}_{\mathbb{R}}^*$) is called the *coadjoint* action.

Let $B(X, Y)$ be an (essentially unique) G-invariant symmetric bilinear form on $\mathfrak{g}$, say $B(X, Y)$ is the Killing form $Tr(ad(X)ad(Y))$. This bilinear form B induces a G-equivariant isomorphism $I : \mathfrak{g} \xrightarrow{\sim} \mathfrak{g}^*$ which restricts to a $G_{\mathbb{R}}$-equivariant isomorphism $\mathfrak{g}_{\mathbb{R}} \xrightarrow{\sim} \mathfrak{g}_{\mathbb{R}}^*$.

We denote by $\mathfrak{g}_{\mathbb{R}}'$ the set of *regular semisimple* elements in $\mathfrak{g}_{\mathbb{R}}$. These are elements $X \in \mathfrak{g}_{\mathbb{R}}$ such that the action of $ad(X)$ on $\mathfrak{g}_{\mathbb{R}}$ can be diagonalized over $\mathbb{C}$ and $ad(X)$ has maximal possible rank. We also call the I-image of $\mathfrak{g}_{\mathbb{R}}'$ the *regular semisimple* elements of $\mathfrak{g}_{\mathbb{R}}^*$ and denote them by $(\mathfrak{g}_{\mathbb{R}}^*)' = I(\mathfrak{g}_{\mathbb{R}}')$.

The set $\mathfrak{g}_{\mathbb{R}}'$ is an open subset of $\mathfrak{g}_{\mathbb{R}}$ and its complement has measure zero. Each regular semisimple $X \in \mathfrak{g}_{\mathbb{R}}'$ lies in a unique Cartan subalgebra $\mathfrak{t}_{\mathbb{R}}(X) \subset \mathfrak{g}_{\mathbb{R}}$. It is important to remember that when $G_{\mathbb{R}}$ is not compact some Cartan subalgebras of $\mathfrak{g}_{\mathbb{R}}$ may not be conjugate by an element from $G_{\mathbb{R}}$.

We pick a regular semisimple element $\lambda' \in (\mathfrak{g}_{\mathbb{R}}^*)'$ and consider its coadjoint orbit $G_{\mathbb{R}} \cdot \lambda' \subset \mathfrak{g}_{\mathbb{R}}^*$. Because we are going to apply results of [L1,L2], it will be more convenient to replace λ' with $\lambda = i\lambda' \in i(\mathfrak{g}_{\mathbb{R}}^*)'$ and consider

$$\Omega_{\lambda, \mathbb{R}} = G_{\mathbb{R}} \cdot \lambda \subset i\mathfrak{g}_{\mathbb{R}}^* \subset \mathfrak{g}^*.$$

The $G_{\mathbb{R}}$-orbit $\Omega_{\lambda, \mathbb{R}}$ is a real submanifold of the complex coadjoint orbit $\Omega_{\lambda} = G \cdot \lambda \subset \mathfrak{g}^*$.

If $X \in \mathfrak{g}$ is an element of the Lie algebra, we denote by $X_{\Omega_{\lambda}}$ the vector field on Ω_{λ} generated by X: if $v \in \Omega_{\lambda}$ and $f \in \mathscr{C}^{\infty}(\Omega_{\lambda})$, then

$$X_{\Omega_{\lambda}}(v)f = \frac{d}{d\varepsilon} f(\exp(\varepsilon X) \cdot v)\Big|_{\varepsilon=0}.$$

We call a point $v \in \Omega_{\lambda}$ a *fixed point of X* if the vector field $X_{\Omega_{\lambda}}$ on Ω_{λ} vanishes at v, i.e. $X_{\Omega_{\lambda}}(v) = 0$.

The complex coadjoint orbit Ω_{λ} has a canonical structure of a complex symplectic manifold. We describe its symplectic form σ_{λ}. The vector fields $X_{\Omega_{\lambda}}$, $X \in \mathfrak{g}$, are sections of the tangent bundle $T\Omega_{\lambda}$ and span the tangent space at each point. Let $v \in \Omega_{\lambda}$, we define σ_{λ} on $T_v\Omega_{\lambda}$ by

$$\sigma_{\lambda}(X_{\Omega_{\lambda}}, Y_{\Omega_{\lambda}})_v =_{def} \langle v, [X, Y] \rangle, \qquad X, Y \in \mathfrak{g}.$$

Then σ_{λ} is a well-defined G-invariant symplectic form on Ω_{λ}, furthermore, the action of G on Ω_{λ} is Hamiltonian and the symplectic moment of X is the function $v \mapsto \langle v, X \rangle$.

We define the *Liouville form* on Ω_λ by (notice the $\frac{1}{(2\pi i)^n}$ factor)

$$d\beta = \frac{1}{(2\pi i)^n n!} \sigma_\lambda^n, \quad n = dim_\mathbb{C}(\Omega_\lambda)/2 = dim_\mathbb{R}(\Omega_{\lambda,\mathbb{R}})/2. \tag{1}$$

The form $d\beta$ restricts to a volume form on $\Omega_{\lambda,\mathbb{R}}$ which determines a canonical orientation on $\Omega_{\lambda,\mathbb{R}}$.

At this point, we need to specify that if M is a complex manifold and $M^\mathbb{R}$ is the underlying real analytic manifold, then there are at least two different but equally natural identifications between the holomorphic cotangent bundle T^*M and the $\mathscr{C}^\infty$ cotangent bundle $T^*M^\mathbb{R}$; we use the convention of [KaScha, Chapter XI]. The same convention is used in [L1,L2,R3,SchV2]. Under this convention, if σ_M is the canonical complex symplectic form on T^*M and $\sigma_{M^\mathbb{R}}$ is the canonical real symplectic form on $T^*M^\mathbb{R}$, then $\sigma_{M^\mathbb{R}}$ gets identified with $2\,\mathrm{Re}\,\sigma_M$.

From now on, we regard $\Omega_{\lambda,\mathbb{R}}$ as a Borel–Moore cycle in Ω_λ. We will use notation $|\Omega_{\lambda,\mathbb{R}}|$ to denote the support of $\Omega_{\lambda,\mathbb{R}}$.

If φ is a smooth compactly supported differential form on $\mathfrak{g}_\mathbb{R}$ of top degree, then we define its Fourier transform as in [L1,L2,SchV2]

$$\hat{\varphi}(\zeta) = \int_{\mathfrak{g}_\mathbb{R}} e^{\langle g,\zeta \rangle} \varphi \quad (g \in \mathfrak{g}_\mathbb{R}, \zeta \in \mathfrak{g}^*)$$

without the customary factor of $i = \sqrt{-1}$ in the exponent. Because $\hat{\varphi}$ decays fast in the imaginary directions, the integral

$$\widehat{\Omega_{\lambda,\mathbb{R}}}(\varphi) =_{def} \int_{\Omega_{\lambda,\mathbb{R}}} \hat{\varphi} \, d\beta = \frac{1}{(2\pi i)^n n!} \int_{\Omega_{\lambda,\mathbb{R}}} \hat{\varphi} \sigma_\lambda^n$$

is finite. Thus, we obtain a distribution on $\mathfrak{g}_\mathbb{R}$ called the *Fourier transform* of the coadjoint orbit $\Omega_{\lambda,\mathbb{R}}$. Since the symplectic form σ_λ is $G_\mathbb{R}$-invariant, so is the distribution $\widehat{\Omega_{\lambda,\mathbb{R}}}$.

Let $\mathscr{Z}(\mathscr{U}(\mathfrak{g}_\mathbb{R}))$ denote the center of the universal enveloping algebra of $\mathfrak{g}_\mathbb{R}$. It is canonically isomorphic to the algebra of conjugate-invariant constant coefficient differential operators on $\mathfrak{g}_\mathbb{R}$. It is not hard to see that $\widehat{\Omega_{\lambda,\mathbb{R}}}$ is an eigendistribution with respect to $\mathscr{Z}(\mathscr{U}(\mathfrak{g}_\mathbb{R}))$, i.e. each element of $\mathscr{Z}(\mathscr{U}(\mathfrak{g}_\mathbb{R}))$ acts on $\widehat{\Omega_{\lambda,\mathbb{R}}}$ by multiplication by some scalar. Then by Harish-Chandra's regularity theorem ([HC] or [A, Theorem 3.3]),

$$\widehat{\Omega_{\lambda,\mathbb{R}}}(\varphi) = \int_{\mathfrak{g}_\mathbb{R}} F_\lambda \varphi, \tag{2}$$

where F_λ is an $Ad(G_\mathbb{R})$-invariant, locally L^1 function on $\mathfrak{g}_\mathbb{R}$ which is represented by a real analytic function on $\mathfrak{g}'_\mathbb{R}$.

3. Harish-Chandra's formula for $\widehat{\Omega_{\lambda,\mathbb{R}}}$

In this section, we derive Harish-Chandra's formula (9) for the Fourier transform of a regular semisimple coadjoint orbit $\widehat{\Omega_{\lambda,\mathbb{R}}}$ as a fixed point integral localization formula.

Let $\mathscr{X}$ denote the *flag variety*, i.e. the variety of Borel subalgebras $\mathfrak{b} \subset \mathfrak{g} = \mathfrak{g}_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$. The space $\mathscr{X}$ is a smooth complex projective variety.

As before, if $X \in \mathfrak{g}$ is an element of the Lie algebra, we denote by $X_{\mathscr{X}}$ the vector field on $\mathscr{X}$ generated by X: if $x \in \mathscr{X}$ and $f \in \mathscr{C}^{\infty}(\mathscr{X})$, then

$$X_{\mathscr{X}}(x)f = \frac{d}{d\varepsilon} f(\exp(\varepsilon X) \cdot x)\Big|_{\varepsilon=0}.$$

The ordinary moment map $\mu : T^* \mathscr{X} \to \mathfrak{g}^*$ is defined by

$$\mu(\zeta) : X \mapsto \langle \zeta, X_{\mathscr{X}} \rangle, \quad \zeta \in T^* \mathscr{X}, \ X \in \mathfrak{g}^*.$$

The moment map μ takes values in the nilpotent cone $\mathscr{N}^* \subset \mathfrak{g}^*$.

Next, we define Rossmann's *twisted moment map*. It is best described in [SchV1, Section 8]. Let $\mathfrak{h}$ be the *universal Cartan algebra* of $\mathfrak{g}$. That is $\mathfrak{h}$ is a Cartan algebra equipped with a choice of positive roots. Let $\tilde{\lambda} \in \mathfrak{h}^*$, and fix a compact real form $U_{\mathbb{R}} \subset G$. The twisted moment map $\mu_{\tilde{\lambda}} : T^* \mathscr{X} \to \mathfrak{g}^*$ can be written as

$$\mu_{\tilde{\lambda}} = \mu + \tilde{\lambda}_x,$$

where $\mu : T^* \mathscr{X} \to \mathfrak{g}^*$ is the ordinary moment map and the map $\tilde{\lambda}_x : \mathscr{X} \to \mathfrak{g}^*$ is defined as follows. For each $x \in \mathscr{X}$, let $B(x) \subset G$ and $T_{\mathbb{R}}(x) \subset U_{\mathbb{R}}$ be the stabilizer groups of x:

$$B(x) = \{g \in G; g \cdot x = x\}, \quad T_{\mathbb{R}}(x) = \{u \in U_{\mathbb{R}}; u \cdot x = x\}. \tag{3}$$

Then $B(x)$ is the Borel subgroup of G corresponding to $x \in \mathscr{X}$, and $T_{\mathbb{R}}(x)$ is the maximal torus of $U_{\mathbb{R}}$ fixing x. Let $\mathfrak{b}_x$ denote the Lie algebra of $B(x)$. The quotient $\mathfrak{b}_x/[\mathfrak{b}_x, \mathfrak{b}_x]$ is canonically isomorphic to the universal Cartan algebra $\mathfrak{h}$ so that the nilpotent algebra $\mathfrak{n}_x = [\mathfrak{b}_x, \mathfrak{b}_x]$ is the vector space direct sum of the negative root spaces. On the other hand, the complexified Lie algebra $\mathfrak{t}_x$ of $T_{\mathbb{R}}(x)$ can be realized as a subalgebra of $\mathfrak{b}_x$ which maps isomorphically onto $\mathfrak{b}_x/[\mathfrak{b}_x, \mathfrak{b}_x]$. Thus, we obtain an identification of $\mathfrak{t}_x$ with the universal Cartan algebra $\mathfrak{h}$. This identification makes $\tilde{\lambda} \in \mathfrak{h}^*$ correspond to a $\tilde{\lambda}_x \in (\mathfrak{t}_x)^*$, which we extend to a linear functional on $\mathfrak{g}$ using the canonical splitting of vector spaces

$$\mathfrak{g} = \mathfrak{t}_x \oplus [\mathfrak{t}_x, \mathfrak{g}] = \mathfrak{t}_x \oplus (\text{root spaces}).$$

The map $x \mapsto \tilde{\lambda}_x \in \mathfrak{g}^*$ is real algebraic and $U_{\mathbb{R}}$-equivariant (but not G-equivariant unless $\tilde{\lambda} = 0$).

If $\tilde{\lambda} = 0$, then $\tilde{\lambda}_x \equiv 0$ and the twisted moment map $\mu_{\tilde{\lambda}} = \mu + \tilde{\lambda}_x$ reduces to the ordinary moment map μ. On the other hand, if $\tilde{\lambda}$ is regular, then $\mu_{\tilde{\lambda}}$ takes values in the coadjoint orbit $\Omega_{\tilde{\lambda}}$ and

$$\mu_{\tilde{\lambda}} : T^*\mathcal{X} \xrightarrow{\sim} \Omega_{\tilde{\lambda}}$$

is a real algebraic diffeomorphism which is equivariant with respect to $U_{\mathbb{R}}$, but never G-equivariant (see [SchV1, Section 8]).

Recall that $\lambda \in i(\mathfrak{g}_{\mathbb{R}}^*)'$ is a regular semisimple element and we are interested in the Fourier transform of $\Omega_{\lambda,\mathbb{R}} = G_{\mathbb{R}} \cdot \lambda$ which is a cycle in $\Omega_{\lambda} = G \cdot \lambda$ such that its support $|\Omega_{\lambda,\mathbb{R}}|$ is a smooth submanifold. Let $\mathfrak{t}$ be the unique Cartan subalgebra of $\mathfrak{g}$ which contains $I^{-1}(i\lambda) \in \mathfrak{g}_{\mathbb{R}}' \subset \mathfrak{g}$, and let us select some positive system of roots Φ^+ on $\mathfrak{t}$. Then we get an identification of $\mathfrak{t}$ with the universal Cartan algebra $\mathfrak{h}$, and λ can be regarded as a regular element of $\mathfrak{h}^*$ and thus we obtain a twisted moment map $\mu_{\lambda} : T^*\mathcal{X} \xrightarrow{\sim} \Omega_{\lambda}$. Because μ_{λ} is a diffeomorphism, we can rewrite the Fourier transform of $\Omega_{\lambda,\mathbb{R}}$ as

$$\widehat{\Omega_{\lambda,\mathbb{R}}}(\varphi) = \frac{1}{(2\pi i)^n n!} \int_{\Omega_{\lambda,\mathbb{R}}} \hat{\varphi} \sigma_{\lambda}^n = \frac{1}{(2\pi i)^n n!} \int_{\mu_{\lambda}^{-1}(\Omega_{\lambda,\mathbb{R}})} \mu_{\lambda}^* \hat{\varphi} \cdot (\mu_{\lambda}^* \sigma_{\lambda})^n. \tag{4}$$

The form $\mu_{\lambda}^* \sigma_{\lambda}$ is known—it is precisely the form that appears in the integral character formula in [L1,L2,R3,SchV2]. In order to make a precise statement, let τ_{λ} be a $U_{\mathbb{R}}$-invariant real algebraic two form on $\mathcal{X}$ defined by the formula

$$\tau_{\lambda}(u_x, v_x) = \lambda_x([u, v]),$$

where $u_x, v_x \in T_x\mathcal{X}$ are the tangent vectors at x induced by $u, v \in \mathfrak{u}_{\mathbb{R}}$ via the infinitesimal action of the Lie algebra $\mathfrak{u}_{\mathbb{R}}$ of $U_{\mathbb{R}}$. Let σ be the canonical complex algebraic G-invariant symplectic form on $T^*\mathcal{X}$, and let $\pi : T^*\mathcal{X} \to \mathcal{X}$ be the natural projection. Then Proposition 3.3 of [SchV2] (which was derived from Lemma 7.2 in [R3]) says that

$$\mu_{\lambda}^* \sigma_{\lambda} = -\sigma + \pi^* \tau_{\lambda}. \tag{5}$$

This form appears in the integral character formula [R1, Theorem 2.1; Sch V2, Theorem 3.8]; [L1 formula (2); L2 formula (1)].

It is a much more difficult task to identify the cycle $\mu_{\lambda}^{-1}(\Omega_{\lambda,\mathbb{R}})$ in $T^*\mathcal{X}$. Because μ_{λ} is not $G_{\mathbb{R}}$-equivariant, this cycle even may not be $G_{\mathbb{R}}$-invariant. But, as we will see soon, $\mu_{\lambda}^{-1}(\Omega_{\lambda,\mathbb{R}})$ is homologous to a conic Lagrangian $G_{\mathbb{R}}$-invariant cycle.

Lemma 1. *The composition map $\mu \circ \mu_{\lambda}^{-1} : \Omega_{\lambda} \to \mathfrak{g}^* \simeq \mathfrak{g}_{\mathbb{R}}^* \oplus i\mathfrak{g}_{\mathbb{R}}^*$ sends $|\Omega_{\lambda,\mathbb{R}}|$ into a set with bounded real part, i.e. the set*

$$\mathrm{Re}(\mu \circ \mu_{\lambda}^{-1}(|\Omega_{\lambda,\mathbb{R}}|))$$

is a bounded subset of $\mathfrak{g}_{\mathbb{R}}^*$.

Proof. Notice that $\mu = \mu_\lambda - \lambda_x$. Hence $\mu \circ \mu_\lambda^{-1} = j_{\Omega_\lambda \hookrightarrow \mathfrak{g}^*} - \lambda_x \circ \pi \circ \mu_\lambda^{-1}$, where $j_{\Omega_\lambda \hookrightarrow \mathfrak{g}^*}$ denotes the inclusion map. Since $|\Omega_{\lambda,\mathbb{R}}| \subset i\mathfrak{g}_\mathbb{R}^*$,

$$\mathrm{Re}(\mu \circ \mu_\lambda^{-1}(|\Omega_{\lambda,\mathbb{R}}|)) = \mathrm{Re}(-\lambda_x \circ \pi \circ \mu_\lambda^{-1}(|\Omega_{\lambda,\mathbb{R}}|)) \subset \mathrm{Re}(-\lambda_x(\mathscr{X})).$$

Because the flag variety $\mathscr{X}$ is compact and the map λ_x is continuous the result follows. $\square$

If s is a positive real number, then the fiber scaling map

$$Scaling_s : T^*\mathscr{X} \to T^*\mathscr{X}, \qquad \zeta \mapsto s\zeta$$

sends the cycle $\mu_\lambda^{-1}(\Omega_{\lambda,\mathbb{R}})$ into a new cycle denoted by $s \cdot \mu_\lambda^{-1}(\Omega_{\lambda,\mathbb{R}})$. We use the notion of families of cycles and their limits described in Section 3 [SchV1]. The cycles $s \cdot \mu_\lambda^{-1}(\Omega_{\lambda,\mathbb{R}})$, $s \in (0,1)$, piece together to form a family of cycles $C_{(0,1)}$ parameterized by an open interval $(0,1)$. Thus, we can take its limit as $s \to 0^+$: let

$$C_\lambda = \lim_{s \to 0^+} s \cdot \mu_\lambda^{-1}(\Omega_{\lambda,\mathbb{R}}). \tag{6}$$

It is clear from the construction that the cycle C_λ is *conic*, i.e. invariant under the fiber scaling maps $Scaling_s : T^*\mathscr{X} \to T^*\mathscr{X}$, $s > 0$.

In general, if $\tilde{C}_{(0,a)}$ is a family of cycles in some space Z parameterized by an open interval $(0,a)$, then $\lim_{s \to 0^+} \tilde{C}_s = \tilde{C}_0$ means that if we regard $\tilde{C}_{(0,a)}$ as a chain in $[0,a) \times Z$, then $\partial \tilde{C}_{(0,a)} = -\tilde{C}_0$. Thus, it is natural to define $\lim_{s \to a^-} \tilde{C}_s$ by setting

$$\lim_{s \to a^-} \tilde{C}_s = \partial \tilde{C}_{(0,a)},$$

where $\tilde{C}_{(0,a)}$ is regarded as a chain in $(0,a] \times Z$.

When $s = 1$, the fiber scaling map $Scaling_1 : T^*\mathscr{X} \to T^*\mathscr{X}$ is the identity map on $T^*\mathscr{X}$, and it is clear that $\lim_{s \to 1^-} s \cdot \mu_\lambda^{-1}(\Omega_{\lambda,\mathbb{R}}) = \mu_\lambda^{-1}(\Omega_{\lambda,\mathbb{R}})$. Next we make the following observation. Recall that the family of cycles $C_{(0,1)}$ is a cycle in $(0,1) \times T^*\mathscr{X}$. We can regard it as a chain in $[0,1] \times T^*\mathscr{X}$. Let $p : [0,1] \times T^*\mathscr{X} \twoheadrightarrow T^*\mathscr{X}$ be the projection. Then $p_*(C_{(0,1)})$ is a Borel–Moore $(2n+1)$-chain in $T^*\mathscr{X}$ such that

$$\partial(p_*(C_{(0,1)})) = \lim_{s \to 1^-} C_s - \lim_{\varepsilon \to 0^+} C_s = \mu_\lambda^{-1}(\Omega_{\lambda,\mathbb{R}}) - C_\lambda.$$

In particular, the cycles C_λ and $\mu_\lambda^{-1}(\Omega_{\lambda,\mathbb{R}})$ are homologous. Moreover, the support of $p_*(C_{(0,1)})$ lies in the closure

$$\overline{\bigcup_{s \in [0,1]} s|\mu_\lambda^{-1}(\Omega_{\lambda,\mathbb{R}})|}.$$

We see from Lemma 1 that the image under $\mathrm{Re}(\mu) : T^*\mathscr{X} \to \mathfrak{g}_\mathbb{R}^*$ of the support $|p_*(C_{(0,1)})|$ is bounded and hence Lemma 3.19 from [SchV2] together with Eq. (5)

implies that

$$\int_{\mu_\lambda^{-1}(\Omega_{\lambda,\mathbb{R}})} \mu_\lambda^* \hat{\varphi} \cdot (\mu_\lambda^* \sigma_\lambda)^n = \int_{C_\lambda} \mu_\lambda^* \hat{\varphi} \cdot (\mu_\lambda^* \sigma_\lambda)^n. \tag{7}$$

The group $G_{\mathbb{R}}$ acts on $\mathscr{X}$ real algebraically with finitely many orbits. The resulting stratification of $\mathscr{X}$ by $G_{\mathbb{R}}$-orbits satisfies the Whitney conditions. We denote by $T^*_{G_{\mathbb{R}}}\mathscr{X}$ the union of the $\mathscr{C}^\infty$ conormal bundles of the $G_{\mathbb{R}}$-action. The set $T^*_{G_{\mathbb{R}}}\mathscr{X}$ is a closed $G_{\mathbb{R}}$-invariant real semialgebraic subset of $T^*\mathscr{X}$ of dimension $2n$. We can also express this set as

$$T^*_{G_{\mathbb{R}}}\mathscr{X} = \mu^{-1}(i\mathfrak{g}^*_{\mathbb{R}}).$$

Clearly, the moment map μ takes only purely imaginary values on the support $|C_\lambda|$. Thus $|C_\lambda|$ is a subset of $T^*_{G_{\mathbb{R}}}\mathscr{X}$. This makes C_λ a top-dimensional cycle in $T^*_{G_{\mathbb{R}}}\mathscr{X}$. Because the group $G_{\mathbb{R}}$ was assumed to be connected, it follows that the cycle C_λ is $G_{\mathbb{R}}$-invariant.

We already know that C_λ is a conic cycle, and since its support $|C_\lambda|$ lies in a finite union of conormal bundles, C_λ is a Lagrangian cycle with respect to the canonical symplectic structure on $T^*\mathscr{X}$, viewed as a $\mathscr{C}^\infty$ (not holomorphic!) cotangent bundle on $\mathscr{X}$. Hence by Theorem 9.7.1 of [KaScha], there exists a bounded complex of sheaves $\mathscr{F}_\lambda$ on $\mathscr{X}$ with $\mathbb{R}$-constructible cohomology such that its characteristic cycle $Ch(\mathscr{F}_\lambda) = C_\lambda$.

Characteristic cycles were introduced by Kashiwara and their definition can be found in [KaScha]. On the other hand, Schmid and Vilonen give a geometric way to understand characteristic cycles in [SchV1]. A comprehensive treatment of characteristic cycles can be found in [Schü].

Combining this observation with Eqs. (4), (5) and (7) we can rewrite the Fourier transform of $\Omega_{\lambda,\mathbb{R}}$ as follows:

$$\widehat{\Omega_{\lambda,\mathbb{R}}}(\varphi) = \frac{1}{(2\pi i)^n n!} \int_{Ch(\mathscr{F}_\lambda)} \mu_\lambda^* \hat{\varphi} \cdot (-\sigma + \pi^* \tau_\lambda)^n. \tag{8}$$

The right-hand side of this formula is precisely the integral character formula from [L1,L2,SchV2]. If we can identify the complex of sheaves $\mathscr{F}_\lambda$, then we can apply the localization argument from [L2] to compute this integral and identify the function F_λ in Eq. (2). (We do not require the complex of sheaves $\mathscr{F}_\lambda$ to be $G_{\mathbb{R}}$-equivariant or to be an element of the "$G_{\mathbb{R}}$-equivariant derived category on $\mathscr{X}$ with twist $(\lambda - \rho)$", $D_{G_{\mathbb{R}}}(\mathscr{X})_\lambda$, as in [SchV2] because the localization argument in [L2] only uses that $Ch(\mathscr{F}_\lambda)$ is $G_{\mathbb{R}}$-invariant.)

If $X \in \mathfrak{g}'_{\mathbb{R}}$ is a regular semisimple element, let $\mathscr{X}^X$ denote the set of zeroes of the vector field $X_{\mathscr{X}}$ on $\mathscr{X}$, and let $\mathfrak{t}(X) \subset \mathfrak{g}$ be the unique Cartan subalgebra containing X.

Then

$$\widehat{\Omega_{\lambda,\mathbb{R}}}(\varphi) = \int_{\mathfrak{g}_{\mathbb{R}}} F_{\lambda}\varphi, \qquad F_{\lambda}(X) = \sum_{x \in \mathscr{X}^X} \frac{d_{X,x} e^{\langle X, \lambda_x \rangle}}{\alpha_{x,1}(X) \ldots \alpha_{x,n}(X)}, \tag{9}$$

where $d_{X,x}$'s are certain integers (to be described below) which are local invariants of the sheaf $\mathscr{F}_{\lambda}$ given by formula (5.25b) in [SchV2] and $\alpha_{x,1}, \ldots, \alpha_{x,n} \in \mathfrak{t}(X)^*$ are the roots of $\mathfrak{t}(X)$ such that, letting $\mathfrak{g}^{\alpha_{x,1}}, \ldots, \mathfrak{g}^{\alpha_{x,n}} \subset \mathfrak{g}$ be the corresponding root spaces and $\mathfrak{b}_x$ be the Borel subalgebra of $\mathfrak{g}$ corresponding to x,

$$\mathfrak{b}_x = \mathfrak{t}(X) \oplus \mathfrak{g}^{\alpha_{x,1}} \oplus \cdots \oplus \mathfrak{g}^{\alpha_{x,n}}$$

as vector spaces. Thus, we obtain Harish-Chandra's formula for the invariant eigendistribution $\widehat{\Omega_{\lambda,\mathbb{R}}}$.

It remains to specify the integer multiplicities $d_{X,x}$'s. Choose a positive root system Φ_x^+ on $\mathfrak{t}(X)$ so that, letting

$$\mathfrak{n}_x^+ = \bigoplus_{\alpha \in \Phi_x^+} \mathfrak{g}^{\alpha},$$

we have

$$\mathfrak{g} = \mathfrak{b}_x \oplus \mathfrak{n}_x^+$$

as vector spaces. Then we can select two subsets $\Phi_x', \Phi_x'' \subset \Phi_x^+$ with the following properties:

(a) for $\alpha \in \Phi_x^+$, if $\operatorname{Re}(\alpha(X)) \neq 0$, then $\alpha \in \Phi_x' \Leftrightarrow \operatorname{Re}(\alpha(X)) < 0$ and $\alpha \in \Phi_x'' \Leftrightarrow \operatorname{Re}(\alpha(X)) > 0$;

(b) for $\alpha_1, \alpha_2 \in \Phi_x^+$, if $\alpha_1 + \alpha_2 \in \Phi_x^+$, then $\alpha_1, \alpha_2 \in \Phi_x' \Rightarrow \alpha_1 + \alpha_2 \in \Phi_x', \alpha_1, \alpha_2 \in \Phi_x'' \Rightarrow \alpha_1 + \alpha_2 \in \Phi_x''$.

Such subsets Φ_x', Φ_x'' always exist: for example, the subsets defined by (a) without the restriction $\operatorname{Re}(\alpha(X)) \neq 0$ satisfy also (b). Because of (b),

$$\mathfrak{n}_x' = \bigoplus_{\alpha \in \Phi_x'} \mathfrak{g}^{\alpha}, \qquad \mathfrak{n}_x'' = \bigoplus_{\alpha \in \Phi_x''} \mathfrak{g}^{\alpha}$$

are subalgebras of $\mathfrak{n}_x^+$. We also define

$$N_x^+ = \exp(\mathfrak{n}_x^+) \cdot x, \quad N_x' = \exp(\mathfrak{n}_x') \cdot x, \quad N_x'' = \exp(\mathfrak{n}_x'') \cdot x.$$

Then $N_x^+ \subset \mathscr{X}$ is an open Schubert cell which contains N_x' and N_x'' as affine linear subspaces. Finally, the coefficient

$$d_{X,x} = \chi(\mathscr{H}_{N_x''}^{\bullet}(\mathbb{D}\mathscr{F}_{\lambda})_x) = \chi(\mathscr{H}_{\{x\}}^{\bullet}(\mathbb{D}\mathscr{F}_{\lambda}|_{N_x''})) \tag{10}$$

is the Euler characteristic, where $\mathbb{D}\mathscr{F}_\lambda$ denotes the Verdier dual sheaf to $\mathscr{F}_\lambda$. This multiplicity $d_{X,x}$ is exactly the local contribution of x to the Lefschetz fixed point formula generalized to sheaf cohomology by Goresky and MacPherson [GM].

Example 2. Suppose that the group $G_\mathbb{R}$ is compact. Then we can choose $U_\mathbb{R} = G_\mathbb{R}$, in which case the twisted moment map $\mu_\lambda : T^*\mathscr{X} \xrightarrow{\sim} \Omega_\lambda$ restricts to a $G_\mathbb{R}$-equivariant diffeomorphism $\mu_\lambda : \mathscr{X} \xrightarrow{\sim} |\Omega_{\lambda,\mathbb{R}}|$. We can choose a positive system of roots of $\mathfrak{t}$ so that this restriction is orientation preserving. Then $C_\lambda = \mathscr{X}$, we can take $\mathscr{F}_\lambda = \mathbb{C}_{\mathscr{X}}$—the constant sheaf on $\mathscr{X}$, all the coefficients $d_{X,x} = 1$, and formula (9) coincides with the result of application of the classical fixed point integral localization formula for equivariant forms as described in Section 7.5 in [BGV].

4. General properties of the cycle C_λ

In this section, we describe some general properties of C_λ. We begin with the following lemma.

The definition of the twisted moment map μ_λ involves choices of a real compact form $U_\mathbb{R}$ and an identification between the Cartan subalgebra $\mathfrak{t} \subset \mathfrak{g}$ containing $I^{-1}(i\lambda)$ and the universal Cartan algebra $\mathfrak{h}$. We fix an isomorphism $\mathfrak{t} \simeq \mathfrak{h}$ and consider two different compact real forms $U_\mathbb{R}, U'_\mathbb{R} \subset G$. We denote by $\mu_{\lambda,U_\mathbb{R}}$ (respectively $\mu_{\lambda,U'_\mathbb{R}}$) and $C_{\lambda,U_\mathbb{R}}$ (respectively $C_{\lambda,U'_\mathbb{R}}$) the twisted moment map and the corresponding cycle defined relatively to $U_\mathbb{R}$ (respectively $U'_\mathbb{R}$).

Lemma 3. *Suppose that* $U'_\mathbb{R} = gU_\mathbb{R}g^{-1}$ *for some* $g \in G_\mathbb{R}$. *Then the cycles* $C_{\lambda,U_\mathbb{R}}$ *and* $C_{\lambda,U'_\mathbb{R}}$ *are the same.*

Proof. One can see from the definition of the twisted moment map that the relation between the two maps $\mu_{\lambda,U_\mathbb{R}}$ and $\mu_{\lambda,U'_\mathbb{R}}$ is

$$\mu_{\lambda,U'_\mathbb{R}}(g \cdot \zeta) = g \cdot \mu_{\lambda,U_\mathbb{R}}(\zeta), \quad \zeta \in T^*\mathscr{X}.$$

Therefore,

$$C_{\lambda,U'_\mathbb{R}} = \lim_{s \to 0^+} s \cdot \mu_{\lambda,U'_\mathbb{R}}^{-1}(\Omega_{\lambda,\mathbb{R}}) = \lim_{s \to 0^+} s \cdot g \cdot \mu_{\lambda,U_\mathbb{R}}^{-1}(g^{-1} \cdot \Omega_{\lambda,\mathbb{R}})$$

$$= g \cdot \lim_{s \to 0^+} s \cdot \mu_{\lambda,U_\mathbb{R}}^{-1}(\Omega_{\lambda,\mathbb{R}}) = g \cdot C_{\lambda,U_\mathbb{R}} = C_{\lambda,U_\mathbb{R}}$$

by $G_\mathbb{R}$-invariance of the cycles $\Omega_{\lambda,\mathbb{R}}$ in Ω_λ and $C_{\lambda,U_\mathbb{R}}$ in $T^*\mathscr{X}$. $\quad\square$

Lemma 4. *The composition map* $\pi \circ \mu_\lambda^{-1} : \Omega_\lambda \to \mathscr{X}$ *is G-equivariant.*

Proof. Recall that $\mathscr{X}$ is the set of all Borel subalgebras $\mathfrak{b} \subset \mathfrak{g}$, $\mathfrak{h}$ is the universal Cartan algebra of $\mathfrak{g}$, and $\lambda \in \mathfrak{h}^*$. Then the map $\pi \circ \mu_\lambda^{-1}$ sends $\lambda \in \mathfrak{h}^*$ into the only Borel algebra

b which contains $\mathfrak{h}$ and all the negative root spaces. If $v \in \Omega_\lambda$ is an arbitrary element, then $v = g \cdot \lambda$ for some $g \in G$, and $g\mathfrak{h}g^{-1}$ is naturally a universal Cartan algebra of $\mathfrak{g}$. On the other hand, $g\mathfrak{b}g^{-1} = g \cdot \mathfrak{b}$ is the Borel algebra containing $g\mathfrak{h}g^{-1}$ and its negative root spaces. Therefore, $\pi \circ \mu_\lambda^{-1}(v) = g \cdot \mathfrak{b}$. This proves that the composition map $\pi \circ \mu_\lambda^{-1} : \Omega_\lambda \to \mathscr{X}$ is G-equivariant. $\quad \square$

Corollary 5. *The composition map $\pi \circ \mu_\lambda^{-1} : \Omega_\lambda \to \mathscr{X}$ sends $|\Omega_{\lambda,\mathbb{R}}|$ into a single $G_\mathbb{R}$-orbit in $\mathscr{X}$.*

Let $\mathcal{O}_\lambda$ denote this $G_\mathbb{R}$-orbit in $\mathscr{X}$:

$$\mathcal{O}_\lambda = \pi \circ \mu_\lambda^{-1}(|\Omega_{\lambda,\mathbb{R}}|).$$

We pick an element $v \in |\Omega_{\lambda,\mathbb{R}}|$, and let $x = \pi \circ \mu_\lambda^{-1}(v) \in \mathscr{X}$. Recall that $\mathfrak{b}_x$ denotes the Borel algebra corresponding to x, i.e. the Lie algebra of the group $B(x)$ defined by (3). Let $\mathfrak{n}_x = [\mathfrak{b}_x, \mathfrak{b}_x]$ be the nilpotent subalgebra of $\mathfrak{b}_x$, and let $\mathfrak{n}_{x,\mathbb{R}} = \mathfrak{n}_x \cap \mathfrak{g}_\mathbb{R}$.

Lemma 6. *The coadjoint orbit $|\Omega_{\lambda,\mathbb{R}}|$ contains the set $v + iI(\mathfrak{n}_{x,\mathbb{R}}) \subset i\mathfrak{g}_\mathbb{R}^*$.*

Proof. Let $N_{x,\mathbb{R}} = \exp(\mathfrak{n}_{x,\mathbb{R}})$ be the subgroup of $G_\mathbb{R}$ generated by $\mathfrak{n}_{x,\mathbb{R}}$. It is enough to show that

$$N_{x,\mathbb{R}} \cdot v \quad \supset \quad v + iI(\mathfrak{n}_{x,\mathbb{R}})$$

as subsets of $i\mathfrak{g}_\mathbb{R}^*$. Moreover, it is enough to show that

$$N_{x,\mathbb{R}} \cdot I^{-1}(v) \quad \supset \quad I^{-1}(v) + i\mathfrak{n}_{x,\mathbb{R}}$$

as subsets of $i\mathfrak{g}_\mathbb{R}$.

First we notice that if $n \in \mathfrak{n}_{x,\mathbb{R}}$, then

$$\exp(n) \cdot I^{-1}(v) = Ad(\exp(n))(I^{-1}(v)) = e^{ad(n)}(I^{-1}(v)) \quad \subset \quad I^{-1}(v) + i\mathfrak{n}_{x,\mathbb{R}},$$

because $ad(n)(I^{-1}(v)) \in i\mathfrak{n}_{x,\mathbb{R}}$.

We recall that the Lie algebra $\mathfrak{n}_{x,\mathbb{R}}$ is nilpotent and introduce its ideals

$$\mathfrak{n}_{x,\mathbb{R}}^0 = \mathfrak{n}_{x,\mathbb{R}}, \quad \mathfrak{n}_{x,\mathbb{R}}^1 = [\mathfrak{n}_{x,\mathbb{R}}, \mathfrak{n}_{x,\mathbb{R}}^0], \quad \mathfrak{n}_{x,\mathbb{R}}^2 = [\mathfrak{n}_{x,\mathbb{R}}, \mathfrak{n}_{x,\mathbb{R}}^1], \dots,$$

$$\mathfrak{n}_{x,\mathbb{R}}^k = [\mathfrak{n}_{x,\mathbb{R}}, \mathfrak{n}_{x,\mathbb{R}}^{k-1}], \quad \mathfrak{n}_{x,\mathbb{R}}^{k+1} = [\mathfrak{n}_{x,\mathbb{R}}, \mathfrak{n}_{x,\mathbb{R}}^k] = 0.$$

Because the element $I^{-1}(v) \in i\mathfrak{g}_\mathbb{R}$ is regular semisimple and the centralizer of a regular semisimple element is a Cartan algebra and hence consists of semisimple elements, each of the maps

$$\mathfrak{n}_{x,\mathbb{R}}^j \to i\mathfrak{n}_{x,\mathbb{R}}^j, \qquad n \mapsto [n, I^{-1}(v)],$$

$j = 0, \ldots, k$ is a vector space isomorphism. In particular, each of the maps

$$\mathfrak{n}^j_{x,\mathbb{R}} / \mathfrak{n}^{j+1}_{x,\mathbb{R}} \to i(\mathfrak{n}^j_{x,\mathbb{R}} / \mathfrak{n}^{j+1}_{x,\mathbb{R}}), \quad n + \mathfrak{n}^{j+1}_{x,\mathbb{R}} \mapsto [n, I^{-1}(v)] + i\mathfrak{n}^{j+1}_{x,\mathbb{R}},$$

$j = 0, \ldots, k$, is a vector space isomorphism. It follows that each of the maps

$$\mathfrak{n}^j_{x,\mathbb{R}} \to I^{-1}(v) + i(\mathfrak{n}^j_{x,\mathbb{R}} / \mathfrak{n}^{j+1}_{x,\mathbb{R}}),$$

$$n \mapsto e^{ad(n)}(I^{-1}(v)) + i\mathfrak{n}^{j+1}_{x,\mathbb{R}} = I^{-1}(v) + [n, I^{-1}(v)] + i\mathfrak{n}^{j+1}_{x,\mathbb{R}},$$

$j = 0, \ldots, k$, is surjective. Therefore, we get the following inclusions:

$$N_{x,\mathbb{R}} \cdot I^{-1}(v) + i\mathfrak{n}^{j+1}_{x,\mathbb{R}} \quad \supset \quad I^{-1}(v) + i\mathfrak{n}^j_{x,\mathbb{R}},$$

$j = 0, \ldots, k$, which together imply

$$N_{x,\mathbb{R}} \cdot I^{-1}(v) \quad \supset \quad I^{-1}(v) + i\mathfrak{n}_{x,\mathbb{R}}.$$

This finishes our proof of the lemma. $\quad \square$

Proposition 7. *The set $\mu_\lambda^{-1}(|\Omega_{\lambda,\mathbb{R}}|) \cap T_x^* \mathscr{X}$ is precisely $\mu_\lambda^{-1}(v) + T_{\mathcal{O}_\lambda,x}^* \mathscr{X}$, where*

$$T_{\mathcal{O}_\lambda,x}^* \mathscr{X} = \{\zeta \in T_x^* \mathscr{X}; \operatorname{Re}(\zeta)|_{T\mathcal{O}_\lambda} = 0\}$$

is the $\mathscr{C}^\infty$ conormal space to $\mathcal{O}_\lambda$ at the point x.

Proof. Pick any element $\zeta \in T_x^* \mathscr{X}$ such that $\mu_\lambda^{-1}(v) + \zeta \in \mu_\lambda^{-1}(|\Omega_{\lambda,\mathbb{R}}|)$. Then

$$i\mathfrak{g}_{\mathbb{R}}^* \supset |\Omega_{\lambda,\mathbb{R}}| \ni \mu_\lambda(\mu_\lambda^{-1}(v) + \zeta) = \lambda_x + \mu(\mu_\lambda^{-1}(v) + \zeta)$$

$$= (\lambda_x + \mu(\mu_\lambda^{-1}(v))) + \mu(\zeta) = v + \mu(\zeta).$$

This shows that $\mu(\zeta) \in i\mathfrak{g}_{\mathbb{R}}^*$ or, equivalently, $\zeta \in T_{\mathcal{O}_\lambda,x}^* \mathscr{X}$, which proves one inclusion

$$\mu_\lambda^{-1}(|\Omega_{\lambda,\mathbb{R}}|) \cap T_x^* \mathscr{X} \quad \subset \quad \mu_\lambda^{-1}(v) + T_{\mathcal{O}_\lambda,x}^* \mathscr{X}.$$

To prove the other inclusion we use the result of Lemma 6 that

$$|\Omega_{\lambda,\mathbb{R}}| \quad \supset \quad v + iI(\mathfrak{n}_{x,\mathbb{R}}).$$

Notice that

$$\mu_\lambda^{-1}(v + iI(\mathfrak{n}_{x,\mathbb{R}})) \quad \subset \quad T_x^* \mathscr{X}$$

and also that

$$\mu_\lambda^{-1}(v + iI(\mathfrak{n}_{x,\mathbb{R}})) - \mu_\lambda^{-1}(v) = \mu^{-1}(iI(\mathfrak{n}_{x,\mathbb{R}})) \cap T_x^* \mathscr{X}$$

is a real vector subspace of $T^*_x \mathcal{X}$. The dimension of this subspace equals

$$dim_{\mathbb{R}}(\mathfrak{n}_{x,\mathbb{R}}) = dim_{\mathbb{R}}(\mathfrak{g}_{\mathbb{R}}) - \tfrac{1}{2} dim_{\mathbb{C}}(\mathfrak{h}) - dim_{\mathbb{R}}(\mathcal{O}_{\lambda})$$

$$= dim_{\mathbb{R}}(\mathcal{X}) - dim_{\mathbb{R}}(\mathcal{O}_{\lambda}) = dim_{\mathbb{R}}(T^*_{\mathcal{O}_{\lambda},x}\mathcal{X}).$$

This forces

$$\mu_{\lambda}^{-1}(v + iI(\mathfrak{n}_{x,\mathbb{R}})) - \mu_{\lambda}^{-1}(v) = T^*_{\mathcal{O}_{\lambda},x}\mathcal{X},$$

which implies the proposition. $\square$

Theorem 8. (1) *The cycle C_{λ} can be written as*

$$C_{\lambda} = T^*_{\mathcal{O}_{\lambda}}\mathcal{X} + c_{\lambda},$$

where the conformal space $T^*_{\mathcal{O}_{\lambda}}\mathcal{X}$ is equipped with an appropriate orientation and regarded as a chain in $T^*\mathcal{X}$ and c_{λ} is a chain supported inside $\pi^{-1}(\overline{\mathcal{O}_{\lambda}}\backslash\mathcal{O}_{\lambda})$. In particular,

$$|C_{\lambda}| \cap \pi^{-1}(\mathcal{O}_{\lambda}) = T^*_{\mathcal{O}_{\lambda}}\mathcal{X}.$$

(2) *The coefficients in the fixed point formula (9) $d_{X,x}$'s given by Eq. (10) are zero unless x lies in the closure $\overline{\mathcal{O}_{\lambda}}$. If $x \in \mathcal{O}_{\lambda}$, then $d_{X,x} = \pm 1$.*

Proof. Part (1) follows form the definition of C_{λ} (Eq. (6)), Corollary 5 and Proposition 7.

Part (1) implies the first assertion of part (2). Let $\mathbb{C}_{\mathcal{O}_{\lambda}}$ denote the constant sheaf on $\mathcal{O}_{\lambda}$, and let $j_{\mathcal{O}_{\lambda} \hookrightarrow \mathcal{X}} : \mathcal{O}_{\lambda} \hookrightarrow \mathcal{X}$ denote the inclusion map. Since the coefficients $d_{X,x}$'s described by Eq. (10) are determined by *local* properties of the cycle C_{λ} at x, as far as the second assertion of part (2) is concerned, we can assume that

$$C_{\lambda} = \pm Ch((Rj_{\mathcal{O}_{\lambda} \hookrightarrow \mathcal{X}})_* \mathbb{C}_{\mathcal{O}_{\lambda}}).$$

(The choice of sign depends on the orientation of $T^*_{\mathcal{O}_{\lambda}}\mathcal{X}$.) Then

$$Ch(\mathbb{D}\mathcal{F}_{\lambda}) = \pm Ch((j_{\mathcal{O}_{\lambda} \hookrightarrow \mathcal{X}})_! \mathbb{C}_{\mathcal{O}_{\lambda}})$$

and, for the purpose of computing the multiplicities $d_{X,x}$'s, we may assume that $\mathbb{D}\mathcal{F}_{\lambda}$ is either

$$(j_{\mathcal{O}_{\lambda} \hookrightarrow \mathcal{X}})_! \mathbb{C}_{\mathcal{O}_{\lambda}} \quad \text{or} \quad (j_{\mathcal{O}_{\lambda} \hookrightarrow \mathcal{X}})_! \mathbb{C}_{\mathcal{O}_{\lambda}}[1],$$

where [1] denotes a shift in degree by 1, so that

$$Ch((j_{\mathcal{O}_{\lambda} \hookrightarrow \mathcal{X}})_! \mathbb{C}_{\mathcal{O}_{\lambda}}) = -Ch((j_{\mathcal{O}_{\lambda} \hookrightarrow \mathcal{X}})_! \mathbb{C}_{\mathcal{O}_{\lambda}}[1]).$$

Then the result follows immediately from (10). $\square$

Remark 9. Part (1) of Theorem 8 is precisely the assertion made by Rossmann in his proof of Theorem 1.4(b) in [R2]. (The proof of this statement was omitted because it was not the point of that paper.)

5. The maximally split Cartan case

In this section, we will identify the cycle C_λ as a characteristic cycle of some sheaf $\mathscr{F}_\lambda$ when $I^{-1}(i\lambda)$ lies in a maximally split Cartan subalgebra of $\mathfrak{g}_\mathbb{R}$. Hence we will compute the function F_λ in formula (2) for the Fourier transform of $\Omega_{\lambda,\mathbb{R}}$.

We will use the *Iwasawa decomposition* of $G_\mathbb{R}$ (see [Kn] for reference). Let $U_\mathbb{R} \subset G$ be a compact real form, and let $\mathfrak{u}_\mathbb{R} \subset \mathfrak{g}$ be its Lie algebra. Let σ and $\tau : \mathfrak{g} \to \mathfrak{g}$ denote the complex conjugations with respect to $\mathfrak{g}_\mathbb{R}$ and $\mathfrak{u}_\mathbb{R}$ respectively. There is a choice of $U_\mathbb{R}$ such that the involutions τ and σ commute. From now on we fix this $U_\mathbb{R}$ and let

$$\theta = \sigma\tau = \tau\sigma : \mathfrak{g} \to \mathfrak{g}$$

be the *Cartan involution* ($\theta^2 = \mathrm{Id}_\mathfrak{g}$). This involution θ preserves $\mathfrak{g}_\mathbb{R}$. We define $\mathfrak{k}$ and $\mathfrak{p}$ as the $(+1)$ eigenspace and the (-1) eigenspace of $\theta : \mathfrak{g} \to \mathfrak{g}$, respectively, and let

$$\mathfrak{k}_\mathbb{R} = \mathfrak{k} \cap \mathfrak{g}_\mathbb{R}, \qquad \mathfrak{p}_\mathbb{R} = \mathfrak{p} \cap \mathfrak{g}_\mathbb{R}.$$

Then

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}, \quad \mathfrak{g}_\mathbb{R} = \mathfrak{k}_\mathbb{R} \oplus \mathfrak{p}_\mathbb{R}, \quad \mathfrak{u}_\mathbb{R} = \mathfrak{k}_\mathbb{R} \oplus i\mathfrak{p}_\mathbb{R}.$$

We define $K_\mathbb{R}$ to be the identity component of $U_\mathbb{R} \cap G_\mathbb{R}$. Then $K_\mathbb{R}$ is a maximal compact subgroup of $G_\mathbb{R}$ with Lie algebra $\mathfrak{k}_\mathbb{R}$. Let $\mathfrak{a}_\mathbb{R} \subset \mathfrak{p}_\mathbb{R}$ be a maximal abelian subspace, let $\mathfrak{m}_\mathbb{R} = \mathscr{Z}_{\mathfrak{k}_\mathbb{R}}(\mathfrak{a}_\mathbb{R})$ be the centralizer of $\mathfrak{a}_\mathbb{R}$ in $\mathfrak{k}_\mathbb{R}$, and let $\mathfrak{a} = \mathfrak{a}_\mathbb{R} \otimes \mathbb{C} \subset \mathfrak{g}$ and $\mathfrak{m} = \mathfrak{m}_\mathbb{R} \otimes \mathbb{C} \subset \mathfrak{g}$ be the complexifications of $\mathfrak{a}_\mathbb{R}$ and $\mathfrak{m}_\mathbb{R}$. Then we obtain a root space decomposition of $\mathfrak{g}$ with respect to $\mathfrak{a}$:

$$\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{a} \oplus \left(\bigoplus_{\alpha \in \Phi} \mathfrak{g}^\alpha \right),$$

where $\Phi \subset \mathfrak{a}^* \backslash \{0\}$ is the reduction of relative roots of the pair $(\mathfrak{g}, \mathfrak{a})$. We select a positive root system $\Phi^+ \subset \Phi$, and let

$$\mathfrak{n}_\mathbb{R} = \mathfrak{g}_\mathbb{R} \cap \bigoplus_{\alpha \in \Phi^+} \mathfrak{g}^{-\alpha}, \quad N_\mathbb{R} = \exp(\mathfrak{n}_\mathbb{R}) \subset G_\mathbb{R}, \quad A_\mathbb{R} = \exp(\mathfrak{a}_\mathbb{R}) \subset G_\mathbb{R}.$$

Then $G_\mathbb{R} = K_\mathbb{R} A_\mathbb{R} N_\mathbb{R}$ is the Iwasawa decomposition of $G_\mathbb{R}$.

We let $\mathfrak{t}_\mathbb{R} \subset \mathfrak{m}_\mathbb{R}$ be a maximal abelian subalgebra consisting of semisimple elements, then $\mathfrak{h}_\mathbb{R} = \mathfrak{t}_\mathbb{R} \oplus \mathfrak{a}_\mathbb{R}$ is a maximally split Cartan subalgebra of $\mathfrak{g}_\mathbb{R}$, and its complexification $\mathfrak{h} = \mathfrak{h}_\mathbb{R} \otimes \mathbb{C}$ is a Cartan subalgebra of $\mathfrak{g}$. We equip $\mathfrak{h}$ with a positive root

system so that $\mathfrak{n}_\mathbb{R}$ lies in the vector space direct sum of the negative root spaces. This way $\mathfrak{h}$ becomes a universal Cartan subalgebra.

In this section, we compute the Fourier transform $\widehat{\Omega_{\lambda,\mathbb{R}}}$ when $\lambda \in i(\mathfrak{g}_\mathbb{R}^*)' \cap iI(\mathfrak{h}_\mathbb{R})$ is a regular semisimple element dual to the maximally split Cartan subalgebra $\mathfrak{h}_\mathbb{R}$.

Notice that by construction the groups $A_\mathbb{R}$ and $N_\mathbb{R}$ fix the element $\pi \circ \mu_\lambda^{-1}(\lambda) \in \mathscr{X}$ and so the $G_\mathbb{R}$-orbit

$$\mathscr{O}_\lambda = G_\mathbb{R} \cdot (\pi \circ \mu_\lambda^{-1}(\lambda)) = K_\mathbb{R} A_\mathbb{R} N_\mathbb{R} \cdot (\pi \circ \mu_\lambda^{-1}(\lambda)) = K_\mathbb{R} \cdot (\pi \circ \mu_\lambda^{-1}(\lambda))$$

is closed. Then the first part of Theorem 8 implies that the cycle C_λ is just the conormal space $T^*_{\mathscr{O}_\lambda}\mathscr{X}$ equipped with some orientation.

As in the proof of Theorem 8, let $\mathbb{C}_{\mathscr{O}_\lambda}$ denote the constant sheaf on $\mathscr{O}_\lambda$, and let $j_{\mathscr{O}_\lambda \hookrightarrow \mathscr{X}} : \mathscr{O}_\lambda \hookrightarrow \mathscr{X}$ denote the inclusion map. Then

$$C_\lambda = \pm Ch(\mathscr{F}_\lambda), \quad \mathscr{F}_\lambda = (j_{\mathscr{O}_\lambda \hookrightarrow \mathscr{X}})_* \mathbb{C}_{\mathscr{O}_\lambda}.$$

Substituting this sheaf into Harish-Chandra's formula (9) and applying the second part of Theorem 8 we obtain the following description of the function F_λ in Eq. (2). It is an $Ad(G_\mathbb{R})$-invariant, locally L^1 function on $\mathfrak{g}_\mathbb{R}$ which is represented by a real analytic function on $\mathfrak{g}'_\mathbb{R}$. If $X \in \mathfrak{g}'_\mathbb{R}$, then $F_\lambda(X) = 0$ unless X is $G_\mathbb{R}$-conjugate to an element of $\mathfrak{h}_\mathbb{R}$. And if $X \in \mathfrak{h}_\mathbb{R} \cap \mathfrak{g}'_\mathbb{R}$, then

$$F_\lambda(X) = \sum_{x \in \mathscr{X}^X \cap \mathscr{O}_\lambda} \frac{\pm e^{\langle X, \lambda_x \rangle}}{\alpha_{x,1}(X) \ldots \alpha_{x,n}(X)},$$

where $\mathscr{X}^X$ denotes the set of zeroes of the vector field on $\mathscr{X}$ generated by X, $\alpha_{x,1}, \ldots, \alpha_{x,n} \in \mathfrak{h}^*$ are the roots of $\mathfrak{h}$ such that, letting $\mathfrak{g}^{\alpha_{x,1}}, \ldots, \mathfrak{g}^{\alpha_{x,n}} \subset \mathfrak{g}$ be the corresponding root spaces and $\mathfrak{b}_x$ be the Borel subalgebra of $\mathfrak{g}$ corresponding to x,

$$\mathfrak{b}_x = \mathfrak{h} \oplus \mathfrak{g}^{\alpha_{x,1}} \oplus \cdots \oplus \mathfrak{g}^{\alpha_{x,n}}$$

as vector spaces.

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