18

Nonlinear progressive waves in deep water

Consider two-dimensional progressive gravity waves moving at a phase speed C on the surface of infinitely deep water. The problem is defined in a coordinate system (x, y) fixed to the waves, with the x-axis positive in the direction of wave propagation and the y-axis pointing vertically upward from the still-water level. Assume that the fluid is inviscid, incompressible, and without surface tension. Let $\phi(x, y)$ denote the velocity potential and $\zeta(x)$ the wave elevation, respectively. The fluid motion can be described by the Laplace equation

$$\nabla^2\phi(x,y) = 0 \qquad \text{for } (x,y) \in \Omega, \tag{18.1}$$

where

$$\Omega = \{(x,y) \mid -\infty < x < +\infty, -\infty < y < \zeta(x)\}.$$

The velocity potential $\phi(x, y)$ is subject to the free surface boundary conditions

$$C^2\phi_{xx} + g\phi_y + \frac{1}{2}\nabla\phi\nabla(\nabla\phi\nabla\phi) - 2C\nabla\phi\nabla\phi_x = 0 \qquad \text{at } y = \zeta(x), \tag{18.2}$$

$$\zeta(x) = \frac{1}{g}\left(C\phi_x - \frac{1}{2}\nabla\phi\nabla\phi\right) \qquad \text{at } y = \zeta(x), \tag{18.3}$$

and the bottom condition

$$\lim_{y\to-\infty}\frac{\partial\phi}{\partial y} = 0, \tag{18.4}$$

where g is the acceleration of gravity and the subscripts x and y denote partial derivatives in the respective directions.

Although the governing equation (18.1) is linear, the free surface boundary conditions (18.2) and (18.3) are nonlinear and are defined on a surface that is unknown *a priori*. This classic water-wave problem does not have a simple solution and has attracted attention from many researchers since the mid-19th century. Stokes [122] first proposed a perturbation technique for this classical problem and later obtained an analytic solution to the fifth order in wave amplitude [123, 124]. Thereafter, researchers have applied Stokes' perturbation approach and derived higher-order solutions [125, 126, 127]. Using computer, Schwartz [128] extended Stokes' perturbation expansion to obtain a solution to the 58th order. The solution is obtained in the complex plane through a mapping function. His perturbation expansion has limited convergence and the Padé technique is employed to derive the solution at the limiting wave condition $(H/L)_{max} = 0.14118$, where H is the wave height and L denotes the wavelength.

Following Schwartz [128], Longuet-Higgins [129] took the Stokes-type expansion in wave amplitude to high orders and obtained stable solutions up to the wave steepness $H/L = 0.1411$. The results show that for a given wavelength the energy and phase speed are not monotonic functions of wave steepness. Besides, Longuet-Higgins [130, 131] investigated the stability of steady gravity waves to infinitesimal disturbances and found that subharmonic modes that become unstable when the wave height reaches a certain value, may become stable and then unstable again as the wave height continues to increase. Chen and Saffman [132] found by numerical techniques that symmetrical steady gravity waves of large amplitudes have bifurcations at $H/L \approx 0.13$. Additional high-order solutions based on Stokes' perturbation approach further illustrate the nonlinear characteristics of steep gravity waves [133, 134, 135, 136].

In this chapter we apply the homotopy analysis method to solve this boundary-value problem with nonlinear conditions on an unknown surface.

18.1 Homotopy analysis solution

18.1.1 Zero-order deformation equation

The velocity potential ϕ satisfies the Laplace equation (18.1) and the bottom boundary condition (18.4). So, it is easily understood that ϕ can be expressed by the set of base functions

$$\{\exp(mky)\ \sin(nkx) \mid m \geq 1, n \geq 1\} \tag{18.5}$$

in the form:

$$\phi(x,y) = \sum_{m=1}^{+\infty}\sum_{n=1}^{+\infty} \alpha_{m,n}\ \exp(mky)\ \sin(nkx), \tag{18.6}$$

where $k = 2\pi/L$ is the wave number, $\alpha_{m,n}$ is a coefficient. This provides us with the *rule of solution expression* for the velocity potential $\phi(x,y)$. Accordingly, the wave elevation $\zeta(x)$ can be expressed by the set of base functions

$$\{\cos(m\ k\ x) \mid m \geq 0\} \tag{18.7}$$

in the form:

$$\zeta(x) = \sum_{m=0}^{+\infty} \beta_m\ \cos(mkx), \tag{18.8}$$

which provides us with the *rule of solution expression* for the wave elevation $\zeta(x)$.

Under the *rule of solution expression* denoted by (18.6), it is expedient to select the solution of the linear Airy wave theory

$$\phi_0(x, y) = A\, C_0\ \exp(ky) \sin(kx), \tag{18.9}$$

$$C_0 = \sqrt{\frac{g}{k}}, \tag{18.10}$$

as the initial guesses of the velocity potential $\phi(x, y)$ and the phase speed C, where A is a constant to be determined later. In spite of the more obvious choice from the linear solution, we choose

$$\zeta_0(x) = 0 \tag{18.11}$$

as the initial guess of the surface elevation $\zeta(x)$ to simplify the subsequent formulation and the solution procedure. Based on the two linear terms of the free surface boundary condition (18.2), we choose an auxiliary linear operator

$$\mathcal{L}\left[\Phi(x, y; q), \Lambda(q)\right] = \Lambda^2(q)\ \frac{\partial^2 \Phi(x, y; q)}{\partial x^2} + g\ \frac{\partial \Phi(x, y; q)}{\partial y}, \tag{18.12}$$

where $q \in [0, 1]$ is an embedding parameter, $\Lambda(q)$ is a real function of q, $\Phi(x, y; q)$ is a real function of x, y, and q. From the two free surface boundary conditions (18.2) and (18.3), we define two nonlinear operators

$$\begin{aligned} &\mathcal{N}\left[\Phi(x, y; q), \Lambda(q)\right] \\ &= \Lambda^2(q)\Phi_{xx}(x, y; q) + g\Phi_y(x, y; q) \\ &+ \frac{1}{2}\nabla\Phi(x, y; q)\nabla\left[\nabla\Phi(x, y; q)\nabla\Phi(x, y; q)\right] \\ &- 2\Lambda(q)\nabla\Phi(x, y; q)\nabla\Phi_x(x, y; q) \end{aligned} \tag{18.13}$$

and

$$\begin{aligned} &\mathcal{Z}\left[\Phi(x, y; q), \Lambda(q)\right] \\ &= \frac{1}{g}\left[\Lambda(q)\ \Phi_x(x, y; q) - \frac{1}{2}\nabla\Phi(x, y; q)\nabla\Phi(x, y; q)\right]. \end{aligned} \tag{18.14}$$

The homotopy analysis method is based on a continuous variation from an initial trial to the exact solution. In the water-wave problem, we construct the mappings $\phi(x, y) \to \Phi(x, y; q)$, $\zeta(x) \to \eta(x; q)$, and $C \to \Lambda(q)$ so that, as the embedding parameter q increases from 0 to 1, $\Phi(x, y; q)$, $\eta(x; q)$, and $\Lambda(q)$ vary from the initial guesses to the exact solution $\phi(x, y)$, $\zeta(x)$, and C respectively. To ensure this, based on Equations (18.1) to (18.4), we construct the zero-order deformation equation

$$\nabla^2\Phi(x, y; q) = 0 \qquad \text{for } (x, y) \in \overline{\Omega}(q), \tag{18.15}$$

subject to the boundary conditions on the unknown free surface $y = \eta(x; q)$,

$$\begin{aligned}&(1-q)\,\mathcal{L}\left[\Phi(x,y;q)-\phi_0(x,y),\Lambda(q)\right]\\&= q\,\hbar_1\;H_1(x)\;\mathcal{N}[\Phi(x,y;q),\Lambda(q)],\end{aligned} \tag{18.16}$$

$$\begin{aligned}&(1-q)\,[\eta(x;q)-\zeta_0(x)]\\&= q\,\hbar_2\;H_2(x)\;\left\{\eta(x;q)-\mathcal{Z}[\Phi(x,y;q),\Lambda(q)]\right\},\end{aligned} \tag{18.17}$$

and the boundary condition on the bottom

$$\lim_{y\to-\infty}\frac{\partial\Phi(x,y;q)}{\partial y}=0, \tag{18.18}$$

where $q \in [0,1]$ is the embedding parameter, $\hbar_1, \hbar_2$ are two nonzero auxiliary parameters, $H_1(x), H_2(x)$ are two nonzero auxiliary functions, the domain

$$\overline{\Omega}(q)=\{(x,y)\mid -\infty<x<+\infty,-\infty<y<\eta(x;q)\}$$

should preserve the connectedness as q spans the interval $[0,1]$.

When $q = 0$, the governing equation (18.15) and the boundary conditions (18.16) to (18.18) yield the initial approximation

$$\Phi(x,y;0)=\phi_0(x,y),\qquad \eta(x,0)=\zeta_0(x),\qquad \Lambda(0)=C_0, \tag{18.19}$$

where C_0 is the initial guess of the phase speed. When $q = 1$, since

$$\hbar_1\neq 0,\hbar_2\neq 0,H_1(x)\neq 0,H_2(x)\neq 0,$$

Equations (18.15) to (18.18) are equivalent to Equations (18.1) to (18.4), provided

$$\Phi(x,y;1)=\phi(x,y),\qquad \eta(x,1)=\zeta(x),\qquad \Lambda(1)=C. \tag{18.20}$$

As q increases from 0 to 1, the boundary-value problem defined by Equations (18.15) to (18.18) thus provides a continuous variation to transform the initial trial into the exact solution.

Using Taylor's theorem and Equation (18.19), we expand $\Phi(x,y;q), \eta(x;q)$, and $\Lambda(q)$ in the power series of q as follows:

$$\Phi(x,y;q)=\phi_0(x,y)+\sum_{m=1}^{+\infty}\frac{\phi_0^{[m]}(x,y)}{m!}\,q^m, \tag{18.21}$$

$$\eta(x;q)=\zeta_0(x)+\sum_{m=1}^{+\infty}\frac{\zeta_0^{[m]}(x)}{m!}\,q^m, \tag{18.22}$$

$$\Lambda(q)=C_0+\sum_{m=1}^{+\infty}\frac{C_0^{[m]}}{m!}\,q^m, \tag{18.23}$$

where

$$\phi_0^{[m]}(x,y) = \left.\frac{\partial^m \Phi(x,y;q)}{\partial q^m}\right|_{q=0}, \tag{18.24}$$

$$\zeta_0^{[m]}(x) = \left.\frac{\partial^m \eta(x;q)}{\partial q^m}\right|_{q=0}, \tag{18.25}$$

$$C_0^{[m]} = \left.\frac{d^m \Lambda(q)}{dq^m}\right|_{q=0}. \tag{18.26}$$

Note that Equations (18.16) and (18.17) contain two auxiliary parameters $\hbar_1, \hbar_2$, and two auxiliary functions $H_1(x), H_2(x)$. Assuming that all of them are correctly chosen so that the above series are convergent at $q = 1$, from (18.20) we have

$$\phi(x,y) = \phi_0(x,y) + \sum_{m=1}^{+\infty} \frac{\phi_0^{[m]}(x,y)}{m!}, \tag{18.27}$$

$$\zeta(x) = \zeta_0(x) + \sum_{m=1}^{+\infty} \frac{\zeta_0^{[m]}(x)}{m!}, \tag{18.28}$$

$$C = C_0 + \sum_{m=1}^{+\infty} \frac{C_0^{[m]}}{m!}. \tag{18.29}$$

18.1.2 High-order deformation equation

For brevity, define the vectors

$$\vec{\phi}_n = \left\{\phi_0(x,y), \phi_0^{[1]}(x,y), \phi_0^{[2]}(x,y), \cdots, \phi_0^{[n]}(x,y)\right\},$$

$$\vec{\zeta}_n = \left\{\zeta_0(x), \zeta_0^{[1]}(x), \zeta_0^{[2]}(x), \cdots, \zeta_0^{[n]}(x)\right\},$$

and

$$\vec{C}_n = \left\{C_0, C_0^{[1]}, C_0^{[2]}, \cdots, C_0^{[n]}\right\}.$$

Besides, define the so-called deformation derivatives

$$\Phi^{[m]}(x,y;q) = \frac{\partial^m \Phi(x,y;q)}{\partial q^m}, \tag{18.30}$$

$$\eta^{[m]}(x;q) = \frac{\partial^m \eta(x;q)}{\partial q^m}, \tag{18.31}$$

$$\Lambda^{[m]} = \frac{d^m \Lambda(q)}{dq^m}. \tag{18.32}$$

Differentiating Equations (18.15) and (18.18) m times with respect to q and setting $q = 0$, we have the high-order deformation equation

$$\nabla^2 \phi_0^{[m]}(x,y) = 0 \quad \text{in } (x,y) \in \Omega_0 \tag{18.33}$$

and the condition on the bottom

$$\lim_{y\to-\infty} \frac{\partial \phi_0^{[m]}(x,y)}{\partial y} = 0, \tag{18.34}$$

where

$$\Omega_0 = \{(x,y) \mid -\infty < x < +\infty, -\infty < y \le \zeta_0(x)\}.$$

It should be emphasized that the free surface boundary conditions (18.16) and (18.17) are satisfied at $y = \eta(x;q)$, which is dependent on q. Thus, it holds for $\Phi(x,y;q)$ at $y = \eta(x;q)$ that

$$\frac{D^m \Phi(x,y;q)}{Dq^m} = \left[\frac{\partial}{\partial p} + \eta^{[1]}(x;q)\frac{\partial}{\partial y}\right]^m \Phi(x,y;q), \tag{18.35}$$

where $\eta^{[1]}(x;q)$ is defined by (18.31). The differential operator D^m/Dq^m, which contains the linear term $\partial^m/\partial q^m$, is determined from a simple procedure described later in this chapter. We simply write

$$\frac{D^m \Phi(x,y;q)}{Dq^m} = \Phi^{[m]}(x,y;q) + \mathcal{R}_m[\Phi(x,y;q),\Lambda(q)], \tag{18.36}$$

where $\mathcal{R}_m$ is a nonlinear operator and $\Phi^{[m]}(x,y;q)$ is defined by (18.30). Note that for functions independent of $y = \eta(x;q)$, such as $\Lambda(q)$ and $\eta(x;q)$, we have

$$\frac{D^m \eta(x;q)}{Dq^m} = \frac{\partial^m \eta(x;q)}{\partial q^m} = \eta^{[m]}(x;q), \tag{18.37}$$

$$\frac{D^m \Lambda(q)}{Dq^m} = \frac{d^m \Lambda(q)}{dq^m} = \Lambda^{[m]}(q), \tag{18.38}$$

which are consistent with (18.31) and (18.32), respectively.

Thereafter, differentiating Equations (18.16) and (18.17) m times with respect to q and setting $q = 0$, we have the respective free surface boundary conditions defined at $y = \zeta_0(x)$ as

$$\begin{aligned}
&\sum_{i=0}^{m} \binom{m}{i} \left.\frac{D^i\left[\Lambda^2(q)\right]}{Dq^i}\right|_{q=0} \left.\frac{D^{m-i}\Phi_{xx}(x,y;q)}{Dq^{m-i}}\right|_{q=0} \\
&+g \left.\frac{D^m \Phi_y(x,y;q)}{Dq^m}\right|_{q=0} \\
&= m\ \chi_m \left.\frac{D^{m-1}\mathcal{L}\left[\Phi(x,y;q),\Lambda(q)\right]}{Dq^{m-1}}\right|_{q=0} \\
&+ m\ \hbar_1\ H_1(x) \left.\frac{D^{m-1}\mathcal{N}\left[\Phi(x,y;q),\Lambda(q)\right]}{Dq^{m-1}}\right|_{q=0}
\end{aligned} \tag{18.39}$$

and

$$\zeta_0^{[m]}(x) = m\, W_m(x, \vec{\zeta}_{m-1}, \vec{C}_{m-1}), \tag{18.40}$$

where χ_m is defined by (2.42) and

$$W_m(x, \vec{\zeta}_{m-1}, \vec{C}_{m-1}) = \chi_m\, \zeta_0^{[m-1]}(x)$$
$$+ \hbar_2\, H_2(x) \left[\zeta_0^{[m-1]}(x) - \left. \frac{D^{m-1} \mathcal{Z}\left[\Phi(x, y; q), \Lambda(q)\right]}{Dq^{m-1}} \right|_{q=0} \right]. \tag{18.41}$$

Substituting Equation (18.36) into (18.39), at $y = \zeta_0(x)$ we have

$$C_0^2\, \frac{\partial^2 \phi_0^{[m]}(x, y)}{\partial x^2} + g\, \frac{\partial \phi_0^{[m]}(x, y)}{\partial y} = S_m(x, \vec{\phi}_{m-1}, \vec{\zeta}_m, \vec{C}_m), \tag{18.42}$$

where

$$S_m(x, \vec{\phi}_{m-1}, \vec{\zeta}_m, \vec{C}_m)$$
$$= \left\{ m\, \chi_m\, \frac{D^{m-1} \mathcal{L}\left[\Phi(x, y; q), \Lambda(q)\right]}{Dq^{m-1}} \right.$$
$$+ m\, \hbar_1\, H_1(x)\, \frac{D^{m-1} \mathcal{N}\left[\Phi(x, y; q), \Lambda(q)\right]}{Dq^{m-1}}$$
$$- C_0^2\, \mathcal{R}_m\left[\Phi_{xx}(x, y; q), \Lambda(q)\right] - g\, \mathcal{R}_m\left[\Phi_y(x, y; q), \Lambda(q)\right]$$
$$\left. \left. - \sum_{i=1}^{m} \binom{m}{i} \frac{D^i\left[\Lambda^2(q)\right]}{Dq^i}\, \frac{D^{m-i}\left[\Phi_{xx}(x, y; q)\right]}{Dq^{m-i}} \right\} \right|_{q=0}. \tag{18.43}$$

Note that the resulting boundary conditions (18.40) and (18.42) are satisfied on the initial approximation of the surface elevation $\zeta_0(x)$ and the reason for choosing $\zeta_0(x) = 0$ is now evident.

The boundary-value problem at the mth-order approximation is defined by the governing equation (18.33) and the boundary conditions (18.34), (18.40), and (18.42). It is clear that the term

$$W_m(x, \vec{\zeta}_{m-1}, \vec{C}_{m-1})$$

is only dependent upon results up to the (m-1)th approximation. Thus, $\zeta_0^{[m]}(x)$ can be directly calculated from Equation (18.40). Thereafter, there exist two unknowns: $\phi_0^{[m]}(x, y)$ and $C_0^{[m]}$. However, we have only one governing equation (18.33) with the boundary conditions (18.34) and (18.42) for $\phi_0^{[m]}(x, y)$. So, the problem is not closed and an additional algebraic equation is needed to determine $C_0^{[m]}$.

Under the *rules of solution expression* denoted by (18.6) and (18.8) and from Equations (18.40) and (18.42), the auxiliary functions $H_1(x)$ and $H_2(x)$ may appear as

$$H_1(x) = \cos(n_1 k x), \qquad H_2(x) = \cos(n_2 k x),$$

where n_1, n_2 are integers. For simplicity, we choose

$$n_1 = n_2 = 0,$$

corresponding to

$$H_1(x) = H_2(x) = 1. \tag{18.44}$$

Then, under the *rules of solution expression* denoted by (18.6) and (18.8), the term $S_m(x, \vec{\phi}_{m-1}, \vec{\zeta}_m, \vec{C}_m)$ can be expressed by

$$S_m(x, \vec{\phi}_{m-1}, \vec{\zeta}_m, \vec{C}_m) = \sum_{n=1}^{m} b_{m,n}(\vec{C}_m)\ \sin(nkx) \quad \text{for } m \geq 1, \tag{18.45}$$

where $b_{m,n}(\vec{C}_m)$ is a coefficient dependent of the vector $\vec{C}_m$. Obviously, when $b_{m,1}(\vec{C}_m) \neq 0$, due to Equation (18.42), the solution $\phi_0^{[m]}(x,y)$ of the high-order deformation equations contains the secular terms, which do not conform to the *rule of solution expression* denoted by (18.6). To avoid this, we must enforce

$$b_{m,1}(\vec{C}_m) = 0 \quad \text{for } m \geq 1, \tag{18.46}$$

which provides us with one additional algebraic equation in the form

$$\alpha_m(\vec{C}_{m-1})\ C_0^{[m]} + \beta_m(\vec{C}_{m-1}) = 0,$$

where $\alpha_m(\vec{C}_{m-1})$ and $\beta_m(\vec{C}_{m-1})$ are coefficients. Using this equation, $C_0^{[m]}$ is obtained. In this way, the problem is closed and the *rule of solution existence* is satisfied.

Thereafter, it is easy to obtain the solution

$$\phi_0^{[m]}(x,y) = \sum_{n=1}^{m} a_{m,n}\ \exp(nky)\ \sin(nkx), \tag{18.47}$$

where

$$a_{m,n} = \frac{b_{m,n}(\vec{C}_m)}{(kn)g - C_0^2 (kn)^2} \qquad \text{for } 2 \leq n \leq m. \tag{18.48}$$

Note that the coefficient $a_{m,1}$ is still unknown. To relate the solution and the wave height H, we use

$$\zeta_0^{[m]}(0) - \zeta_0^{[m]}(L/2) = \begin{cases} H & \text{for } m = 1 \\ 0 & \text{for } m \geq 2. \end{cases} \tag{18.49}$$

This relationship provides a linear algebraic equation in the form

$$\gamma_m \, a_{m,1} + \delta_m = 0,$$

where γ_m and δ_m are coefficients, from which the solution of $a_{m,1}$ can be evaluated. The value of A in the initial approximation $\phi_0(x, y)$ given by (18.9) is determined from (18.40) and (18.49) as

$$A = -\left(\frac{g \, H}{2 \, \hbar_2 \, k \, C_0^2} \right). \tag{18.50}$$

The nonlinear water-wave problem is now reduced to the two linear algebraic equations for $C_0^{[m]}$ and $a_{m,1}$. The solutions of the two equations complete the expression for $\phi_0^{[m]}(x, y)$ as well as the mth-order approximation of the solution. The formulation can be easily adapted for symbolic computation. In this way we obtain the high-order approximations $\zeta_0^{[m]}(x), C_0^{[m]}$, and $\phi_0^{[m]}(x, y)$, successively, in the order $m = 1, 2, 3, \cdots$.

The operator D^m/Dq^m for $m \geq 1$ can be determined by following the procedure outlined here. The potential $\Phi(x, y; q)$ on the free surface at $y = \eta(x; q)$ can be expanded about $q = 0$ by a Taylor series to give

$$\Phi(x, y; q) = \sum_{m=0}^{+\infty} \left. \frac{D^m \Phi(x, y; q)}{Dq^m} \right|_{q=0} \left(\frac{q^m}{m!} \right). \tag{18.51}$$

Similarly, this can be expanded by a Taylor series about the free surface at $y = \eta(x; 0)$ as

$$\Phi(x, y; q) = \sum_{n=0}^{+\infty} \sum_{r=0}^{+\infty} \left. \frac{\partial^n \Phi^{[r]}(x, y; q)}{\partial y^n} \right|_{q=0} \left(\frac{q^r}{n! \; r!} \right) [\eta(x; q) - \eta(x, 0)]^n . \tag{18.52}$$

Equating the two expressions for $\Phi(x, y; q)$ and invoking (18.19) and (18.22), we obtain

$$\begin{aligned} &\sum_{m=0}^{+\infty} \left. \frac{D^m \Phi(x, y; q)}{Dq^m} \right|_{q=0} \left(\frac{q^m}{m!} \right) \\ &= \sum_{n=0}^{+\infty} \sum_{r=0}^{+\infty} \left. \frac{\partial^n \Phi^{[r]}(x, y; q)}{\partial y^n} \right|_{q=0} \left(\frac{q^r}{n! \; r!} \right) \left[\sum_{s=1}^{+\infty} \left(\frac{q^s}{s!} \right) \zeta_0^{[s]}(x) \right]^n . \end{aligned} \tag{18.53}$$

Expanding the right-hand side of the above equation and comparing the coefficients of the same power of q give the definition of the operator D^m/Dq^m for $m \geq 1$. This can be accomplished by symbolic computation. For details, the reader is referred to Liao and Cheung [50].

18.2 Result analysis

There exist two auxiliary parameters: $\hbar_1$ and $\hbar_2$. We have therefore a two-parameter family of solution expressions. For simplicity, we set

$$\hbar_1 = \hbar_2 = \hbar.$$

Physically, the phase speed C, velocity potential $\phi(x, y)$, and surface elevation $\zeta(x)$ are dependent upon the wave steepness. All of them are mathematically dependent on $\hbar$, which influences the convergence rate and region of the solution series (18.27), (18.28), and (18.29). In practice, a finite number of terms are used in the solution series. The Mth-order approximation of (18.27), (18.28), and (18.29) becomes

$$\phi(x, y) \approx \phi_0(x, y) + \sum_{m=1}^{M} \frac{\phi_0^{[m]}(x, y)}{m!}, \tag{18.54}$$

$$\zeta(x) \approx \zeta_0(x) + \sum_{m=1}^{M} \frac{\zeta_0^{[m]}(x)}{m!}, \tag{18.55}$$

$$C \approx C_0 + \sum_{m=1}^{M} \frac{C_0^{[m]}}{m!}. \tag{18.56}$$

Most researchers focus their attention on the dispersion relationship between the phase speed C and the wave height H. Schwartz [128] formulated the Mth-order approximation of the phase speed as

$$\left(\frac{C}{C_0}\right)^2 \approx \sum_{j=0}^{M} a_j \, (kH)^{2j}, \tag{18.57}$$

where a_j is a coefficient. Schwartz applied the Padé technique to improve the convergence and obtained the solution with the maximum wave steepness $(H/L)_{max} = 0.14118$. In our approach, the Mth-order approximation of the phase speed is

$$\frac{C}{C_0} \approx \sum_{j=0}^{M} b_j \, (kH)^{2j}, \tag{18.58}$$

where b_j is a coefficient. In general, for a given value of kH, the influence of $\hbar$ on the convergence of the above series can be investigated by plotting the so-called $\hbar$-curves (see page 26 and §3.5.1) of C/C_0. As long as the series of the phase speed is convergent, the corresponding series of the velocity potential $\phi(x, y)$ and wave elevation $\zeta(x)$ also converge.

The accuracy and convergency of the phase speed can be enhanced by the homotopy-Padé technique (see page 38 and §3.5.2). It is found that the $[\kappa, \kappa]$ homotopy-Padé approximant of the phase speed is expressed by

$$\frac{C}{C_0} \approx \frac{1 + \sum_{n=1}^{\kappa(\kappa+1)/2} \Gamma_{2\kappa,n}\, (kH)^{2n}}{1 + \sum_{n=1}^{\kappa(\kappa+1)/2} \Delta_{2\kappa,n}\, (kH)^{2n}}, \tag{18.59}$$

where $\Gamma_{2\kappa,j}$ and $\Delta_{2\kappa,j}$ are coefficients independent of $\hbar$. Note that the $[\kappa, \kappa]$ homotopy-Padé expression (18.59) is to $O(H^{2\kappa^2+2\kappa})$, which is considerably higher than $O(H^{2\kappa})$ achieved by the $[\kappa, \kappa]$ Padé expansion used by Schwartz [128].

Table 1 lists the dimensionless phase speed, C^2/C_0^2, computed at various levels of the homotopy-Padé approximation (18.59) and from Schwartz's perturbation solution to $O(H^{116})$ [128]. For wave steepness up to $H/L = 0.10$, the homotopy-Padé approximation converges at [6,6] and yields results identical to Schwartz's for the number of decimals considered. The computed dimensionless phase speed at this level of approximation is to $O(H^{82})$, which is lower than that considered by Schwartz. At the 20th-order approximation of the solution series, C^2 given by the [10,10] homotopy-Padé approximation is to $O(H^{220})$ and converges to slightly different results in comparison to Schwartz's for wave steepness $H/L > 0.12$. The homotopy-Padé approximation converges rapidly with the number of terms and the 20th- and 22nd-order approximations of the solution give identical or similar results over the range of wave steepness considered, indicating reasonable convergence at the 20th order and the validity of the proposed homotopy-Padé technique.

The [10,10] and [11,11] homotopy-Padé approximations of C/C_0 are compared with Longuet-Higgins' perturbation solution [129] in Table 2. The two homotopy-Padé approximations and Longuet-Higgins' results are identical for wave steepness up to $H/L = 0.121921$, whereas the [10,10] Homotopy-Padé approximation remains convergent up to $H/L = 0.137249$ for the number of decimals considered. The phase speeds computed by the various methods as the wave steepness approaches the limiting condition are compared in Figure 18.1. Both the present and Longuet-Higgins' approaches gives the maximum phase speed at the same wave steepness $H/L = 0.138712$ and demonstrate that phase speed is not a monotonic function of wave steepness. The homotopy-Padé approximations agree with Longuet-Higgins' results up to $H/L = 0.14$, but show more rapid decrease of the phase speed toward the limiting wave condition beyond that. The phase speed given by Schwartz [128] at $H/L = 0.14$ is slightly lower in comparison to the other predictions.

As observed in the previous and present studies, the physics of steep gravity waves is complicated and different approaches produce different solutions toward the limiting wave condition. The limiting wave is physically unstable and might be mathematically as well. It would be interesting to employ our approach to investigate the bifurcations of gravity waves for $H/L \approx 0.13$, found numerically by Chen and Saffman [132].

TABLE 18.1
Comparison of the $[\kappa, \kappa]$ homotopy-Padé approximation of C^2/C_0^2 with results given by Schwartz [128].

H/L	Schwartz's result	$\kappa = 6$	$\kappa = 8$	$\kappa = 10$	$\kappa = 11$
0.040	1.01592	1.01592	1.01592	1.01592	1.01592
0.070	1.04955	1.04955	1.04955	1.04955	1.04955
0.100	1.10367	1.10367	1.10367	1.10367	1.10367
0.120	1.15182	1.15190	1.15184	1.15182	1.15181
0.130	1.17820	1.17865	1.17834	1.17821	1.17821
0.135	1.18996	1.19148	1.19061	1.19003	1.19003
0.140	1.1930	1.20150	1.19833	1.19369	1.19385

Source: Kluwer Academic Publishers, *Journal of Engineering Mathematics*, vol. 45, No. 2, 2003, pp. 105-116, "Homotopy analysis of nonlinear progressive waves in deep water", Liao and Cheung, Table 1, Kluwer Academic Publishers Copyright ©2003 Kluwer Academic Publishers, with kind permission of Kluwer Academic Publishers.

TABLE 18.2
Comparison of the $[\kappa, \kappa]$ homotopy-Padé approximation of C/C_0 with result given by Longuet-Higgins [129].

H/L	Longuet-Higgins' result	$\kappa = 10$	$\kappa = 11$
0	1.00000	1.00000	1.00000
0.045266	1.01016	1.01016	1.01016
0.064351	1.02065	1.02065	1.02065
0.079187	1.03143	1.03143	1.03143
0.091809	1.04247	1.04247	1.04247
0.102959	1.05366	1.05366	1.05366
0.108093	1.05926	1.05926	1.05926
0.112962	1.06482	1.06482	1.06482
0.117572	1.07029	1.07029	1.07029
0.121921	1.07558	1.07558	1.07558
0.125993	1.08059	1.08060	1.08060
0.129760	1.08516	1.08517	1.08517
0.133178	1.08904	1.08906	1.08906
0.136178	1.09184	1.09188	1.09188
0.136723	1.09222	1.09228	1.09228
0.137249	1.09255	1.09260	1.09260
0.137755	1.09275	1.09284	1.09285
0.138242	1.09290	1.09300	1.09301
0.138712	1.09295	1.09306	1.09308
0.139170	1.09291	1.09302	1.09305
0.139610	1.09279	1.09285	1.09290
0.140060	1.09258	1.09250	1.09258
0.140530	1.09240	1.09189	1.09202
0.141100	1.09230	1.09066	1.09089

Source: Kluwer Academic Publishers, *Journal of Engineering Mathematics*, vol. 45, No. 2, 2003, pp. 105-116, "Homotopy analysis of nonlinear progressive waves in deep water", Liao and Cheung, Table 2, Kluwer Academic Publishers Copyright ©2003 Kluwer Academic Publishers, with kind permission of Kluwer Academic Publishers.

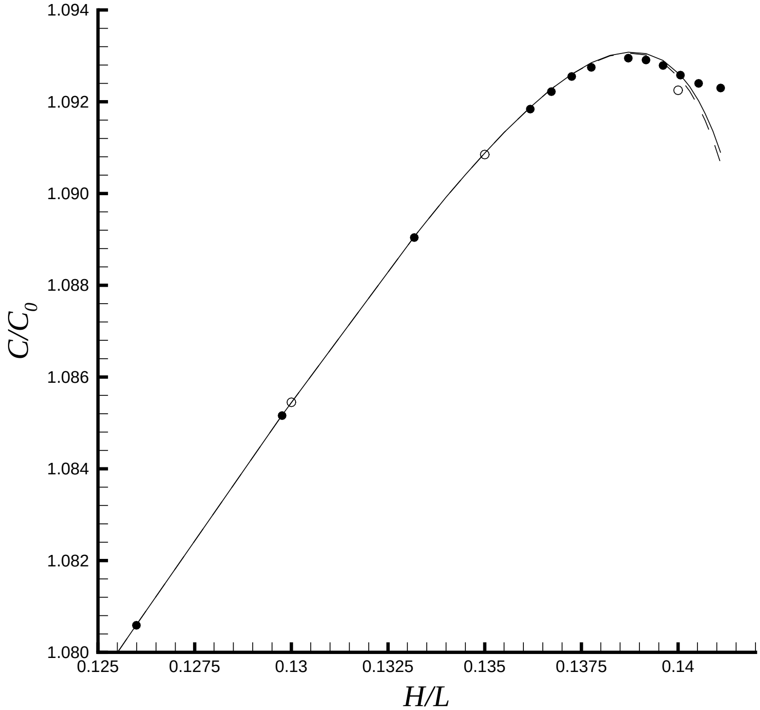

FIGURE 18.1
Phase speed C/C_0 versus wave steepness H/L for nonlinear progressive waves in deep water. Dashed line: [10,10] homotopy-Padé approximation; solid line: [11,11] homotopy-Padé approximation; open circle: Schwartz' results [128]; filled circle: Longuet-Higgins' results [129]. (From Kluwer Academic Publishers, *Journal of Engineering Mathematics*, vol. 45, No. 2, 2003, pp. 105-116, "Homotopy analysis of nonlinear progressive waves in deep water", Liao and Cheung, Figure 1, Kluwer Academic Publishers Copyright ©2003 Kluwer Academic Publishers, with kind permission of Kluwer Academic Publishers.)