17

Von Kármán swirling viscous flow

Consider the steady, laminar, axially symmetric viscous flow of an incompressible fluid induced by an infinite disk rotating steadily with angular velocity Ω about the z-axis in a cylindrical coordinate system (r, θ, z). The motion of the fluid is governed by the continuity equation

$$\frac{1}{r}\frac{\partial (rV_r)}{\partial r} + \frac{1}{r}\frac{\partial V_\theta}{\partial \theta} + \frac{\partial V_z}{\partial z} = 0 \tag{17.1}$$

and the Navier-Stokes equations

$$V_r \frac{\partial V_r}{\partial r} + V_z \frac{\partial V_r}{\partial z} - \frac{V_\theta^2}{r} = \nu \left[\frac{\partial^2 V_r}{\partial r^2} + \frac{1}{r}\frac{\partial V_r}{\partial r} + \frac{\partial^2 V_r}{\partial z^2} - \frac{V_r}{r^2} \right] - \frac{1}{\rho}\frac{\partial p}{\partial r}, \tag{17.2}$$

$$V_r \frac{\partial V_\theta}{\partial r} + V_z \frac{\partial V_\theta}{\partial z} + \frac{V_r V_\theta}{r} = \nu \left[\frac{\partial^2 V_\theta}{\partial r^2} + \frac{1}{r}\frac{\partial V_\theta}{\partial r} + \frac{\partial^2 V_\theta}{\partial z^2} - \frac{V_\theta}{r^2} \right], \tag{17.3}$$

$$V_r \frac{\partial V_z}{\partial r} + V_z \frac{\partial V_z}{\partial z} = \nu \left[\frac{\partial^2 V_z}{\partial r^2} + \frac{1}{r}\frac{\partial V_z}{\partial r} + \frac{\partial^2 V_z}{\partial z^2} \right] - \frac{1}{\rho}\frac{\partial p}{\partial z}, \tag{17.4}$$

subject to the nonslip boundary conditions

$$V_\theta = r\Omega, \quad V_r = V_z = 0, \qquad \text{when } z = 0, \tag{17.5}$$

and the conditions at infinity

$$V_r = V_\theta = 0, \qquad \text{when } z = +\infty, \tag{17.6}$$

where ρ denotes the fluid density, ν the kinematic viscosity coefficient, p the pressure, and V_r, V_θ, V_z the velocity components in the radial, azimuthal, and axial directions, respectively. Defining the similarity variable

$$\eta = z\sqrt{\frac{\Omega}{\nu}} \tag{17.7}$$

and using the similarity transformation

$$V_r = (r\Omega)\ f(\eta), \tag{17.8}$$

$$V_\theta = (r\Omega)\ g(\eta), \tag{17.9}$$

$$V_z = \sqrt{\nu\Omega}\ w(\eta), \tag{17.10}$$

$$p = -\rho\nu\Omega\ P(\eta), \tag{17.11}$$

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Von Kármán [112] devised the governing partial differential equations (17.1) to (17.6) to a set of ordinary differential equations

$$f'' = f^2 - g^2 + f'\ w, \tag{17.12}$$

$$g'' = g'\ w + 2f\ g, \tag{17.13}$$

$$w\ w' = P' + w'', \tag{17.14}$$

$$2f + w' = 0, \tag{17.15}$$

subject to the boundary conditions

$$f(0) = f(+\infty) = 0,\ g(0) = 1,\ g(+\infty) = 0,\ w(0) = 0, \tag{17.16}$$

where the prime denotes the derivative with respect to η. From (17.15),

$$f = -\frac{w'}{2}. \tag{17.17}$$

Substituting it into Equations (17.12) and (17.13), we have

$$w''' - w''\ w + \frac{1}{2}w'\ w' - 2g^2 = 0, \tag{17.18}$$

$$g'' - w\ g' + w'\ g = 0, \tag{17.19}$$

subject to the boundary conditions

$$w(0) = w'(0) = w'(+\infty) = 0,\ g(0) = 1,\ g(+\infty) = 0. \tag{17.20}$$

For details the reader is referred to Von Kármán [112] and Zandbergen and Dijkstra [113].

The above equations are coupled and strongly nonlinear. They were investigated by many researchers including Von Kármán [112], Cochran [114], Fettis [115], Rogers and Lance [116], Benton [117], McLeod [118], Zandbergen and Dijkstra [119], Ackroyd [120], and Hulzen [121]. These solutions are either numerical or numerical-analytic. In this chapter the homotopy analysis method is employed to give a purely analytic solution.

17.1 Homotopy analysis solution

Note that the velocity component at infinity in the axial direction is unknown, which has significant physical meaning. So, as Benton [117] did, we define

$$\gamma = -w(+\infty) \tag{17.21}$$

and use the transformation

$$w(\eta) = -\gamma\,[1 - s(\eta)]\,. \tag{17.22}$$

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However, unlike Benton [117], we introduce another transformation

$$\xi = \lambda\, \eta, \tag{17.23}$$

where λ is the so-called spatial-scale parameter. Using (17.22) and (17.23), Equations (17.18) and (17.19) become

$$\gamma\lambda^3 s''' + \gamma^2\lambda^2(1-s)s'' + \frac{1}{2}\gamma^2\lambda^2 s's' - 2g^2 = 0, \tag{17.24}$$

$$\lambda g'' + \gamma(1-s)g' + \gamma s' g = 0, \tag{17.25}$$

subject to the boundary conditions

$$s(0) = g(0) = 1,\ s(+\infty) = g(+\infty) = 0,\ s'(0) = s'(+\infty) = 0, \tag{17.26}$$

where the prime denotes differentiation with respect to ξ. Note that we have the freedom to choose the value of the spatial-scale parameter λ, but γ defined by (17.21) is unknown.

17.1.1 Zero-order deformation equation

According to Rogers and Lance's investigation [116], Von Kármán's swirling flow has exponential property at infinity. In 1978, Dijkstra gave a full asymptotic expansion that contains only exponentials. Hulzen [121] computed the series that consists of exponentials multiplied by polynomials. Recently, Yang and Liao [43] applied the homotopy analysis method to give, for the first time, an explicit, purely analytic solution expressed by exponentials multiplied by polynomials. Here, we give the solutions $s(\xi)$ and $g(\xi)$ expressed by the set of base functions

$$\{\exp(-n\,\xi) \mid n \geq 1\} \tag{17.27}$$

in the forms:

$$s(\xi) = \sum_{n=1}^{+\infty} a_n \exp(-n\xi), \qquad g(\xi) = \sum_{n=1}^{+\infty} b_n \exp(-n\xi), \tag{17.28}$$

where a_n and b_n are coefficients. The above expressions provide the so-called *rules of solution expression* for $s(\xi)$ and $g(\xi)$, respectively.

Let ϵ denote an auxiliary parameter. Under the *rules of solution expression* denoted by (17.28) and using (17.26), it is easy to choose the initial guesses

$$s_0(\xi) = 2\exp(-\xi) - \exp(-2\xi), \tag{17.29}$$

$$g_0(\xi) = \exp(-\xi) + \epsilon\,[\exp(-2\xi) - \exp(-\xi)] \tag{17.30}$$

of $s(\xi)$ and $g(\xi)$, respectively. Under the *rules of solution expression* denoted by (17.28) and from Equations (17.24) and (17.25), we choose the auxiliary

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linear operators

$$\mathcal{L}_s f = \frac{\partial^3 f}{\partial \xi^3} + 2\,\frac{\partial^2 f}{\partial \xi^2} - \frac{\partial f}{\partial \xi} - 2f, \tag{17.31}$$

$$\mathcal{L}_g f = \frac{\partial^2 f}{\partial \xi^2} - f \tag{17.32}$$

with the properties

$$\mathcal{L}_s\left[C_1 \exp(\xi) + C_2 \exp(-\xi) + C_3 \exp(-2\xi)\right] = 0 \tag{17.33}$$

and

$$\mathcal{L}_g\left[C_1 \exp(\xi) + C_2 \exp(-\xi)\right] = 0, \tag{17.34}$$

where C_1, C_2, and C_3 are coefficients. For conciseness, from Equations (17.24) and (17.25) we define the two nonlinear operators

$$\begin{aligned} &\mathcal{N}_s\left[S(\xi;q), G(\xi;q), \Lambda(q), \Gamma(q)\right] \\ &= \Gamma(q)\Lambda^3(q)\frac{\partial^3 S(\xi;q)}{\partial \xi^3} + \Gamma^2(q)\Lambda^2(q)\left[1 - S(\xi;q)\right]\frac{\partial^2 S(\xi;q)}{\partial \xi^2} \\ &+ \left(\frac{1}{2}\right)\Gamma^2(q)\Lambda^2(q)\left[\frac{\partial S(\xi;q)}{\partial \xi}\right]^2 - 2G(\xi;q)^2 \end{aligned} \tag{17.35}$$

and

$$\begin{aligned} &\mathcal{N}_g\left[S(\xi;q), G(\xi;q), \Lambda(q), \Gamma(q)\right] \\ &= \Lambda(q)\frac{\partial^2 G(\xi;q)}{\partial \xi^2} + \Gamma(q)\left[1 - S(\xi;q)\right]\frac{\partial G(\xi;q)}{\partial \xi} \\ &+ \Gamma(q)G(\xi;q)\frac{\partial S(\xi;q)}{\partial \xi}, \end{aligned} \tag{17.36}$$

where $q \in [0,1]$ is an embedding parameter, $S(\xi;q)$ and $G(\xi;q)$ are real functions of ξ and q, $\Lambda(q)$ and $\Gamma(q)$ are real functions of q, respectively. Let $\hbar_g$ and $\hbar_s$ denote two nonzero auxiliary parameters, $H_s(\xi)$ and $H_g(\xi)$ two nonzero auxiliary functions, $q \in [0,1]$ the embedding parameter, respectively. We construct the zero-order deformation equations

$$\begin{aligned} &(1-q)\ \ \mathcal{L}_s\left[S(\xi;q) - s_0(\xi)\right] \\ &= q\,\hbar_s\ \ H_s(\xi)\ \ \mathcal{N}_s[S(\xi;q), G(\xi;q), \Lambda(q), \Gamma(q)], \end{aligned} \tag{17.37}$$

$$\begin{aligned} &(1-q)\ \mathcal{L}_g\left[G(\xi;q) - g_0(\xi)\right] \\ &= q\,\hbar_g\ H_g(\xi)\ \ \mathcal{N}_g[S(\xi;q), G(\xi;q), \Lambda(q), \Gamma(q)], \end{aligned} \tag{17.38}$$

subject to the boundary conditions

$$S(0;q) = 1,\ S(+\infty;q) = 0,\ \left.\frac{\partial S(\xi;q)}{\partial \xi}\right|_{\xi=0} = \left.\frac{\partial S(\xi;q)}{\partial \xi}\right|_{\xi=+\infty} = 0 \tag{17.39}$$

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and

$$G(0;q) = 1,\ G(+\infty;q) = 0. \tag{17.40}$$

When $q = 0$, it is clear from (17.29), (17.30), and Equations (17.37) to (17.40) that

$$S(\xi;0) = s_0(\xi),\ \ G(\xi;0) = g_0(\xi). \tag{17.41}$$

When $q = 1$, since $\hbar_s \neq 0, \hbar_g \neq 0, H_s(\xi) \neq 0$ and $H_g(\xi) \neq 0$, Equations (17.37) to (17.40) are equivalent to the original equations (17.24) to (17.26), provided

$$S(\xi;1) = s(\xi),\ \ G(\xi;1) = g(\xi),\ \ \Lambda(1) = \lambda,\ \ \Gamma(1) = \gamma. \tag{17.42}$$

By Taylor's theorem and using (17.41), we have the power series in the expansion of q as follows:

$$S(\xi;q) = s_0(\xi) + \sum_{n=1}^{+\infty} s_n(\xi)\ q^n, \tag{17.43}$$

$$G(\xi;q) = g_0(\xi) + \sum_{n=1}^{+\infty} g_n(\xi)\ q^n, \tag{17.44}$$

$$\Lambda(q) = \lambda_0 + \sum_{n=1}^{+\infty} \lambda_n\ q^n, \tag{17.45}$$

$$\Gamma(q) = \gamma_0 + \sum_{n=1}^{+\infty} \gamma_n\ q^n, \tag{17.46}$$

where λ_0 and γ_0 are initial guesses of λ and γ, and

$$s_n(\xi) = \frac{1}{n!}\left.\frac{\partial^n S(\xi;q)}{\partial q^n}\right|_{q=0}, \tag{17.47}$$

$$g_n(\xi) = \frac{1}{n!}\left.\frac{\partial^n G(\xi;q)}{\partial q^n}\right|_{q=0}, \tag{17.48}$$

$$\lambda_n = \frac{1}{n!}\left.\frac{\partial^n \Lambda(q)}{\partial q^n}\right|_{q=0}, \tag{17.49}$$

$$\gamma_n = \frac{1}{n!}\left.\frac{\partial^n \Gamma(q)}{\partial q^n}\right|_{q=0}. \tag{17.50}$$

Note that Equations (17.37) and (17.38) contain two auxiliary parameters $\hbar_s$ and $\hbar_g$, and two auxiliary functions $H_s(\xi)$ and $H_g(\xi)$. Assuming that all of them are correctly chosen so that the above series are convergent at $q = 1$, we have, using (17.42), the solution series

$$s(\xi) = s_0(\xi) + \sum_{n=1}^{+\infty} s_n(\xi), \tag{17.51}$$

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$$g(\xi) = g_0(\xi) + \sum_{n=1}^{+\infty} g_n(\xi), \tag{17.52}$$

$$\lambda = \lambda_0 + \sum_{n=1}^{+\infty} \lambda_n, \tag{17.53}$$

$$\gamma = \gamma_0 + \sum_{n=1}^{+\infty} \gamma_n. \tag{17.54}$$

17.1.2 High-order deformation equation

For conciseness, define the vectors

$$\vec{s}_k = \{s_0(\xi), s_1(\xi), s_2(\xi), \cdots, s_k(\xi)\},$$

$$\vec{g}_k = \{g_0(\xi), g_1(\xi), g_2(\xi), \cdots, g_k(\xi)\},$$

and

$$\vec{\lambda}_k = \{\lambda_0, \lambda_1, \lambda_2, \cdots, \lambda_k\}, \qquad \vec{\gamma}_k = \{\gamma_0, \gamma_1, \gamma_2, \cdots, \gamma_k\}.$$

Differentiating the zero-order deformation equations (17.37) to (17.40) n times with respect to q, then dividing by $n!$, and finally setting $q = 0$, we have the high-order deformation equations

$$\mathcal{L}_s\left[s_n(\xi) - \chi_n\ s_{n-1}(\xi)\right] = \hbar_s\ H_s(\xi)\ R_n^s(\vec{s}_{n-1}, \vec{g}_{n-1}, \vec{\lambda}_{n-1}, \vec{\gamma}_{n-1}), \tag{17.55}$$

$$\mathcal{L}_g\left[g_n(\xi) - \chi_n\ g_{n-1}(\xi)\right] = \hbar_g\ H_g(\xi)\ R_n^g(\vec{s}_{n-1}, \vec{g}_{n-1}, \vec{\lambda}_{n-1}, \vec{\gamma}_{n-1}), \tag{17.56}$$

subject to the boundary conditions

$$s_n(0) = g_n(0) = s_n(+\infty) = g_n(+\infty) = 0, \quad s_n'(0) = s_n'(+\infty) = 0, \tag{17.57}$$

where χ_n is defined by (2.42),

$$\begin{aligned} & R_n^s(\vec{s}_{n-1}, \vec{g}_{n-1}, \vec{\lambda}_{n-1}, \vec{\gamma}_{n-1}) \\ =& \frac{1}{(n-1)!}\ \frac{\partial^{n-1}\mathcal{N}_s\left[S(\xi;q), G(\xi;q), \Lambda(q), \Gamma(q)\right]}{\partial q^{n-1}}\bigg|_{q=0} \\ =& \sum_{k=0}^{n-1}\left[\alpha_{n-1-k}\ s_k'''(\xi) + \beta_{n-1-k}\ s_k''(\xi)\right] \\ & -\sum_{k=0}^{n-1}\beta_{n-1-k}\left[\sum_{j=0}^{k} s_j(\xi)\ s_{k-j}''(\xi)\right] \\ & +\frac{1}{2}\sum_{k=0}^{n-1}\beta_{n-1-k}\left[\sum_{j=0}^{k} s_j'(\xi)\ s_{k-j}'(\xi)\right] \\ & -2\sum_{k=0}^{n-1} g_{n-1-k}(\xi)\ g_k(\xi), \end{aligned} \tag{17.58}$$

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and

$$
\begin{aligned}
& R_n^g(\vec{s}_{n-1}, \vec{g}_{n-1}, \vec{\lambda}_{n-1}, \vec{\gamma}_{n-1}) \\
= & \frac{1}{(n-1)!} \left. \frac{\partial^{n-1} \mathcal{N}_g \left[S(\xi;q), G(\xi;q), \Lambda(q), \Gamma(q) \right]}{\partial q^{n-1}} \right|_{q=0} \\
= & \sum_{k=0}^{n-1} \left[\lambda_{n-1-k} \; g_k''(\xi) + \gamma_{n-1-k} \; g_k'(\xi) \right] \\
& + \sum_{k=0}^{n-1} \gamma_{n-1-k} \sum_{j=0}^{k} \left[s_j'(\xi) \; g_{k-j}(\xi) - s_j(\xi) \; g_{k-j}'(\xi) \right]
\end{aligned}
\tag{17.59}
$$

under the definitions

$$
\alpha_n = \sum_{k=0}^{n} \lambda_{n-k} \; \delta_k, \tag{17.60}
$$

$$
\beta_n = \sum_{k=0}^{n} \gamma_{n-k} \; \delta_k, \tag{17.61}
$$

$$
\delta_n = \sum_{k=0}^{n} \gamma_{n-k} \sum_{j=0}^{k} \lambda_j \; \lambda_{k-j}. \tag{17.62}
$$

Note that the high-order deformation equations (17.55) and (17.56) are linear and uncoupled, subject to the linear boundary conditions (17.57). It is easy to successively solve them, using symbolic computation software.

Note that there exist four unknowns: $s_n(\xi)$, $g_n(\xi)$, λ_{n-1}, and γ_{n-1}. However, we have only two differential equations (17.55) and (17.56) for $s_n(\xi)$ and $g_n(\xi)$. Thus, the problem is not closed and two additional algebraic equations are needed to determine λ_{n-1} and γ_{n-1}. Under the *rules of solution expression* denoted by (17.28) and from Equations (17.55) and (17.56), the auxiliary functions should be

$$
H_s(\xi) = \exp(\kappa_s \; \xi), \qquad H_g(\xi) = \exp(\kappa_g \; \xi), \tag{17.63}
$$

where κ_s and κ_g are integers. Using (17.29) and (17.30), we have

$$
R_1^s(\vec{s}_0, \vec{g}_0, \vec{\lambda}_0, \vec{\gamma}_0) = \sum_{k=1}^{4} c_{1,k}(\lambda_0, \gamma_0) \; \exp(-k\xi), \tag{17.64}
$$

$$
R_1^g(\vec{s}_0, \vec{g}_0, \vec{\lambda}_0, \vec{\gamma}_0) = \sum_{k=1}^{3} d_{1,k}(\lambda_0, \gamma_0) \; \exp(-k\xi), \tag{17.65}
$$

where $c_{1,k}(\lambda_0, \gamma_0)$ and $d_{1,k}(\lambda_0, \gamma_0)$ are coefficients independent of ξ. There are two ways to solve the problem. One is to enforce the coefficients $c_{1,1}(\lambda_0, \gamma_0)$ and $d_{1,1}(\lambda_0, \gamma_0)$ to be zero, which gives

$$
2\gamma_0(\gamma_0 - \lambda_0)\lambda_0^2 = 0, \qquad \gamma_0 - \lambda_0 = 0. \tag{17.66}
$$

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The above set of algebraic equations has an infinite number of solutions

$$\gamma_0 = \lambda_0. \tag{17.67}$$

It implies that $d_{1,1}(\lambda_0, \gamma_0) = 0$ is true as long as $c_{1,1}(\lambda_0, \gamma_0) = 0$. Thus, it does not work. The other is to make the coefficients $c_{1,1}(\lambda_0, \gamma_0)$ and $c_{1,2}(\lambda_0, \gamma_0)$ zero, which gives

$$2\gamma_0(\gamma_0 - \lambda_0)\lambda_0^2 = 0, \qquad (1-\epsilon)^2 + 3\gamma_0^2\lambda_0^2 - 4\gamma_0\lambda_0^3 = 0. \tag{17.68}$$

This set of algebraic equations has the unique nonzero solution

$$\gamma_0 = \sqrt{|1-\epsilon|}, \qquad \lambda_0 = \sqrt{|1-\epsilon|}. \tag{17.69}$$

In this way, the problem is closed and the *rule of solution existence* is satisfied. In this case, if $\kappa_s > 0$, the right-hand side of Equation (17.55) contains the term $\exp(-2\xi)$ and/or $\exp(-\xi)$. Thus, according to the property (17.33), $s_1(\xi)$ contains the term $\xi\ \exp(-2\xi)$ and/or $\xi\ \exp(-\xi)$, which however does not conform to the *rules of solution expression* denoted by (17.28). So, we have

$$\kappa_s \leq 0.$$

On the other hand, when $\kappa_s \leq -1$, the coefficient of the term $\exp(-3\xi)$ of $s(\xi)$ cannot be modified; and this however is not in accordance with the *rule of coefficient ergodicity*. So, to ensure that both of the *rules of solution expression* denoted by (17.28) and the *rule of coefficient ergodicity* hold, we must choose

$$\kappa_s = 0, \tag{17.70}$$

corresponding to $H_s(\xi) = 1$. Similarly, we have

$$\kappa_g = 0, \tag{17.71}$$

which gives $H_g(\xi) = 1$.

In summary, under the *rules of solution expression* denoted by (17.28) and the *rule of coefficient ergodicity*, and in order to work out the high-order deformation equations, we choose the auxiliary functions

$$H_s(\xi) = H_g(\xi) = 1 \tag{17.72}$$

and solve the set of two algebraic equations

$$c_{n,1}(\vec{\lambda}_{n-1}, \vec{\gamma}_{n-1}) = 0, \qquad c_{n,2}(\vec{\lambda}_{n-1}, \vec{\gamma}_{n-1}) = 0 \tag{17.73}$$

to obtain λ_{n-1} and γ_{n-1}, where $c_{n,1}(\vec{\lambda}_{n-1}, \vec{\gamma}_{n-1})$ and $c_{n,2}(\vec{\lambda}_{n-1}, \vec{\gamma}_{n-1})$ are coefficients of the terms

$$R_n^s(\vec{s}_{n-1}, \vec{g}_{n-1}, \vec{\lambda}_{n-1}, \vec{\gamma}_{n-1}) = \sum_{k=1}^{2n+2} c_{n,k}(\vec{\lambda}_{n-1}, \vec{\gamma}_{n-1})\ \exp(-k\xi), \tag{17.74}$$

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$$R_n^g(\vec{s}_{n-1}, \vec{g}_{n-1}, \vec{\lambda}_{n-1}, \vec{\gamma}_{n-1}) = \sum_{k=1}^{2n+2} d_{n,k}(\vec{\lambda}_{n-1}, \vec{\gamma}_{n-1}) \ \exp(-k\xi). \qquad (17.75)$$

Thus, we can successively solve the linear differential equations (17.55) and (17.56) under the linear boundary conditions (17.57), together with the set of two algebraic equations (17.73) that are linear when $n \geq 2$. Then,

$$s_n(\xi) = \sum_{k=1}^{2n+2} a_{n,k} \ \exp(-k\xi), \qquad (17.76)$$

$$g_n(\xi) = \sum_{k=1}^{2n+2} b_{n,k} \ \exp(-k\xi), \qquad (17.77)$$

where $a_{n,k}$ and $b_{n,k}$ are coefficients. The recursive formulae for the coefficients $a_{n,k}$ and $b_{n,k}$ may be obtained if the above expressions are substituted into Equations (17.55) to (17.57).

17.1.3 Convergence theorem

THEOREM 17.1
If the solution series (17.51), (17.52), (17.53), and (17.54) are convergent, where $s_n(\xi)$ and $g_n(\xi)$ are governed by (17.55), (17.56), and (17.57) under the definitions (17.31), (17.32), (17.58), (17.59), and (2.42), they must be the solution of Equations (17.24) and (17.25) under the boundary conditions (17.26) .

Proof: If the series (17.51) and (17.52) are convergent, it is necessary that

$$\lim_{m\to+\infty} s_m(\xi) = 0, \qquad \lim_{m\to+\infty} g_m(\xi) = 0. \qquad (17.78)$$

Then, using (17.31), (17.32), (2.42) and from Equations (17.55) and (17.56), we have

$$\hbar_s \ H_s(\xi) \ \sum_{n=1}^{+\infty} R_n^s(\vec{s}_{n-1}, \vec{g}_{n-1}, \vec{\lambda}_{n-1}, \vec{\gamma}_{n-1})$$
$$= \lim_{m\to+\infty} \mathcal{L}_s\left[s_m(\xi)\right] = \mathcal{L}_s\left[\lim_{m\to+\infty} s_m(\xi)\right] = 0 \qquad (17.79)$$

and

$$\hbar_g \ H_g(\xi) \ \sum_{n=1}^{+\infty} R_n^g(\vec{s}_{n-1}, \vec{g}_{n-1}, \vec{\lambda}_{n-1}, \vec{\gamma}_{n-1})$$
$$= \lim_{m\to+\infty} \mathcal{L}_g\left[g_m(\xi)\right] = \mathcal{L}_g\left[\lim_{m\to+\infty} g_m(\xi)\right] = 0. \qquad (17.80)$$

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Since $\hbar_s \neq 0, \hbar_g \neq 0, H_s(\xi) \neq 0$ and $H_g(\xi) \neq 0$, the above equations yield

$$\sum_{n=1}^{+\infty} R_n^s(\vec{s}_{n-1}, \vec{g}_{n-1}, \vec{\lambda}_{n-1}, \vec{\gamma}_{n-1}) = 0 \tag{17.81}$$

and

$$\sum_{n=1}^{+\infty} R_n^g(\vec{s}_{n-1}, \vec{g}_{n-1}, \vec{\lambda}_{n-1}, \vec{\gamma}_{n-1}) = 0. \tag{17.82}$$

Substituting the definitions (17.58) and (17.59) into the above expressions and then simplifying them, due to the convergence of the series (17.51) to (17.54), we have

$$\begin{aligned} &\left(\sum_{i=0}^{+\infty} \gamma_i\right)\left(\sum_{j=0}^{+\infty} \lambda_j\right)^3 \frac{d^3}{d\xi^3}\left[\sum_{k=0}^{+\infty} s_k(\xi)\right] \\ &+ \left(\sum_{i=0}^{+\infty} \gamma_i\right)^2 \left(\sum_{j=0}^{+\infty} \lambda_j\right)^2 \left[1 - \sum_{k=0}^{+\infty} s_k(\xi)\right] \frac{d^2}{d\xi^2}\left[\sum_{k=0}^{+\infty} s_k(\xi)\right] \\ &+ \frac{1}{2}\left(\sum_{i=0}^{+\infty} \gamma_i\right)^2 \left(\sum_{j=0}^{+\infty} \lambda_j\right)^2 \frac{d}{d\xi}\left[\sum_{k=0}^{+\infty} s_k(\xi)\right] \frac{d}{d\xi}\left[\sum_{k=0}^{+\infty} s_k(\xi)\right] \\ &- 2\left[\sum_{k=0}^{+\infty} g_k(\xi)\right]^2 = 0 \end{aligned} \tag{17.83}$$

and

$$\begin{aligned} &\left(\sum_{j=0}^{+\infty} \lambda_j\right) \frac{d^2}{d\xi^2}\left[\sum_{k=0}^{+\infty} g_k(\xi)\right] + \left(\sum_{i=0}^{+\infty} \gamma_i\right)\left(1 - \sum_{j=0}^{+\infty} s_j(\xi)\right) \frac{d}{d\xi}\left[\sum_{k=0}^{+\infty} g_k(\xi)\right] \\ &+ \left(\sum_{i=0}^{+\infty} \gamma_i\right)\left[\sum_{j=0}^{+\infty} g_j(\xi)\right] \frac{d}{d\xi}\left[\sum_{k=0}^{+\infty} s_k(\xi)\right] = 0. \end{aligned} \tag{17.84}$$

Furthermore, using (17.29), (17.30), and (17.57), we have

$$\sum_{n=0}^{+\infty} s_n(0) = \sum_{n=0}^{+\infty} g_n(0) = 1, \qquad \sum_{n=0}^{+\infty} s_n(+\infty) = \sum_{n=0}^{+\infty} g_n(+\infty) = 0, \tag{17.85}$$

and

$$\sum_{n=0}^{+\infty} s'_n(0) = \sum_{n=0}^{+\infty} s'_n(+\infty) = 0. \tag{17.86}$$

Comparing the above four expressions with Equations (17.24), (17.25), and (17.26), the convergent series (17.51), (17.52), (17.53), and (17.54) must be the solution of Von Kármán's swirling flow. This ends the proof.

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17.2 Result analysis

According to Theorem 17.1, we need only to focus on ensuring that the solution series (17.51) to (17.54) are convergent. There exist three auxiliary parameters: ϵ, $\hbar_s$, and $\hbar_g$. We have therefore a three-parameter family of solution expressions. For simplicity, consider the case of

$$\hbar_s = \hbar_g = \hbar$$

and investigate first the influence of ϵ and $\hbar$ on the convergence of $\gamma = -w(+\infty)$ that has a clear physical meaning.

Obviously, for any chosen value of ϵ, γ is a power series of $\hbar$; thus we can investigate the influence of $\hbar$ on the convergence of γ by plotting the so-called $\hbar$-curves (see page 26 and §3.5.1) of γ, as shown in Figure 17.1. From these $\hbar$-curves, it is clear that the series of γ converges when $\epsilon = 0$ and $-1/5 \leq \hbar < 0$, or $\epsilon = 1/4$ and $-3/5 \leq \hbar < 0$. For example, when $\epsilon = 0$ and $\hbar_s = \hbar_g = -1/5$ or $\epsilon = 1/4$ and $\hbar_s = \hbar_g = -1/2$, the corresponding series of γ converges to Benton's [117] numerical result, as shown in Table 17.1. When $\epsilon = 1/4$ and $-3/5 \leq \hbar < 0$ the series converges faster than when $\epsilon = 0$ and $-1/5 \leq \hbar < 0$, indicating that the auxiliary parameters ϵ and $\hbar = \hbar_s = \hbar_g$ may control the convergence rate of the solution series. The homotopy-Padé technique (see page 38 and §3.5.2) can be employed to accelerate the convergence, as shown in Table 17.2.

From Figure 17.1, as ϵ increases from 0 to 0.5, the valid region of $\hbar$ enlarges but then reduces, indicating that there should exist a value of ϵ that corresponds to the largest valid region of $\hbar$. To obtain this value, we choose ϵ so that $\gamma_1 = \lambda_1 = 0$, corresponding to

$$\sqrt{1-\epsilon}\left(119 - 328\epsilon + 193\epsilon^2\right) - 5(3 + 19\epsilon) = 0,$$

whose solution is

$$\epsilon \approx 0.26167. \tag{17.87}$$

The valid region of $\hbar$ corresponding to $\epsilon = 0.26167$ seems to be longest, as shown in Figure 17.1. Moreover, when $\epsilon = 0.26167$ and $\hbar_s = \hbar_g = -1/2$, the series of γ converges even faster, as shown in Tables 17.1 and 17.2.

It is found that

$$\lambda_n = \gamma_n$$

for any integers $n \geq 0$. Thus, we have the relationship

$$\lambda = \gamma,$$

which agrees with the asymptotic expansion for large ξ given by Cochran [114]. Indeed, Von Kármán's swirling flow contains an elegant mathematical structure.

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As long as the series of γ is convergent, the corresponding series of $s(\xi)$ and $g(\xi)$ also converge in the whole region $0 \le \xi < +\infty$. For example, when $\epsilon = 0$ and $\hbar_s = \hbar_g = -1/5$, the analytic approximations of $w(\eta)$ and $g(\eta)$ converge to Benton's [117] numerical result, as shown in Figures 17.2 and 17.3. Moreover, when $\epsilon = 1/4$ and $\hbar_g = \hbar_s = -1/2$, even the [1,1] homotopy-Páde approximants of $g(\eta)$ and $w(\eta)$, i.e.

$$g(\eta) \approx \frac{\Delta_1(\eta)}{\Pi_1(\eta)}, \qquad w(\eta) \approx -\gamma \left[\frac{\Delta_2(\eta)}{\Pi_2(\eta)}\right], \tag{17.88}$$

agree with Benton's numerical result [117], as shown in Figures 17.4 and 17.5, where

$$\begin{aligned}
\Delta_1(\eta) &= \left(935649 + 3881640\sqrt{3}\right)\exp(-\gamma\eta) \\
&+ \left(456252 + 2097200\sqrt{3}\right)\exp(-2\gamma\eta) \\
&+ \left(785007 - 311640\sqrt{3}\right)\exp(-3\gamma\eta) \\
&+ \left(220464 - 317520\sqrt{3}\right)\exp(-4\gamma\eta) \\
&- \left(22212 + 25200\sqrt{3}\right)\exp(-5\gamma\eta) - 4608\exp(-6\gamma\eta), \\
\Pi_1(\eta) &= \left(1247532 + 3022880\sqrt{3}\right) \\
&+ \left(192492 + 1858080\sqrt{3}\right)\exp(-\gamma\eta) \\
&+ \left(982512 + 544320\sqrt{3}\right)\exp(-2\gamma\eta) \\
&- \left(33552 + 100800\sqrt{3}\right)\exp(-3\gamma\eta) - 18432\exp(-4\gamma\eta), \\
\Delta_2(\eta) &= \left(1364904 - 477008\sqrt{3}\right) \\
&- \left(2855992 - 962752\sqrt{3}\right)\exp(-\gamma\eta) \\
&+ \left(1612135 - 497280\sqrt{3}\right)\exp(-2\gamma\eta) \\
&- \left(115206 - 14336\sqrt{3}\right)\exp(-3\gamma\eta) \\
&- \left(6545 + 2800\sqrt{3}\right)\exp(-4\gamma\eta) + 704\exp(-5\gamma\eta), \\
\Pi_2(\eta) &= \left(1364904 - 477008\sqrt{3}\right) \\
&+ \left(28446 + 8736\sqrt{3}\right)\exp(-\gamma\eta) \\
&- \left(57777 + 2800\sqrt{3}\right)\exp(-2\gamma\eta) + 5184\exp(-3\gamma\eta),
\end{aligned}$$

in which $\gamma = 0.884474$. This provides us with a simple but accurate analytic expression of Von Kármán's swirling flow.

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In this chapter we illustrate that the homotopy analysis method is valid for some three-dimensional viscous flows governed by the exact Navier-Stokes equations. The reader is referred to Liao [41] for viscous flows past a sphere in a uniform stream, a well known classical problem in fluid mechanics.

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TABLE 17.1
The mth-order homotopy analysis approximations of $\gamma = -w(+\infty)$.

m	$\epsilon = 0$ $\hbar_s = \hbar_g = -1/5$	$\epsilon = 1/4$ $\hbar_s = \hbar_g = -1/2$	$\epsilon = 0.26167$ $\hbar_s = \hbar_g = -1/2$
10	0.879446	0.882352	0.882977
20	0.881898	0.884437	0.884454
30	0.883607	0.884477	0.884477
40	0.884173	0.884474	0.884474
50	0.884337	0.884474	0.884474

TABLE 17.2
The $[m, m]$ homotopy-Páde approximations of $\gamma = -w(+\infty)$.

$[m, m]$	$\epsilon = 0$ $\hbar_s = \hbar_g = -1/5$	$\epsilon = 1/4$ $\hbar_s = \hbar_g = -1/2$	$\epsilon = 0.26167$ $\hbar_s = \hbar_g = -1/2$
$[5, 5]$	0.879337	0.883856	0.885038
$[10, 10]$	0.884502	0.884482	0.884475
$[15, 15]$	0.884436	0.884474	0.884474
$[20, 20]$	0.884474	0.884474	0.884474
$[25, 25]$	0.884474	0.884474	0.884474

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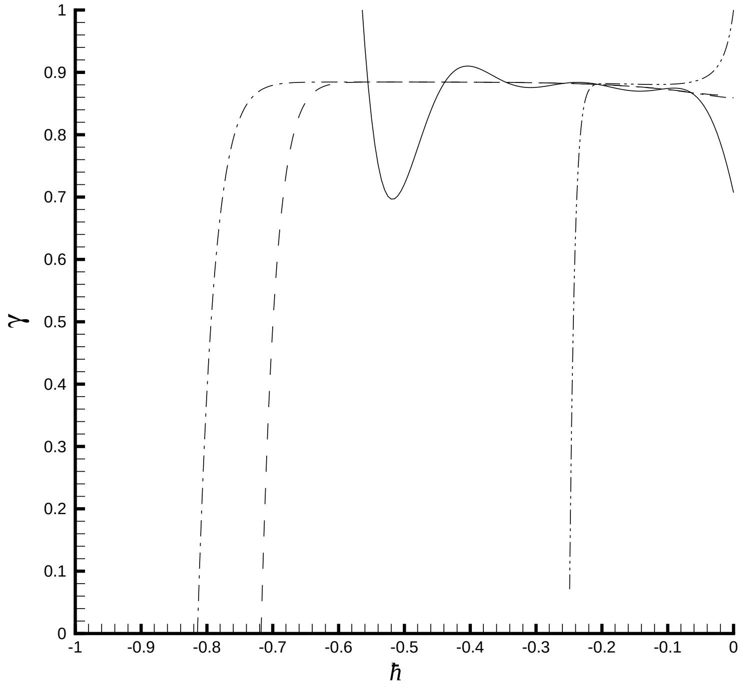

FIGURE 17.1

$\hbar$-curves of $\gamma = -w(+\infty)$ at the 19th order of approximation. Dash-dot-dotted line: $\epsilon = 0$; dashed line: $\epsilon = 1/4$; dash-dotted line: $\epsilon = 0.26167$; solid line: $\epsilon = 1/2$.

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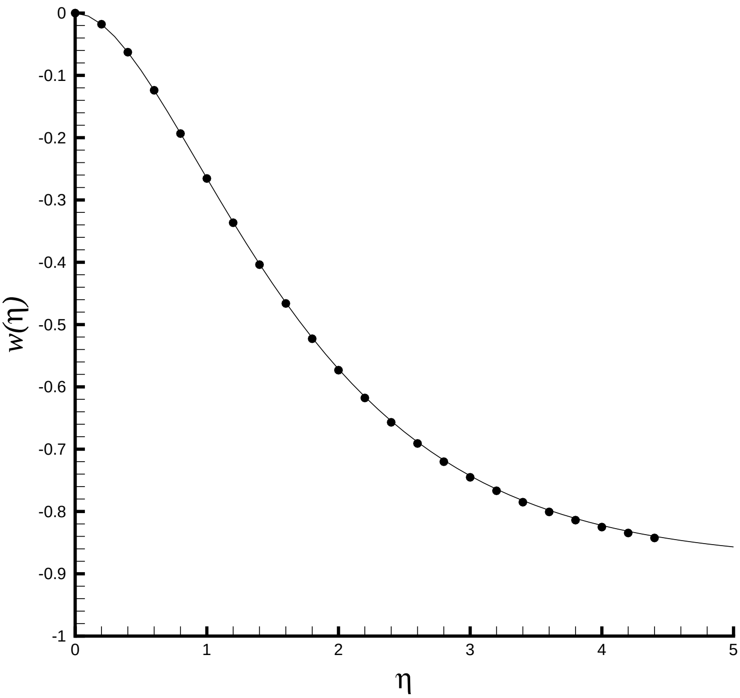

FIGURE 17.2
Comparison of the analytic approximation of $w(\eta)$ with the numerical result given by Benton [117]. Symbol: numerical result; solid line: 20th-order approximation by means of $\epsilon = 0$ and $\hbar_s = \hbar_g = -1/5$.

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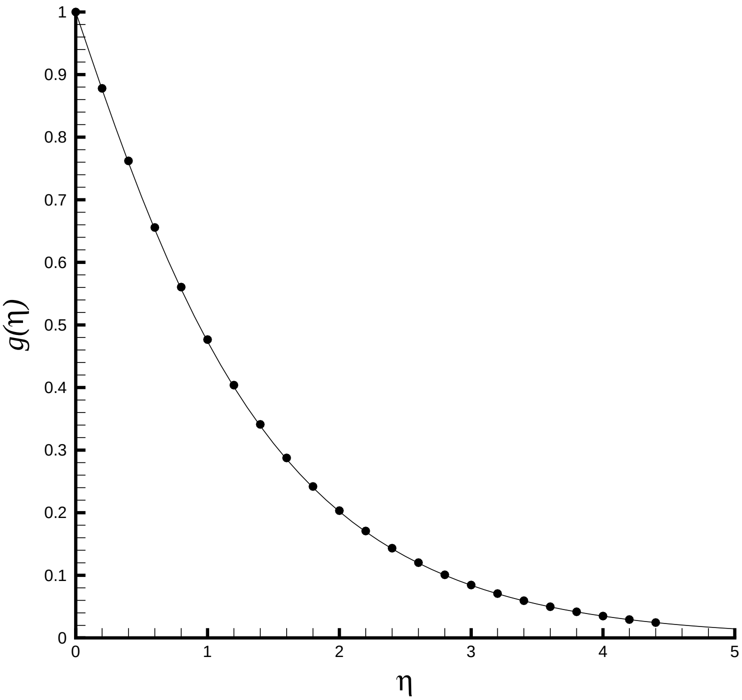

FIGURE 17.3
Comparison of the analytical approximation of $g(\eta)$ with the numerical result given by Benton [117]. Symbol: numerical result; solid line: 20th-order approximation by means of $\epsilon = 0$ and $\hbar_s = \hbar_g = -1/5$.

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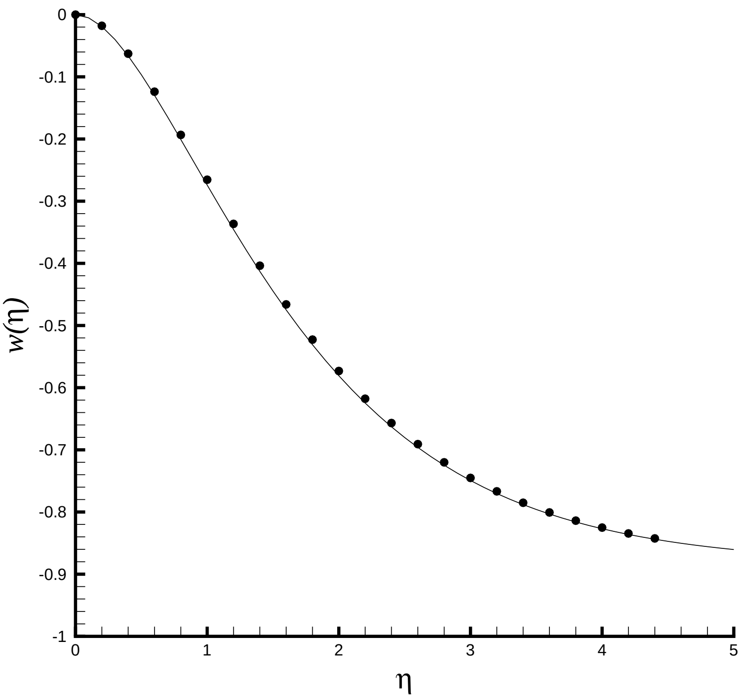

FIGURE 17.4
Comparison of the [1,1] homotopy-Páde approximation of $w(\eta)$ with the numerical result given by Benton [117]. Symbol: numerical result; solid line: [1,1] homotopy-Páde approximation (17.88).

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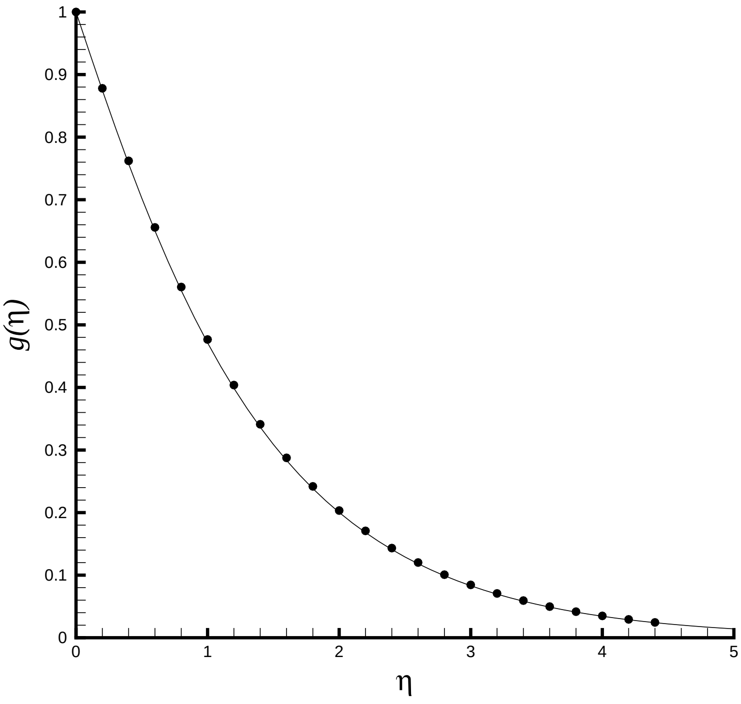

FIGURE 17.5
Comparison of the [1,1] homotopy-Páde approximation of $g(\eta)$ with the numerical result given by Benton [117]. Symbol: numerical result; solid line: [1,1] homotopy-Páde approximation (17.88).

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