15

Boundary-layer flows with exponential property

Consider the two-dimensional laminar viscous flow over a semi-infinite flat plate. The family of similar solutions of the incompressible boundary layers was first obtained by Falkner and Skan [106] in 1931. Let x denote distance from the leading edge of a semi-infinite flat plate and y distance normal to the plate, U the velocity of the fluid in the mainstream, ν the kinematic viscosity, and u and v the components of the velocity of the fluid in the directions of x, y respectively. Falkner and Skan [106] demonstrated that, if $U \propto x^\kappa$, where κ is a constant, there exist solutions of the boundary layer equation

$$f'''(\eta) + f(\eta)f''(\eta) + \beta[1 - f'^2(\eta)] = 0, \tag{15.1}$$

subject to the boundary conditions

$$f(0) = f'(0) = 0, \qquad f'(+\infty) = 1, \tag{15.2}$$

where

$$\beta = \frac{2\kappa}{\kappa + 1}, \qquad \eta = y\sqrt{\frac{(1+\kappa)U}{2\nu x}} \tag{15.3}$$

and the prime denotes differentiation with respect to the similarity variable η. The components u, v of the fluid velocity are given by

$$u = Uf'(\eta), \qquad v = [f(\eta) - (\kappa - 1)\eta f'(\eta)]\sqrt{\frac{\nu U}{2(\kappa+1)x}}\ . \tag{15.4}$$

Note that $f(\eta)$ depends on the physical parameter β only. When $\kappa \geq 0$, from (15.3), it is easy to see that

$$0 \leq \beta \leq 2.$$

When $\kappa < 0$, the mainstream velocity $U \propto x^\kappa$ is singular at $x = 0$ so that Falkner-Skan's solution $f(\eta)$ cannot be taken right back to $x = 0$. This is a general difference between the solutions with a positive and negative β. In 1937 Hartree [107] numerically solved the Falkner-Skan's equations. For large η, Hartree [107] provided the asymptotic expression

$$1 - f'(\eta) \approx A\ \exp(-f^2/2)\ f^{-(2\beta+1)} + B\ f^{2\beta}, \tag{15.5}$$

where A and B are coefficients. From the boundary condition $f'(+\infty) = 1$, it is clear that $f \sim \eta$ as $\eta \to +\infty$ so that $f \to +\infty$ as $\eta \to +\infty$.

Hartree [107] showed that, when β is positive, only one of the solutions, namely that with $B = 0$ in (15.5), would satisfy the conditions at infinity. For positive β, there exists a unique solution $f(\eta)$ so that $f'(\eta) \to 1$ exponentially as $\eta \to +\infty$. Hartree [107] numerically obtained the family of the unique solutions for $0 \le \beta \le 2$. However, when β is negative, any expression of the form (15.5) tends to 0 as η (and so f) tends to ∞, so that any value of $f''(0)$ yields a solution satisfying the condition at infinity. Thus, when β is negative, the boundary conditions (15.2) do not specify a unique solution. To make the solution for $\beta < 0$ unique, Hartree [107] replaced the condition at infinity by

$$f'(\eta) \to 1 \text{ from below as } \eta \to +\infty, \text{ and } f''(0) \text{ as large as possible}$$

and produced a family of numerical results for $\beta_0 \le \beta \le 2$, where $\beta_0 = -0.198$ corresponds to $f''(0) = 0$. The family of Hartree's solutions for $\beta_0 \le \beta \le 2$ has such properties that $f''(0) \ge 0$ and $f'(\eta) \to 1$ *exponentially* as $\eta \to +\infty$, and indicates neither reversed flow nor velocity overshoot.

Stewartson [108] proved that the Falkner-Skan equation (15.1) with the boundary conditions (15.2) has a unique solution when $\beta \ge 0$. To make the solution for $\beta < 0$ unique, Stewartson [108] replaced $f(\eta)$ by $F_\alpha(\eta)$ that satisfies the equation

$$F_\alpha'''(\eta) + F_\alpha(\eta)F_\alpha''(\eta) + \beta[1 - F_\alpha'^2(\eta)] = 0 \tag{15.6}$$

and the boundary conditions

$$F_\alpha(0) = F_\alpha'(0) = 0, F_\alpha'(\alpha) = 1. \tag{15.7}$$

Obviously,

$$f(\eta) = \lim_{\alpha \to +\infty} F_\alpha(\eta).$$

In this explanation Stewartson [108] found in the region $\beta_0 \le \beta < 0$ another new family of numerical solutions exhibiting the property $f''(0) < 0$, demonstrating reversed flow.

Stewartson [108] proved a theorem that, if $\beta < \beta_0 = -0.1988$, then in all the solutions of the Falkner-Skan equation with $f(0) = f'(0) = 0$, there is a range of values of η for which $f'(\eta) > 1$, expressing velocity overshoot in some regions. Unlike Hartree [107] and Stewartson [108], Libby and Liu [109] believed that the overshoot velocity profile might have physical definitions; therefore, they defined other breaches of numerical solutions for $\beta < \beta_0$. Their numerical calculations showed that when $\beta < \beta_0$ multiple (probably an infinite) number of solutions to (15.1) and (15.2) exist for any given values of $f''(0)$.

Note that all of above-mentioned solutions are either numerical or analytic-numerical. In this chapter the homotopy analysis method is applied to present an explicit, purely analytic solution of the Falkner-Skan boundary layer flow.

15.1 Homotopy analysis solution

15.1.1 Zero-order deformation equation

From (15.5), $f'(\eta) \to 1$ exponentially as $\eta \to +\infty$ if $\beta \geq 0$ or $B = 0$ in the case of $\beta < 0$. Thus, it is natural to express $f(\eta)$ by the set of base functions

$$\{\eta^m \ \exp(-n\,\lambda\,\eta) \mid m \geq 0, n \geq 0, \lambda > 0\} \tag{15.8}$$

in the form:

$$f(\eta) = \sum_{m=0}^{+\infty} \sum_{n=0}^{+\infty} a_{m,n}\,\eta^m \exp(-n\,\lambda\,\eta), \tag{15.9}$$

where $a_{m,n}$ is a coefficient and λ is the so-called spatial-scale parameter. This provides us with the *rule of solution expression* for the Falkner-Skan boundary layer flows.

Under the *rule of solution expression* and using (15.2), it is obvious to select

$$f_0(\eta) = \eta - \frac{1 - \exp(-\lambda\,\eta)}{\lambda} + \frac{\gamma[1 - (1 + \lambda\,\eta)\exp(-\lambda\,\eta)]}{\lambda^2} \tag{15.10}$$

as the initial guess of $f(\eta)$, where γ is an auxiliary parameter. Note that

$$f_0''(0) = \lambda + \gamma. \tag{15.11}$$

Under the *rule of solution expression* denoted by (15.9) and from Equations (15.1) and (15.2), we choose the auxiliary linear operator

$$\mathcal{L}[\Phi(\eta;q)] = \frac{\partial^3 \Phi(\eta;q)}{\partial \eta^3} + \lambda\,\frac{\partial^2 \Phi(\eta;q)}{\partial \eta^2} \tag{15.12}$$

with the property

$$\mathcal{L}\left[C_0 + C_1\eta + C_2\exp(-\lambda\,\eta)\right] = 0, \tag{15.13}$$

where C_0, C_1, and C_2 are coefficients, $\Phi(\eta;q)$ is a real function of η and q. From Equation (15.1), we define the nonlinear operator

$$\mathcal{N}[\Phi(\eta;q)] = \frac{\partial^3 \Phi(\eta;q)}{\partial \eta^3} + \Phi(\eta;q)\,\frac{\partial^2 \Phi(\eta;q)}{\partial \eta^2} + \beta\left\{1 - \left[\frac{\partial \Phi(\eta;q)}{\partial \eta}\right]^2\right\}. \tag{15.14}$$

Let $\hbar$ denote a nonzero auxiliary parameter and $H(\eta)$ a nonzero auxiliary function. We construct the so-called zero-order deformation equation

$$(1-q)\,\mathcal{L}\left[\Phi(\eta;q) - f_0(\eta)\right] = q\,\hbar\,H(\eta)\,\mathcal{N}[\Phi(\eta;q)], \tag{15.15}$$

subject to the boundary conditions

$$\Phi(0;q) = 0, \qquad \left.\frac{\partial \Phi(\eta;q)}{\partial \eta}\right|_{\eta=0} = 0, \qquad \left.\frac{\partial \Phi(\eta;q)}{\partial \eta}\right|_{\eta=+\infty} = 1, \tag{15.16}$$

where $q \in [0,1]$ is an embedding parameter.

When $q = 0$, it is easy to demonstrate that

$$\Phi(\eta;0) = f_0(\eta). \tag{15.17}$$

When $q = 1$, since $q \neq 0$ and $H(\eta) \neq 0$, Equations (15.15) and (15.16) are equivalent to Equations (15.1) and (15.2), respectively, provided

$$\Phi(\eta;1) = f(\eta). \tag{15.18}$$

Thus, as the embedding parameter q increases from 0 to 1, $\Phi(\eta;q)$ varies from the initial guess $f_0(\eta)$ to the exact solution $f(\eta)$ of Equations (15.1) and (15.2). Then, by Taylor's theorem and using (15.17), we expand $\Phi(\eta;q)$ in the power series

$$\Phi(\eta;q) = f_0(\eta) + \sum_{k=1}^{+\infty} f_k(\eta)\, q^k, \tag{15.19}$$

where

$$f_k(\eta) = \frac{1}{k!} \left.\frac{\partial^k \Phi(\eta;q)}{\partial q^k}\right|_{q=0}. \tag{15.20}$$

Note that the zero-order deformation equation (15.15) contains the auxiliary parameter $\hbar$ and the auxiliary functions $H(\eta)$. The initial guess $f_0(\eta)$ contains the auxiliary parameter γ. Assuming that all of them are correctly chosen so that the series (15.19) converges when $q = 1$, we have, using (15.18),

$$f(\eta) = f_0(\eta) + \sum_{k=1}^{+\infty} f_k(\eta). \tag{15.21}$$

This provides us with a relationship between the initial approximation $f_0(\eta)$ and the exact solution $f(\eta)$.

15.1.2 High-order deformation equation

For conciseness, define the vector

$$\vec{f}_n = \{f_0(\eta), f_1(\eta), f_2(\eta), \cdots, f_n(\eta)\}.$$

Differentiating the zero-order deformation equations (15.15) and (15.16) k times with respect to q, then setting $q = 0$, and finally dividing them by $k!$, we obtain the high-order deformation equation

$$\mathcal{L}\left[f_k(\eta) - \chi_k f_{k-1}(\eta)\right] = \hbar\, H(\eta)\, R_k(\vec{f}_{k-1}), \tag{15.22}$$

subject to the boundary conditions

$$f_k(0) = f'_k(0) = f'_k(+\infty) = 0, \tag{15.23}$$

where χ_k is defined by (2.42) and

$$R_k(\vec{f}_{k-1}) = f'''_{k-1}(\eta) + \sum_{n=0}^{k-1} \left[f_n(\eta) f''_{k-1-n}(\eta) - \beta f'_n(\eta) f'_{k-1-n}(\eta) \right] + \beta\,(1-\chi_k). \tag{15.24}$$

It is easy to solve the linear differential equations (15.22) and (15.23), using symbolic calculation software.

Under the *rule of solution expression* denoted by (15.9) and from Equation (15.22), the auxiliary function $H(\eta)$ may be in the form

$$H(\eta) = \eta^{\kappa_1}\ \exp(-\lambda\,\kappa_2\,\eta),$$

where $\kappa_1 \geq 0$ and $\kappa_2 \geq 0$ are integers. For simplicity, we choose $\kappa_1 = \kappa_2 = 0$, corresponding to

$$H(\eta) = 1. \tag{15.25}$$

Then, let $f_k^*(\eta)$ denote a special solution of the equation

$$\mathcal{L}[f_k^*(\eta)] = \hbar\, R_k(\vec{f}_{k-1}).$$

Then, from (15.13), we gain the solution

$$f_k(\eta) = \chi_k\ f_{k-1}(\eta) + f_k^*(\eta) + C_0 + C_1\eta + C_2 \exp(-\lambda\,\eta), \tag{15.26}$$

where C_0, C_1, and C_2 are determined by the boundary conditions (15.23).

15.1.3 Recursive formulae

By solving the first several high-order deformation equations (15.22) and (15.23), $f_m(\eta)$ can be expressed by

$$f_m(\eta) = \sum_{k=0}^{m+1} \Psi_{m,k}(\eta) \exp(-k\lambda\,\eta), \qquad m \geq 0, \tag{15.27}$$

where

$$\Psi_{0,0}(\eta) = b_0^{0,0} + b_1^{0,0}\ \eta, \tag{15.28}$$

$$\Psi_{0,1}(\eta) = b_0^{0,1} + b_1^{0,1}\ \eta, \tag{15.29}$$

$$\Psi_{m,0}(\eta) = b_0^{m,0}, \qquad m \geq 1 \tag{15.30}$$

and

$$\Psi_{m,k}(\eta) = \sum_{n=0}^{2(m+1)-k} b_n^{m,k}\, \eta^n, \qquad m \geq 1, 1 \leq k \leq m+1. \tag{15.31}$$

Substituting the above expressions into Equations (15.22) and (15.23), Liao [40] gained the recursive expressions of each coefficient $b_k^{m,n}$, where $m \geq 1$, $0 \leq n \leq m+1$ and $0 \leq k \leq 2(m+1) - n$, as follows:

$$\begin{aligned} b_0^{m,0} &= \chi_m b_0^{m-1,0} - \lambda^{-1} \sum_{r=0}^{2m} \Gamma_r^{m,1} \Pi_r^{1,1} - \sum_{n=2}^{m+1} (n-1) \Gamma_0^{m,n} \Pi_0^{n,0} \\ &\quad + \sum_{n=2}^{m+1} \sum_{r=1}^{2(m+1)-n} \Gamma_r^{m,n} \left(n\Pi_r^{n,0} - \Pi_r^{n,0} - \lambda^{-1} \Pi_r^{n,1} \right), \end{aligned} \tag{15.32}$$

$$b_1^{m,0} = 0, \tag{15.33}$$

$$\begin{aligned} b_0^{m,1} &= \chi_m b_0^{m-1,1} + \lambda^{-1} \sum_{r=0}^{2m} \Gamma_r^{m,1} \Pi_r^{1,1} + \sum_{n=2}^{m+1} n \Gamma_0^{m,n} \Pi_0^{n,0} \\ &\quad + \sum_{n=2}^{m+1} \sum_{r=1}^{2(m+1)-n} \Gamma_r^{m,n} \left(n\Pi_r^{n,0} - \lambda^{-1} \Pi_r^{n,1} \right), \end{aligned} \tag{15.34}$$

$$\begin{aligned} b_k^{m,1} &= \chi_m (1 - \chi_{k+2-2m})\, b_k^{m-1,1} + \sum_{r=k-1}^{2m} \Gamma_r^{m,1} \Pi_r^{1,k}, \\ &\qquad 1 \leq k \leq 2m+1, \end{aligned} \tag{15.35}$$

$$\begin{aligned} b_k^{m,n} &= \chi_m (1 - \chi_{k+1-2m+n})\, b_k^{m-1,n} - \sum_{r=k}^{2(m+1)-n} \Gamma_r^{m,n} \Pi_r^{n,k}, \\ &\qquad 2 \leq n \leq m, 0 \leq k \leq 2(m+1) - n \end{aligned} \tag{15.36}$$

and

$$b_k^{m,m+1} = - \sum_{r=k}^{m+1} \Gamma_r^{m,m+1} \Pi_r^{m+1,k}, \qquad 1 \leq k \leq m+1, \tag{15.37}$$

where

$$\Pi_r^{1,k} = \frac{r!\,(r-k+2)}{k!\,\lambda^{r-k+3}}, \qquad 0 \leq k \leq r+1,$$

$$\Pi_r^{n,k} = \frac{r!}{k!(n-1)^{r-k+1} \lambda^{r-k+3}} \left[1 - \left(1 - \frac{1}{n}\right)^{r-k+1} \left(1 + \frac{r-k+1}{n}\right) \right]$$

$$n \geq 2,\ 0 \leq k \leq r,$$

$$\Gamma_r^{m,n} = \hbar \left[(1 - \chi_{r+1-2m+n})\, d_r^{m-1,n} + \delta_r^{m,n} + \Delta_r^{m,n} \right]$$
$$1 \leq n \leq m, 0 \leq r \leq 2(m+1) - n,$$

$$\Gamma_r^{m,m+1} = \hbar(\delta_r^{m,m+1} + \Delta_r^{m,m+1}),$$

in which

$$\Delta_r^{m,n} = -\beta \sum_{k=0}^{m-1} \sum_{j=\max\{0,n+k-m\}}^{\min\{n,k+1\}} \sum_{i=\max\{0,r-2(m-k)+n-j)\}}^{\min\{r,2(k+1)-j\}} a_i^{k,j}\, a_{r-i}^{m-1-k,n-j},$$

$$\delta_r^{m,n} = \sum_{k=0}^{m-1} \sum_{j=\max\{1,n+k-m\}}^{\min\{n,k+1\}} \sum_{i=\max\{0,r-2(m-k)+n-j)\}}^{\min\{r,2(k+1)-j\}} c_i^{k,j} \times b_{r-i}^{m-1-k,n-j}\, \Lambda_{r-i}^{m-1-k,n-j},$$

$$m \geq 1, 0 \leq n \leq m+1, 0 \leq r \leq 2(m+1) - n$$

under the definitions

$$a_n^{m,k} = (n+1) b_{n+1}^{m,k} \Lambda_{n+1}^{m,k} - (k\lambda) b_n^{m,k} \Lambda_n^{m,k}, \tag{15.38}$$

$$c_n^{m,k} = (n+1)(n+2)\, b_{n+2}^{m,k} \Lambda_{n+2}^{m,k} - 2(k\lambda)(n+1) b_{n+1}^{m,k} \Lambda_{n+1}^{m,k} + (k\lambda)^2\, b_n^{m,k} \Lambda_n^{m,k}, \tag{15.39}$$

$$d_n^{m,k} = (n+1)\, c_{n+1}^{m,k} \Lambda_{n+1}^{m,k} - (k\lambda) c_n^{m,k} \Lambda_n^{m,k} \tag{15.40}$$

and

$$\Lambda_k^{i,j} = \begin{cases} 0, \text{ when } i = j = 0, k \geq 2, \\ 0, \text{ when } i > 0, j = 0, k \geq 1, \\ 0, \text{ when } j > i+1, \\ 0, \text{ when } k > 2(i+1) - j, \\ 1, \text{ otherwise.} \end{cases} \tag{15.41}$$

From (15.10), we obtain the first four coefficients

$$b_0^{0,0} = \frac{\gamma - \lambda}{\lambda^2}, \quad b_1^{0,0} = 1, \quad b_0^{0,1} = \frac{\lambda - \gamma}{\lambda^2}, \quad b_1^{0,1} = -\frac{\gamma}{\lambda} \tag{15.42}$$

from which, we can calculate all other coefficients $b_k^{m,n}$ using the above recursive expressions.

The Mth-order approximation of Equations (15.1) and (15.2) is given by

$$f(\eta) \approx \eta + \left(\sum_{m=0}^{M} b_0^{m,0} \right) + \sum_{n=1}^{M+1} \exp(-n\lambda\,\eta) \left(\sum_{m=n-1}^{M} \sum_{k=0}^{2(m+1)-n} b_k^{m,n}\, \eta^k \right). \qquad (15.43)$$

Therefore, we have the explicit, purely analytic solution of Falkner-Skan laminar viscous flow over a semi-infinite flat plate

$$f(\eta) = \eta + \left(\sum_{m=0}^{+\infty} b_0^{m,0} \right) + \lim_{M \to +\infty} \sum_{n=1}^{M+1} \exp(-n\lambda\,\eta) \left(\sum_{m=n-1}^{M} \sum_{k=0}^{2(m+1)-n} b_k^{m,n}\, \eta^k \right). \qquad (15.44)$$

This exact solution obviously has the asymptotic property $f' \to 1$ exponentially as $\eta \to +\infty$.

15.1.4 Convergence theorem

THEOREM 15.1
The series

$$f_0(\eta) + \sum_{k=1}^{+\infty} f_k(\eta)$$

must be the exact solution of Equations (15.1) and (15.2) as long as it is convergent, where $f_k(\eta)$ is governed by Equations (15.22) and (15.23) under the definitions (15.10), (15.12), (15.24), and (2.42).

Proof: If the series is convergent, we have

$$\lim_{m \to +\infty} f_m(\eta) = 0.$$

From (15.22) and (2.42),

$$\hbar\, H(\eta) \sum_{k=1}^{m} R_k(\vec{f}_{k-1}) = \mathcal{L}[f_m(\eta)].$$

Using (15.12), we gain

$$\hbar\, H(\eta) \sum_{k=1}^{+\infty} R_k(\vec{f}_{k-1}) = \lim_{m \to +\infty} \mathcal{L}[f_m(\eta)] = \mathcal{L}\left[\lim_{m \to +\infty} f_m(\eta) \right] = 0,$$

which gives, since $\hbar \neq 0$ and $H(\eta) \neq 0$,

$$\sum_{k=1}^{+\infty} R_k(\vec{f}_{k-1}) = 0.$$

Substituting (15.24) into the above expression and simplifying it, we obtain

$$\frac{d^3}{d\eta^3}\left[\sum_{k=0}^{+\infty} f_k(\eta)\right] + \left[\sum_{k=0}^{+\infty} f_k(\eta)\right]\frac{d^2}{d\eta^2}\left[\sum_{k=0}^{+\infty} f_k(\eta)\right]$$
$$+\beta\left\{1 - \left[\frac{d}{d\eta}\sum_{k=0}^{+\infty} f_k(\eta)\right]^2\right\} = 0.$$

Furthermore, from (15.10) and (15.23),

$$\sum_{k=0}^{+\infty} f_k(0) = \sum_{k=0}^{+\infty} f'_k(0) = 0, \qquad \sum_{k=0}^{+\infty} f'_k(+\infty) = 1.$$

Therefore, if the series

$$f_0(\eta) + \sum_{k=1}^{+\infty} f_k(\eta)$$

is convergent, it must be an exact solution of Equations (15.1) and (15.2). This ends the proof.

15.2 Result analysis

According to Theorem 15.1, we need only ensure that the solution series (15.21) converges. Note that the solution (15.44) contains three auxiliary parameters $\hbar$, λ, and γ. We have therefore a three-parameter family of solution expressions. The spatial-scale parameter λ affects the rate of $f'(\eta) \to 0$ as $\eta \to +\infty$. From (15.11), the auxiliary parameter γ affects $f''_0(0)$, and we can investigate the relationship between the exact solution and different initial guesses $f_0(\eta)$, which becomes interesting when there exist multiple solutions in the case of $\beta < 0$. It should be emphasized that, for any given values of λ and γ, we still have the freedom to choose a proper value of the auxiliary parameter $\hbar$ to control and adjust the convergence region and rate of the solution (15.44), when necessary.

Physically, $f''(0)$ is related to the friction of the fluid on the plate and therefore has important physical meanings. From (15.21), we have

$$f''(0) = f''_0(0) + \sum_{k=1}^{+\infty} f''_k(0), \tag{15.45}$$

which is dependent of the physical parameter β and the three auxiliary parameters $\hbar$, λ, and γ. First, consider the case of $\hbar = -1, \gamma = 0$ and regard λ as an unknown variable. For given values of β, $f''(0)$ converges to the same corresponding value, provided λ is large enough, as shown in Figure 15.1. Note that, when $\lambda \geq 5$, we can obtain convergent results of $f''(0)$ for $0 \leq \beta \leq 2$, corresponding to $0 \leq \kappa < +\infty$. It is therefore reasonable to choose $\lambda \geq 5$. Then, we consider the case of $\lambda = 5$ and $\gamma = 0$. The influence of $\hbar$ on the convergence of $f''(0)$ can be investigated by plotting the so-called $\hbar$-curves (see page 26 and §3.5.1) of $f''(0)$, as shown in Figure 15.2. Obviously, when $-5/4 \leq \hbar \leq -3/4$, we can obtain convergent results of $f''(0)$ for $0 \leq \beta \leq 2$. Furthermore, when $\lambda = 5, \gamma = 0$, and $\hbar = -1$, $f''(0)$ converges for $\beta_0 \leq \beta \leq 2$, where $\beta_0 = -0.1988$. The related 10th-, 20th- and 30th-order approximations of $f''(0)$ are given by

$$
\begin{aligned}
& f''(0) \\
\approx\ & 0.466892061269575 + 1.270377798259161\,\beta \\
& -0.9366061372519299\,\beta^2 + 0.6565444804810052\,\beta^3 \\
& -0.2989667156611743\,\beta^4 + 8.714746301295173 \times 10^{-2}\,\beta^5 \\
& -1.646263530984164 \times 10^{-2}\,\beta^6 + 2.009360736004046 \times 10^{-3}\,\beta^7 \\
& -1.532383017041316 \times 10^{-4}\,\beta^8 + 6.654735449025599 \times 10^{-6}\,\beta^9 \\
& -1.259647041638817 \times 10^{-7}\,\beta^{10},
\end{aligned}
\tag{15.46}
$$

$$
\begin{aligned}
& f''(0) \\
\approx\ & 0.469470560483573 + 1.295165031248947\,\beta \\
& -1.37974417506381\,\beta^2 + 2.191127183953301\,\beta^3 \\
& -3.010696768394167\,\beta^4 + 3.217599178710972\,\beta^5 \\
& -2.637727245237923\,\beta^6 + 1.672089788089693\,\beta^7 \\
& -0.8288927042391463\,\beta^8 + 0.3244321617350418\,\beta^9 \\
& -0.1009610729534239\,\beta^{10} + 2.508145750314333 \times 10^{-2}\,\beta^{11} \\
& -4.979128795292073 \times 10^{-3}\,\beta^{12} + 7.879252632693684 \times 10^{-4}\,\beta^{13} \\
& -9.872764016512919 \times 10^{-5}\,\beta^{14} + 9.675273712687701 \times 10^{-6}\,\beta^{15} \\
& -7.265429168115138 \times 10^{-7}\,\beta^{16} + 4.042349583833827 \times 10^{-8}\,\beta^{17} \\
& -1.573031139104484 \times 10^{-9}\,\beta^{18} + 3.831177670499221 \times 10^{-11}\,\beta^{19} \\
& -4.409794935993077 \times 10^{-13}\,\beta^{20},
\end{aligned}
\tag{15.47}
$$

and

$$
\begin{aligned}
& f''(0) \\
\approx\ & 0.4695903615312177 + 1.298441994559965\,\beta \\
& -1.491321283547855\,\beta^2 + 3.075663557218445\,\beta^3
\end{aligned}
$$

$$
\begin{aligned}
&-6.529797437132239\,\beta^4 + 12.02830971699564\,\beta^5 \\
&-18.1643591437081\,\beta^6 + 22.24132274854839\,\beta^7 \\
&-22.20184453035751\,\beta^8 + 18.25679986443915\,\beta^9 \\
&-12.5044109522257\,\beta^{10} + 7.205799232502405\,\beta^{11} \\
&-3.523684854407225\,\beta^{12} + 1.472301275851469\,\beta^{13} \\
&-0.5283860769496572\,\beta^{14} + 0.1634753390904293\,\beta^{15} \\
&-4.369783644567797\ \times 10^{-2}\,\beta^{16} + 1.010062677099088\ \times 10^{-2}\,\beta^{17} \\
&-2.018052269391153\ \times 10^{-3}\,\beta^{18} + 3.479126853101193\ \times 10^{-4}\,\beta^{19} \\
&-5.159762018077551\ \times 10^{-5}\,\beta^{20} + 6.552517694061283\ \times 10^{-6}\,\beta^{21} \\
&-7.079292502238022\ \times 10^{-7}\,\beta^{22} + 6.449249702165323\ \times 10^{-8}\,\beta^{23} \\
&-4.894052058789618\ \times 10^{-9}\,\beta^{24} + 3.041591067712079\ \times 10^{-10}\,\beta^{25} \\
&-1.510937645005482\ \times 10^{-11}\,\beta^{26} + 5.783596085142942\ \times 10^{-13}\,\beta^{27} \\
&-1.606885758239585\ \times 10^{-14}\,\beta^{28} + 2.896755640650334\ \times 10^{-16}\,\beta^{29} \\
&-2.560020533395366\ \times 10^{-18}\,\beta^{30}, \qquad (15.48)
\end{aligned}
$$

respectively. When $0 \le \beta \le 2$, the series of $f''(0)$ converges quickly and even the 10th-order approximation agrees with numerical results given by Hartree [107] and White [20], as shown in Table 15.1 and Figure 15.3, respectively. However, when $\beta_0 \le \beta < 0$, the series of $f''(0)$ converges slowly. We fail to accelerate the convergence of $f''(0)$ by means of the Padé technique in the traditional way. But, using the homotopy-Padé technique (see page 38 and §3.5.2), the convergence of the series (15.45) is greatly accelerated, especially when $\beta_0 \le \beta < 0$, as shown in Table 15.2 and Figure 15.3. Also, it is found that the $[m, m]$ homotopy-Padé approximant of $f''(0)$ does not depend upon the auxiliary parameter $\hbar$.

It is found that, as long as the series (15.45) of $f''(0)$ is convergent, the corresponding series of $f(\eta)$ and $f'(\eta)$ also converge in the whole region $0 \le \eta < +\infty$. When $0 \le \beta \le 2$, the series of $f'(\eta)$ converges quickly and the 20th-order approximations agree with Hartree's numerical results [107], as shown in Figure 15.4. However, when $\beta_0 \le \beta < 0$, the closer the value of β is to β_0, the more slowly the series of $f'(\eta)$ converges; thus higher-order approximations are necessary to get accurate enough results, as shown in Figure 15.4.

These verify Hartree's [107] conclusions that there exists a unique solution when $0 \le \beta \le 2$ and a solution when $\beta_0 \le \beta < 0$, with the properties $f''(0) \ge 0$ and $f'(\eta) \to 1$ exponentially as $\eta \to +\infty$. When $\beta_0 \le \beta < 0$, Stewartson [108] numerically found a kind of solution with the property $f''(0) < 0$ but still $f'(\eta) \to 1$ exponentially as $\eta \to +\infty$, showing reversed flow. To check Stewartson's [108] numerical results, we choose a negative value of γ so that

$$f_0''(0) = \gamma + \lambda < 0.$$

For given λ and γ, by means of plotting the corresponding $\hbar$-curves of $f''(0)$, a negative $\hbar$ with a small enough value of $|\hbar|$ may be chosen to ensure that the corresponding solution series converge. Several dozen cases are investigated. However, it is found that, as long as a solution series is convergent, it always converges to Hartree's family of solution. For example, when $\gamma = -5.5, \beta = -15/100, \lambda = 2$, and $\hbar = -1/10$, the approximate series converges to Hartree's family of solution. Thus, it seems that the solution (15.44) might not give Stewartson's [108] family of solutions with reversed flows. Similarly, it seems that the solution (15.44) might not give Libby and Liu's [109] family of solution with velocity overshoot. Note that Stewartson [108], and Libby and Liu [109] found their solutions by numerical techniques, and it is difficult to rigorously check if $f'(\eta) \to 1$ exponentially as $\eta \to +\infty$ by means of numerical methods, because all numerical methods are restricted in a finite domain and thus cannot correctly treat the quantity of infinity. It may therefore be doubtful that solutions given by Stewartson [108] and Libby and Liu [109] have the property $f'(\eta) \to 1$ exponentially as $\eta \to +\infty$ in rigorous mathematical meaning. More evidence is necessary to support this point of view. It is still an open question whether there exist multiple solutions of the Falkner-Skan viscous flow with the exponential property at infinity when $\beta < 0$. If the multiple solutions exist, it would be a challenge to apply the homotopy analysis method to discover all these solutions.

TABLE 15.1
Comparison of the analytic approximations of $f''(0)$ given by (15.45) when $\lambda = 5, \gamma = 0$, and $\hbar = -1$ with White's [20] and Hartree's [107] numerical results.

β	10th-order approx.	20th-order approx.	25th-order approx.	30th-order approx.	numerical result
2.0	1.68647	1.68719	1.68721	1.68722	1.6872
1.6	1.51709	1.52148	1.52152	1.52152	1.5215
1.2	1.33147	1.33578	1.33572	1.33572	1.3357
1.0	1.23079	1.23266	1.23258	1.23259	1.2326
0.8	1.12210	1.12027	1.12028	1.12027	1.1203
0.6	1.00107	0.99572	0.99585	0.99584	0.9958
0.5	0.93379	0.92755	0.92767	0.92768	0.9277
0.4	0.86038	0.85435	0.85440	0.85442	0.8544
0.3	0.77922	0.77483	0.77474	0.77475	0.7748
0.2	0.68830	0.68691	0.68674	0.68671	0.6867
0.1	0.59519	0.58711	0.58707	0.58705	0.5870
0.0	0.46689	0.46947	0.46956	0.46959	0.4696
-0.1	0.32980	0.32363	0.32197	0.32096	0.319
-0.14	0.26876	0.25374	0.24960	0.24682	0.239
-0.16	0.23676	0.21559	0.20947	0.20515	0.190
-0.18	0.20372	0.17499	0.16622	0.15971	0.128
-0.19	0.18679	0.15369	0.14329	0.13537	0.086
-0.198	0.17306	0.13614	0.12426	0.11504	0

Source: Shi-Jun Liao, "A uniformly valid analytic solution of two-dimensional viscous flow past a semi-infinite flat plate", *Journal of Fluid Mechanics* (1999), 385:101-128 Cambridge University Press Copyright ©1999 Cambridge University Press, reprinted with permission.

TABLE 15.2
Comparison of the $[m, m]$ homotopy-Padé approximations of $f''(0)$ when $\lambda = 5, \gamma = 0$, and $\hbar = -1$ with White's [20] and Hartree's [107] numerical results.

β	[5,5]	[10,10]	[15,15]	Numerical result
2.0	1.68636	1.68722	1.68722	1.6872
1.6	1.52026	1.52151	1.52151	1.5215
1.2	1.33399	1.33571	1.33572	1.3357
1.0	1.23063	1.23260	1.23259	1.2326
0.8	1.11816	1.12027	1.12027	1.1203
0.6	0.99372	0.99584	0.99584	0.9958
0.5	0.92563	0.92769	0.92768	0.9277
0.4	0.85246	0.85443	0.85442	0.8544
0.3	0.77287	0.77474	0.77476	0.7748
0.2	0.68478	0.68670	0.68671	0.6867
0.1	0.58484	0.58697	0.58704	0.5870
0.0	0.46736	0.46960	0.46960	0.4696
-0.1	0.32291	0.31935	0.31927	0.319
-0.14	0.25476	0.24074	0.23984	0.239
-0.16	0.21813	0.19380	0.19128	0.190
-0.18	0.17980	0.13870	0.13160	0.128
-0.19	0.16004	0.10677	0.09455	0.086
-0.198	0.14398	0.07833	0.05912	0

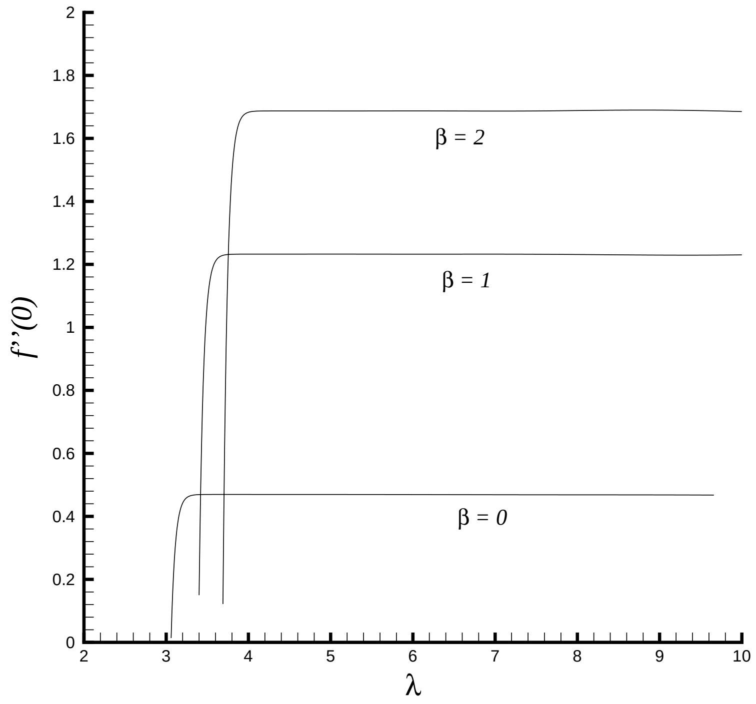

FIGURE 15.1
The 20th-order approximation of $f''(0)$ versus λ when $\hbar = -1$, $\gamma = 0$, and $\beta = 0, 1, 2$.

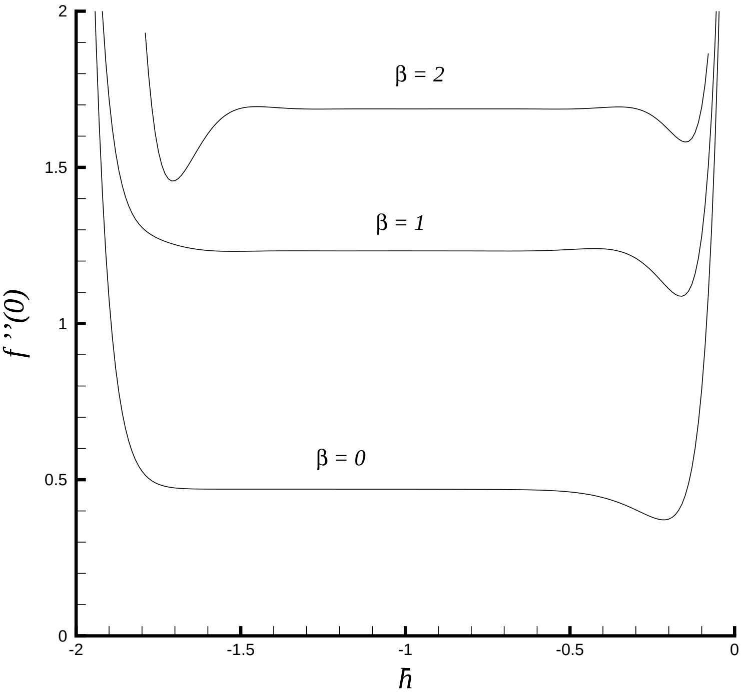

FIGURE 15.2
The $\hbar$-curves of $f''(0)$ at the 20th order of approximation when $\lambda = 5$, $\gamma = 0$, and $\beta = 0, 1, 2$.

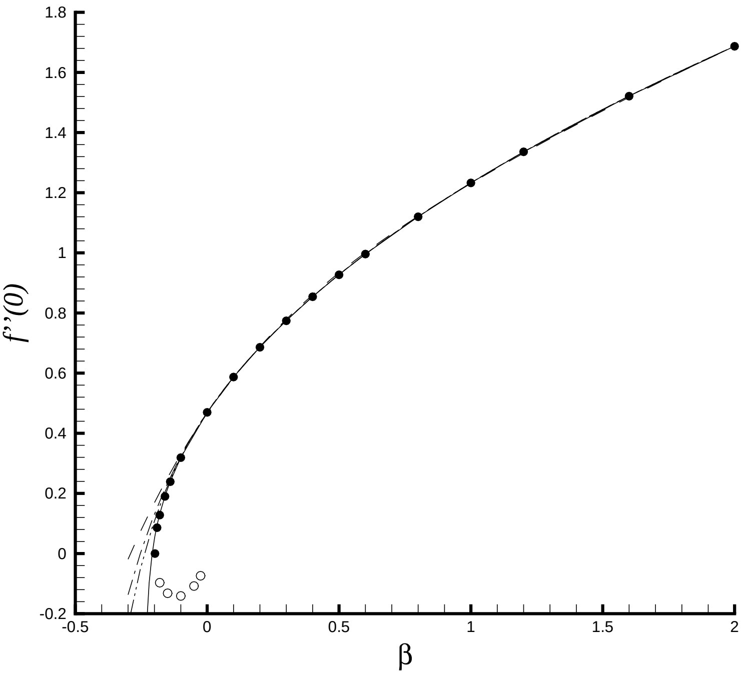

FIGURE 15.3
Comparison of the analytic approximations $f''(0)$ when $\lambda = 5, \gamma = 0$, and $\hbar = -1$ with the numerical results. Dashed line: 10th-order analytic approximations (15.46); dash-dotted line: 20th-order analytic approximations (15.47); dash-dot-dotted line: 30th-order analytic approximations (15.48); solid line: [15,15] homotopy-Padé approximation; filled circle: numerical results given by Hartree [107]; open circle: numerical results given by Stewartson [108].

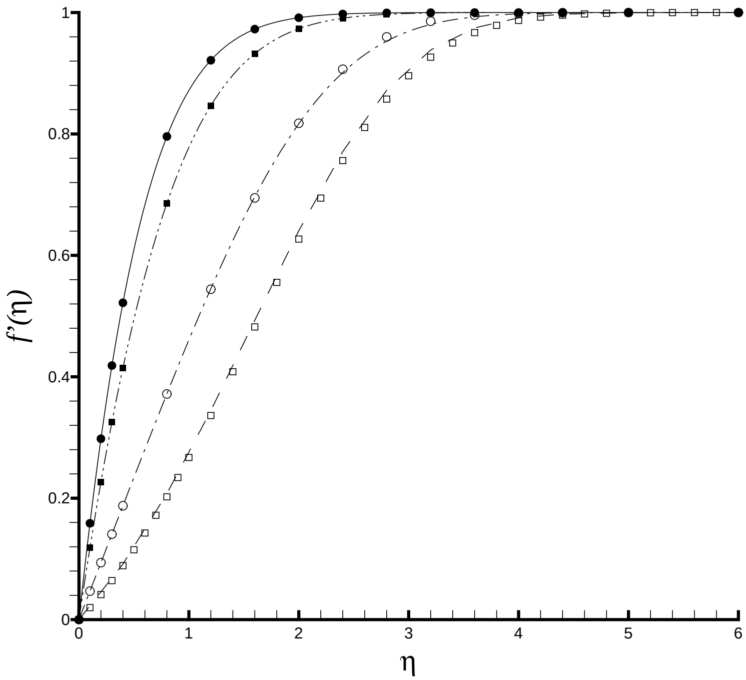

FIGURE 15.4
Comparison of the analytic approximation of $f'(\eta)$ given by (15.43) when $\lambda = 5, \gamma = 0$, and $\hbar = -1$ with Hartree's [107] numerical results. Solid line: numerical result when $\beta = 2$; dash-dot-dotted line: numerical result when $\beta = 1$; dash-dotted line: numerical result when $\beta = 0$; dashed line: numerical result when $\beta = -0.16$; filled circle: 20th-order analytic approximations when $\beta = 2$; filled square: 20th-order analytic approximations when $\beta = 1$; open circle: 20th-order analytic approximation when $\beta = 0$; open square: 50th-order analytic approximations when $\beta = -0.16$.