# 12

# *Free oscillation systems with quadratic nonlinearity*

Consider free oscillations of a conservative system with quadratic nonlinearity, governed by

$$\ddot{U}(t) = f[U(t), \dot{U}(t), \ddot{U}(t)], \tag{12.1}$$

where $t$ denotes the time, the dot denotes derivative with respect to $t$, and $f[U(t), \dot{U}(t), \ddot{U}(t)]$ is a known function of $U(t), \dot{U}(t)$, and $\ddot{U}(t)$. Physically, free oscillation of conservative systems is a periodic motion. Let $\omega$ and $a$ denote the frequency and amplitude of oscillation, respectively. Define the mean of motion

$$\delta = \frac{1}{T}\int_0^T U(t)dt, \tag{12.2}$$

where $T = 2\pi/\omega$ is the period of oscillation. For conservative systems with quadratic nonlinearity, the mean of motion $\delta$ is generally nonzero. This is the main difference between free oscillations of conservative system with odd nonlinearity and those with quadratic one. Obviously, both $\delta$ and $\omega$ have clear physical meanings. For conservative systems, the oscillation amplitude $a$ is determined by initial conditions and is related to the total kinetic energy. Both $\omega$ and $\delta$ are dependent of $a$. Without the loss of generality, we consider oscillations with amplitude $a$ under the initial conditions

$$\dot{U}(0) = 0 \qquad U(0) = a + \delta. \tag{12.3}$$

Unlike perturbation techniques, we need not assume that Equation (12.1) contains any small/large parameters.

## 12.1 Homotopy analysis solution

### 12.1.1 Zero-order deformation equation

Obviously, the free oscillation can be described by the base functions

$$\{\cos(m\omega t) \mid m = 0, 1, 2, 3, \cdots\} \tag{12.4}$$

in the form:

$$U(t) = \delta + \sum_{m=1}^{+\infty} c_m \cos(m\omega t), \tag{12.5}$$

where $c_m$ is a coefficient. Under the transformation

$$\tau = \omega t, \; U(t) = \delta + u(\tau), \tag{12.6}$$

Equations (12.2) and (12.3) become

$$\omega^2 u''(\tau) = f[\delta + u(\tau), \omega u'(\tau), \omega^2 u''(\tau)], \tag{12.7}$$

and

$$u(0) = a, \qquad u'(0) = 0, \tag{12.8}$$

respectively, where the prime denotes derivative with respect to $\tau$. Obviously, $u(\tau)$ can be expressed by the base functions

$$\{\cos(m\tau) \mid m = 1, 2, 3, \cdots\} \tag{12.9}$$

in the form:

$$u(\tau) = \sum_{m=1}^{+\infty} c_m \cos(m\tau). \tag{12.10}$$

This provides us with the *rule of solution expression* for free oscillations of conservative systems with quadratic nonlinearity.

Note that the frequency $\omega$ and the mean of motion $\delta$ are unknown. Let $\omega_0$, $\delta_0$ denote the initial guesses of $\omega$ and $\delta$, respectively. Under the *rule of solution expression* denoted by (12.10) and from (12.8), it is easy to choose

$$u_0(\tau) = a \cos\tau \tag{12.11}$$

as the initial guess of $u(\tau)$, where $a$ is the amplitude of oscillation. Moreover, under the *rule of solution expression* denoted by (12.10) and from Equation (12.7), we choose the auxiliary linear operator

$$\mathcal{L}[\Phi(\tau; q)] = \omega_0^2 \left[ \frac{\partial^2 \Phi(\tau; q)}{\partial \tau^2} + \Phi(\tau; q) \right], \tag{12.12}$$

with the property

$$\mathcal{L}\left(C_1 \sin\tau + C_2 \cos\tau\right) = 0. \tag{12.13}$$

where $q$ is an embedding parameter, $\Phi(\tau; q)$ is a function of $\tau$ and $q$, $C_1$ and $C_2$ are coefficients. From Equation (12.7), we define the nonlinear operator

$$\begin{aligned} &\mathcal{N}\left[\Phi(\tau; q), \Omega(q), \Delta(q)\right] \\ &= \Omega^2(q)\, \frac{\partial^2 \Phi(\tau; q)}{\partial \tau^2} \\ &- f\left[ \Delta(q) + \Phi(\tau; q), \Omega(q)\, \frac{\partial \Phi(\tau; q)}{\partial \tau}, \Omega^2(q)\, \frac{\partial^2 \Phi(\tau; q)}{\partial \tau^2} \right], \end{aligned} \tag{12.14}$$

where $\Omega(q)$ and $\Delta(q)$ are functions of the embedding parameter $q \in [0,1]$, corresponding to the frequency $\omega$ and the mean of motion $\delta$, respectively.

The homotopy analysis method is based on such continuous variations $\Phi(\tau;q)$, $\Omega(q)$, and $\Delta(q)$ that, as the embedding parameter $q$ increases from 0 to 1 , $\Phi(\tau;q)$ varies from the initial guess $u_0(\tau)$ to the exact solution $u(\tau)$, so does $\Omega(q)$ from the initial guess $\omega_0$ to the exact frequency $\omega$, and $\Delta(q)$ from the initial guess $\delta_0$ to the exact mean of motion $\delta$, respectively. To ensure this, we construct such a homotopy in a more general form (see §3.6):

$$\begin{aligned}
&\mathcal{H}[\Phi(\tau;q), \Omega(q), \Delta(q), H(\tau), H_2(\tau), \hbar, \hbar_2, q] \\
&= (1-q)\,\mathcal{L}\left[\Phi(\tau;q) - u_0(\tau)\right] - q\,\hbar\,H(\tau)\,\mathcal{N}[\Phi(\tau;q), \Omega(q), \Delta(q)] \\
&- \hbar_2\,H_2(\tau)\,(1-q)\left\{(f[\Delta(q),0,0] - f[\delta_0,0,0]) + \left[\Omega^2(q) - \omega_0^2\right] u_0''(\tau)\right\}
\end{aligned}$$

where $q \in [0,1]$ is the embedding parameter, $\hbar$ and $\hbar_2$ are nonzero auxiliary parameters, $H(\tau)$ and $H_2(\tau)$ are nonzero auxiliary functions, respectively.

Writing

$$\mathcal{H}[\Phi(\tau;q), \Omega(q), \Delta(q), H(\tau), H_2(\tau), \hbar, \hbar_2, q] = 0,$$

we have the zero-order deformation equation

$$\begin{aligned}
&(1-q)\,\mathcal{L}\left[\Phi(\tau;q) - u_0(\tau)\right] \\
&= q\,\hbar\,H(\tau)\,\mathcal{N}[\Phi(\tau;q), \Omega(q), \Delta(q)] \\
&+ \hbar_2\,H_2(\tau)\,(1-q)\,(f[\Delta(q),0,0] - f[\delta_0,0,0]) \\
&+ \hbar_2\,H_2(\tau)\,(1-q)\left[\Omega^2(q) - \omega_0^2\right] u_0''(\tau),
\end{aligned} \tag{12.15}$$

subject to the initial conditions

$$\Phi(0;q) = a, \qquad \left.\frac{\partial \Phi(\tau;q)}{\partial \tau}\right|_{\tau=0} = 0. \tag{12.16}$$

When $q = 0$ it is easy using (12.11) and (12.15) to show that

$$\Phi(\tau;0) = u_0(\tau), \qquad \Omega(0) = \omega_0, \qquad \Delta(0) = \delta_0. \tag{12.17}$$

When $q = 1$, since $\hbar \neq 0$ and $H(\tau) \neq 0$, Equations (12.15) and (12.16) are equivalent to the original ones (12.7) and (12.8), respectively, provided

$$\Phi(\tau;1) = u(\tau), \qquad \Omega(1) = \omega, \qquad \Delta(1) = \delta. \tag{12.18}$$

Therefore, as $q$ increases from 0 to 1, $\Phi(\tau;q)$ varies from the initial guess $u_0(\tau) = a\cos\tau$ to the exact solution $u(\tau)$, so does $\Omega(q)$ from the initial guess $\omega_0$ to the exact frequency $\omega$, and $\Delta(q)$ from the initial guess $\delta_0$ to the exact mean of motion $\delta$, respectively.

Note that the zero-order deformation equation (12.15) contains the auxiliary parameters $\hbar, \hbar_2$ and the auxiliary functions $H(\tau)$ and $H_2(\tau)$. Assume that all of them are properly chosen so that Equations (12.15) and (12.16) have solutions $\Phi(\tau;q)$, $\Omega(q)$, and $\Delta(q)$ for all $q \in [0,1]$ and, in addition, the so-called high-order deformation derivatives

$$u_0^{[m]}(\tau) = \left.\frac{\partial^m \Phi(\tau;q)}{\partial q^m}\right|_{q=0}, \qquad \omega_0^{[m]} = \left.\frac{\partial^m \Omega(q)}{\partial q^m}\right|_{q=0}, \qquad \delta_0^{[m]} = \left.\frac{\partial^m \Delta(q)}{\partial q^m}\right|_{q=0}$$

exist for $m \geq 1$. Then, by Taylor's theorem and using (12.17), we expand $\Phi(\tau;q)$, $\Omega(q)$, and $\Delta(q)$ in the power series of $q$ as follows:

$$\Phi(\tau;q) = u_0(\tau) + \sum_{m=1}^{+\infty} u_m(\tau)\, q^m, \tag{12.19}$$

$$\Omega(q) = \omega_0 + \sum_{m=1}^{+\infty} \omega_m\, q^m, \tag{12.20}$$

$$\Delta(q) = \delta_0 + \sum_{m=1}^{+\infty} \delta_m\, q^m, \tag{12.21}$$

where

$$u_m(\tau) = \frac{u_0^{[m]}(\tau)}{m!}, \qquad \omega_m = \frac{\omega_0^{[m]}}{m!}, \qquad \delta_m = \frac{\delta_0^{[m]}}{m!}. \tag{12.22}$$

Assuming that the auxiliary parameters $\hbar, \hbar_2$ and the auxiliary functions $H(\tau)$ and $H_2(\tau)$ are properly chosen so that the above series converge at $q = 1$, we have using (12.18) the solution series

$$u(\tau) = u_0(\tau) + \sum_{m=1}^{+\infty} u_m(\tau), \tag{12.23}$$

$$\omega = \omega_0 + \sum_{m=1}^{+\infty} \omega_m, \tag{12.24}$$

$$\delta = \delta_0 + \sum_{m=1}^{+\infty} \delta_m. \tag{12.25}$$

### 12.1.2 High-order deformation equation

For brevity, define the vectors

$$\vec{u}_n = \{u_0(\tau), u_1(\tau), \cdots, u_n(\tau)\}, \qquad \vec{\omega}_n = \{\omega_0, \omega_1, \cdots, \omega_n\}$$

and

$$\vec{\delta}_n = \{\delta_0, \delta_1, \cdots, \delta_n\}.$$

Differentiating Equations (12.15) and (12.16) $m$ times with respect to $q$, then setting $q = 0$, and finally dividing it by $m!$, we have the so-called high-order deformation equation

$$\mathcal{L}\left[u_m(\tau) - \chi_m u_{m-1}(\tau)\right] = \hbar\, H(\tau)\, R_m(\vec{u}_{m-1}, \vec{\omega}_{m-1}, \vec{\delta}_{m-1}) \\ + \hbar_2\, H_2(\tau)\, S_m(\tau, \vec{\omega}_m, \vec{\delta}_m), \tag{12.26}$$

subject to the initial conditions

$$u_m(0) = u_m'(0) = 0, \tag{12.27}$$

where $\chi_m$ is defined by (2.42),

$$R_m(\vec{u}_{m-1}, \vec{\omega}_{m-1}, \vec{\delta}_{m-1}) \\ = \frac{1}{(m-1)!} \left. \frac{d^{m-1}\mathcal{N}[\Phi(\tau;q), \Omega(q), \Delta(q)]}{dq^{m-1}} \right|_{q=0}, \tag{12.28}$$

$$S_m(\tau, \vec{\omega}_m, \vec{\delta}_m) \\ = -\left( \sum_{i=0}^{m} \omega_i \omega_{m-i} - \chi_m \sum_{i=0}^{m-1} \omega_i \omega_{m-1-i} \right) a\ \cos\tau \\ + \left[ Q_m(\vec{\delta}_m) - \chi_m Q_{m-1}(\vec{\delta}_{m-1}) \right], \tag{12.29}$$

and

$$Q_m(\vec{\delta}_m) = \frac{1}{m!} \left. \frac{d^m f[\Delta(q), 0, 0]}{dq^m} \right|_{q=0}. \tag{12.30}$$

Note that there exist three unknowns: $u_m(\tau), \omega_{m-1}$ and $\delta_{m-1}$ (when $\hbar_2 = 0$), or $u_m(\tau), \omega_m$ and $\delta_m$ (when $\hbar_2 \neq 0$). However, we have only Equations (12.26) and (12.27) for $u_m(\tau)$. So, the problem is not closed and two additional algebraic equations are needed to determined $\omega_{m-1}$ and $\delta_{m-1}$ (when $\hbar_2 = 0$), or $\omega_m$ and $\delta_m$ (when $\hbar_2 \neq 0$).

Under the *rule of solution expression* denoted by (12.10) and from Equation (12.26), the auxiliary functions $H(\tau)$ and $H_2(\tau)$ might appear as

$$H(\tau) = \cos(2\kappa_1\tau), \qquad H_2(\tau) = \cos(2\kappa_2\tau),$$

where $\kappa_1$ and $\kappa_2$ are integers. For simplicity, we choose $\kappa_1 = \kappa_2 = 0$, corresponding to

$$H(\tau) = 1, \qquad H_2(\tau) = 1. \tag{12.31}$$

Then, under *the rule of solution expression* denoted by (12.10) and due to the quadratic nonlinearity of conservative systems, the term on the right-hand side of Equation (12.26) may be expressed by

$$b_{m,0} + \sum_{n=1}^{\varphi(m)} b_{m,n} \cos(n\tau), \tag{12.32}$$

where the integer $\varphi(m)$ is dependent of $m$ and the form of the original equation (12.1), and the coefficient $b_{m,n}$ becomes zero when $n > \varphi(m)$. From the property (12.13) of $\mathcal{L}$, the solution of the $m$th-order deformation equation (12.26) contains the so-called secular term $\tau \cos \tau$ if $b_{m,1} \neq 0$. Besides, when $b_{m,0} \neq 0$, its solution $u_m(\tau)$ contains a constant term $b_{m,0}/\omega_0^2$. However, these two terms do not conform to the *rule of solution expression* denoted by (12.10). Therefore, we enforce the coefficients $b_{m,0}$ and $b_{m,1}$ to be zero:

$$b_{m,0} = 0, \qquad b_{m,1} = 0, \qquad (m = 1, 2, 3, \cdots). \tag{12.33}$$

This provides a set of two algebraic equations for $\omega_{m-1}$ and $\delta_{m-1}$ (when $\hbar_2 = 0$), or for $\omega_m$ and $\delta_m$ (when $\hbar_2 \neq 0$). In this way, the problem is closed and the *rule of solution existence* is obeyed. Notice that Equations (12.33) are often nonlinear when $\hbar_2 = 0$ and $m = 1$, but are always linear otherwise. When $\hbar_2 = 0$ and $m = 1$, we must solve a set of nonlinear algebraic equations (12.33) to gain $\omega_0$ and $\delta_0$. However, when $\hbar_2 \neq 0$, we have the freedom to choose the initial guesses $\omega_0$ and $\delta_0$. It is advisable to first consider the case of $\hbar_2 = 0$, because this often gives accurate results even at low order of approximations, as shown by the illustrative examples.

Thereafter, it is easy to gain the solution of the $m$th-order deformation equation (12.26)

$$u_m(\tau) = \chi_m u_{m-1}(\tau) + \sum_{n=2}^{\varphi(m)} \frac{b_{m,n}}{\omega_0^2(1-n^2)} \cos(n\tau) + C_1 \sin\tau + C_2 \cos\tau, \tag{12.34}$$

where $C_1$ and $C_2$ are two integral constants. From (12.27), we have $C_1 = 0$. To ensure that the amplitude of oscillation equals to $a$, we use

$$u_m(0) - u_m(\pi) = 0, \qquad m = 1, 2, 3, \cdots, \tag{12.35}$$

which determines the coefficient $C_2$. We gain $u_m(\tau)$ $(m = 1, 2, 3, \cdots)$ and $\omega_{m-1}, \delta_{m-1}$ (when $\hbar_2 = 0$) or $\omega_m, \delta_m$ (when $\hbar_2 \neq 0$), successively. At the $M$th-order approximation we have

$$u(\tau) \approx \sum_{m=0}^{M} u_m(\tau), \tag{12.36}$$

$$\omega \approx \sum_{m=0}^{M} \omega_m, \tag{12.37}$$

$$\delta \approx \sum_{m=0}^{M} \delta_m. \tag{12.38}$$

## 12.2 Illustrative examples

### 12.2.1 Example 12.2.1

Consider

$$\ddot{U}(t) + U(t) + \gamma\, U^2(t) = 0, \tag{12.39}$$

where $\gamma$ is a constant. Under the transformation $\tau = \omega t$ and $U(t) = \delta + u(\tau)$,

$$\omega^2 u''(\tau) + \delta + u(\tau) + \gamma\, [\delta + u(\tau)]^2 = 0. \tag{12.40}$$

All related formulae are the same as those given in §12.1. From (12.28) and (12.30),

$$R_m = \sum_{n=0}^{m-1} \left( \sum_{j=0}^{n} \omega_j\, \omega_{n-j} \right) u''_{m-1-n}(\tau) + v_{m-1}(\tau) + \gamma \sum_{n=0}^{m-1} v_n(\tau)\, v_{m-1-n}(\tau) \tag{12.41}$$

and

$$Q_m = \delta_m + \gamma \sum_{n=0}^{m} \delta_n\, \delta_{m-n}, \tag{12.42}$$

where

$$v_k(\tau) = \delta_k + u_k(\tau). \tag{12.43}$$

Note that there exist two auxiliary parameters $\hbar$ and $\hbar_2$. First, let us consider the case of $\hbar_2 = 0$. In this case we gain from (12.33) the set of algebraic equations for $\omega_0$ and $\delta_0$,

$$a + 2a\gamma\delta_0 - a\omega_0^2 = 0, \tag{12.44}$$

and

$$\frac{\gamma a^2}{2} + \delta_0 + \gamma\delta_0^2 = 0, \tag{12.45}$$

which yield

$$\omega_0 = \left(1 - 2a^2\gamma^2\right)^{1/4}, \qquad \delta_0 = \frac{\omega_0^2 - 1}{2\gamma}. \tag{12.46}$$

When $\hbar_2 = 0$, we have the first-order approximation

$$\omega \approx \omega_0 - \frac{\hbar (a\gamma)^2}{12\omega_0^3}, \qquad \delta \approx \delta_0, \tag{12.47}$$

the second-order approximation

$$\omega \approx \omega_0 - \frac{\hbar (a\gamma)^2}{6\omega_0^3}\left(1 + \frac{\hbar}{2}\right) + \frac{\hbar^2 (a\gamma)^4}{288\omega_0^7}, \quad \delta \approx \delta_0 + \frac{\hbar^2 a^4 \gamma^3}{144\omega_0^6}, \tag{12.48}$$

the third-order approximation

$$\omega \approx \omega_0 - \frac{\hbar(a\gamma)^2}{4\omega_0^3}\left(1+\hbar+\frac{\hbar^2}{3}\right) + \frac{\hbar^2(a\gamma)^2}{1728\omega_0^7}(18+41\hbar) + \frac{\hbar^3(a\gamma)^6}{3456\omega_0^{11}}, \tag{12.49}$$

$$\delta \approx \delta_0 + \frac{\hbar^2 a^4\gamma^3}{48\omega_0^6}\left(1+\frac{2\hbar}{3}\right), \tag{12.50}$$

and so on. These results are dependent upon the auxiliary parameter $\hbar$. For any given $a$ and $\gamma$ we can investigate the influence of $\hbar$ on the convergence by plotting the so-called $\hbar$-curves (see page 26 or §3.5.1). The series for $\omega$ and $\delta$ are convergent when $-2 \le \hbar < 0$. However, the convergence region depends upon the value of $\hbar$. We can adjust the convergence regions by choosing a proper value of $\hbar$. For example, when $\hbar = -4/5$ or $\hbar = -\omega_0^2$, the third-order approximation of $\omega$ agrees with the exact results in the region $|a\gamma| \le 1/\sqrt{2}$ and is much better than the perturbation approximation, as shown in Figure 12.1. When $\hbar = -1/5$ or $\hbar = -\omega_0^2$, the third-order approximation of $\gamma$ $\delta$ yields good agreement with the numerical results in the region $|a\gamma| \le 1/\sqrt{2}$, as shown in Figure 12.2.

When $|a\gamma| > 1/\sqrt{2}$, the initial approximations $\omega_0$ and $\delta_0$ given by (12.46) have no physical meanings. By some simple calculations, we deduce from Equation (12.39) that solutions exist in the region $|a\gamma| \le 3/4$. Using (12.46) as initial approximations, we certainly cannot gain results valid in the region $1/\sqrt{2} \le |a\gamma| \le 3/4$. To gain approximations valid in the region $0 \le |a\gamma| \le 3/4$, we must choose initial approximations that have physical meanings in the whole region. Fortunately, when $\hbar_2 \ne 0$, the proposed approach provides the freedom to choose such an initial approximation. Now, let us consider the case of $\hbar_2 \ne 0$. Notice that $\omega_0$ defined by (12.46) gives good approximation for $|a\gamma| < 1/\sqrt{2}$. More importantly, it provides valuable information about the mathematical structure of the frequency. Therefore, it is reasonable to select the initial approximation

$$\tilde{\omega}_0 = \left(1 - \frac{16}{9}a^2\gamma^2\right)^{1/4}, \tag{12.51}$$

which is valid in the whole region $0 \le |a\gamma| \le 3/4$. From (12.46), we choose the initial approximation

$$\tilde{\delta}_0 = \frac{\tilde{\omega}_0^2 - 1}{2\gamma}, \tag{12.52}$$

which is also valid in the whole region $0 \le |a\gamma| \le 3/4$. Note that $\omega$ and $\delta$ now contain two auxiliary parameters: $\hbar$ and $\hbar_2$. For simplicity, let us consider the special case of $\hbar_2 = -1$. We successively gain the first-order approximation

$$\omega \approx \tilde{\omega}_0, \qquad \delta \approx \tilde{\delta}_0 + \frac{\hbar}{4\gamma\tilde{\omega}_0^2}\left(\tilde{\omega}_0^4 - 1 + 2a^2\gamma^2\right), \tag{12.53}$$

the second-order approximation

$$\omega \approx \tilde{\omega}_0 - \frac{\hbar^2}{12\tilde{\omega}_0^3}\left(3\,\tilde{\omega}_0^4 - 3 + 5\,a^2\gamma^2\right),$$

$$\delta \approx \tilde{\delta}_0 + \frac{\hbar}{16\gamma\tilde{\omega}_0^6}\left(\tilde{\omega}_0^4 - 1 + 2a^2\gamma^2\right)\left[8\tilde{\omega}_0^4 + \hbar\left(3\,\tilde{\omega}_0^4 + 1 - 2\,a^2\gamma^2\right)\right],$$

the third-order approximation

$$\begin{aligned}\omega \approx\ & \tilde{\omega}_0 - \frac{\hbar^2}{48\tilde{\omega}_0^7}\left\{12\tilde{\omega}_0^4\left(3\tilde{\omega}_0^4 - 3 + 5a^2\gamma^2\right)\right.\\ & \left. + \hbar\left[(21\tilde{\omega}_0^8 - 18\tilde{\omega}_0^4 - 3) + 4\left(7\tilde{\omega}_0^4 + 3\right)a^2\gamma^2 - 12a^4\gamma^4\right]\right\}, \end{aligned} \tag{12.54}$$

$$\begin{aligned}\delta \approx\ & \tilde{\delta}_0 + \frac{\hbar}{288\gamma\tilde{\omega}_0^{10}}\left\{216\,\tilde{\omega}_0^8\,(\tilde{\omega}_0^4 - 1 + 2a^2\gamma^2)\right.\\ & +54\,\tilde{\omega}_0^4\hbar\left[(3\,\tilde{\omega}_0^8 - 2\,\tilde{\omega}_0^4 - 1) + 4(\tilde{\omega}_0^4 + 1)a^2\gamma^2 - 4a^4\gamma^4\right]\\ & +\hbar^2\left[9(5\,\tilde{\omega}_0^{12} - 3\,\tilde{\omega}_0^8 - \tilde{\omega}_0^4 - 1) + 18(3\,\tilde{\omega}_0^8 + 2\,\tilde{\omega}_0^4 + 3)a^2\gamma^2\right.\\ & \left.\left. -2(19\,\tilde{\omega}_0^4 + 54)a^4\gamma^4 + 72\,a^6\gamma^6\right]\right\}, \end{aligned} \tag{12.55}$$

and so on. These results depend upon the auxiliary parameter $\hbar$. Its influence on the convergence regions can be investigated by plotting the so-called $\hbar$-curves (see page 26 and §3.5.1). We see that, at the third order of approximation, the frequency $\omega$ when $\hbar = -\tilde{\omega}_0$ and the mean of motion $\delta$ when $\hbar = -\tilde{\omega}_0/2$ agree with the numerical results in the whole region $0 \le |a\gamma| \le 3/4$, as shown in Figures 12.3 and 12.4, respectively. Using the better initial approximations (12.51) and (12.52) and properly choosing the two auxiliary parameters $\hbar$ and $\hbar_2$, we gain analytic results valid in the whole region $0 \le |a\gamma| \le 3/4$.

### 12.2.2 Example 12.2.2

Consider

$$\ddot{U}(t) - U(t) + U^4(t) = 0. \tag{12.56}$$

Under the transformation $U(t) = \delta + u(\tau)$ and $\tau = \omega t$, we see that

$$\omega^2 u''(\tau) - [u(\tau) + \delta] + [\delta + u(\tau)]^4 = 0. \tag{12.57}$$

All related formulae are the same as those given in §12.1. From (12.28) and (12.30),

$$\begin{aligned} R_m =\ & \sum_{n=0}^{m-1}\left(\sum_{j=0}^{n}\omega_j\omega_{n-j}\right)u''_{m-1-n}(\tau) - v_{m-1}(\tau)\\ & + \sum_{n=0}^{m-1}\left[\sum_{i=0}^{n}v_i(\tau)v_{n-i}(\tau)\right]\left[\sum_{j=0}^{m-1-n}v_j(\tau)v_{m-1-n-j}(\tau)\right], \end{aligned} \tag{12.58}$$

and

$$Q_m = -\delta_m + \sum_{n=0}^{m} \left( \sum_{i=0}^{n} \delta_i \delta_{n-i} \right) \left( \sum_{j=0}^{m-n} \delta_j \delta_{m-n-j} \right), \qquad (12.59)$$

where $v_k(\tau)$ is defined by (12.43).

When $\hbar_2 = 0$, we have from (12.33) the algebraic equations for $\omega_0$ and $\delta_0$,

$$a - 3a^3\delta_0 - 4a\delta_0^3 + a\omega_0^2 = 0, \qquad (12.60)$$

$$\frac{3}{8}a^4 - \delta_0 + 3a^2\delta_0^2 + \delta_0^4 = 0, \qquad (12.61)$$

which give

$$\omega_0 = \sqrt{4\delta_0^3 + 3a^2\delta_0 - 1}, \qquad (12.62)$$

and

$$\delta_0 = \frac{1}{2}\left( \sqrt{\mu_1} + \sqrt{\frac{2}{\sqrt{\mu_1}} - \mu_1 - 6a^2} \right), \qquad (12.63)$$

where

$$\mu_1 = -2a^2 + \frac{3a^4}{\mu_0} + \frac{\mu_0}{2}, \qquad (12.64)$$

$$\mu_0 = \left( 4 - 4a^6 + 2\sqrt{4 - 8a^6 - 50a^{12}} \right)^{1/3}. \qquad (12.65)$$

We have therefore the first-order approximation

$$\begin{aligned} \omega \approx \omega_0 + \frac{\hbar a^2}{(4\delta_0^3 + 6a^2\delta_0 - 1)\omega_0^3} \Bigg[ & \frac{27}{160}a^4 + \left(\frac{1}{16} - \frac{9}{20}a^6\right)\delta_0 \\ & + \frac{3}{4}a^2\delta_0^2 - \frac{9}{5}a^4\delta_0^3 + \frac{5}{2}\delta_0^4 - \frac{15}{2}a^2\delta_0^5 - 11\delta_0^7 \\ & + \left(\frac{1}{16}\delta_0 - \frac{3}{8}a^2\delta_0^2 - \frac{1}{4}\delta_0^4\right)\omega_0^2 \Bigg], \end{aligned} \qquad (12.66)$$

$$\delta \approx \delta_0 + \frac{\hbar a^4 \delta_0}{(4\delta_0^3 + 6a^2\delta_0 - 1)\omega_0^2} \left( \frac{3}{8}a^2 + \frac{9}{4}\delta_0^2 \right), \qquad (12.67)$$

and so on, where $\omega_0, \delta_0$ are given by (12.62) and (12.63), respectively. Similarly, the influence of $\hbar$ on the convergence region can be investigated by plotting the corresponding $\hbar$-curves (see page 26 and §3.5.1). It is found that the series of $\omega$ is convergent when $-2 < \hbar < 0$, so does the series of $\delta$. Even at the first order of approximation, the frequency $\omega$ when $\hbar = -1$ and the mean of motion $\delta$ when $\hbar = -3/4$ agree with the numerical results, as shown in Figures 12.5 and 12.6. It is unnecessary to consider the case $\hbar_2 \neq 0$ for the second illustrative problem.

In this chapter we illustrate how to get better approximations by means of zero-order deformation equations in a more general form as mentioned in §3.6. The illustrative examples demonstrate the flexibility and potential of the homotopy analysis method.

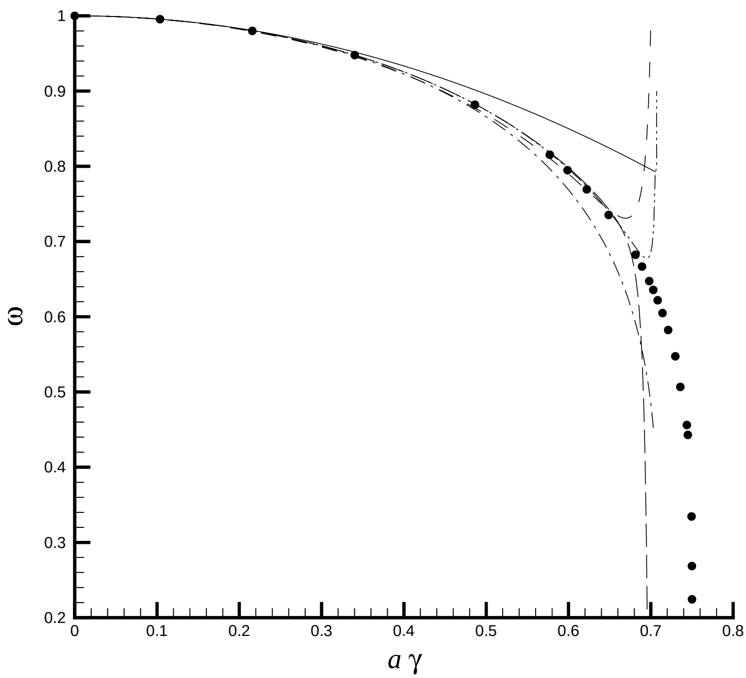


**FIGURE 12.1**
Comparison of the exact frequency $\omega$ of Example 12.2.1 with the approximate results when $\hbar_2 = 0$. Symbols: exact result; solid line: first-order perturbation approximation $\omega = 1 - 5a^2\gamma^2/12$; dashed line: first-order approximation (12.47) when $\hbar = -4/5$; long-dashed line: third-order approximation (12.49) when $\hbar = -4/5$; dash-dotted line: first-order approximation (12.47) when $\hbar = -\omega_0^2$; dash-dot-dotted line: third-order approximation (12.49) when $\hbar = -\omega_0^2$.

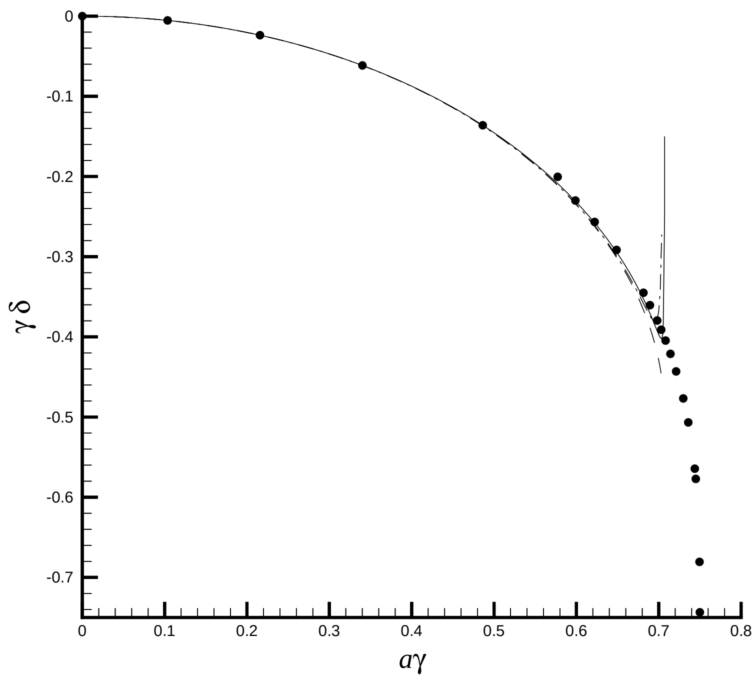


**FIGURE 12.2**
Comparison of the exact mean of motion $\delta$ of Example 12.2.1 with the approximate results when $\hbar_2 = 0$. Symbols: exact result; dashed line: first-order approximation (12.47); dash-dotted line: third-order approximation (12.50) when $\hbar = -1/5$; solid line: third-order approximation (12.50) when $\hbar = -\omega_0^2$.

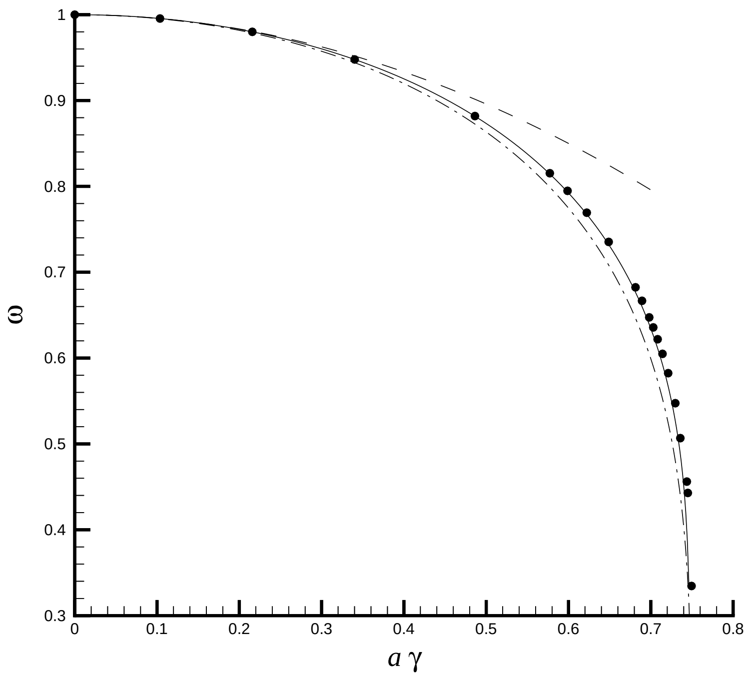


**FIGURE 12.3**

Comparison of the exact frequency $\omega$ of Example 12.2.1 with the approximate results when $\hbar_2 = -1$, $\tilde{\omega}_0 = (1 - 16a^2\gamma^2/9)^{1/4}$ and $\hbar = -\tilde{\omega}_0$. Symbols: exact result; dashed line: perturbation solution $\omega = 1 - 5a^2\gamma^2/12$; dash-dotted line: first-order approximation (12.53); solid line: third-order approximation (12.54).

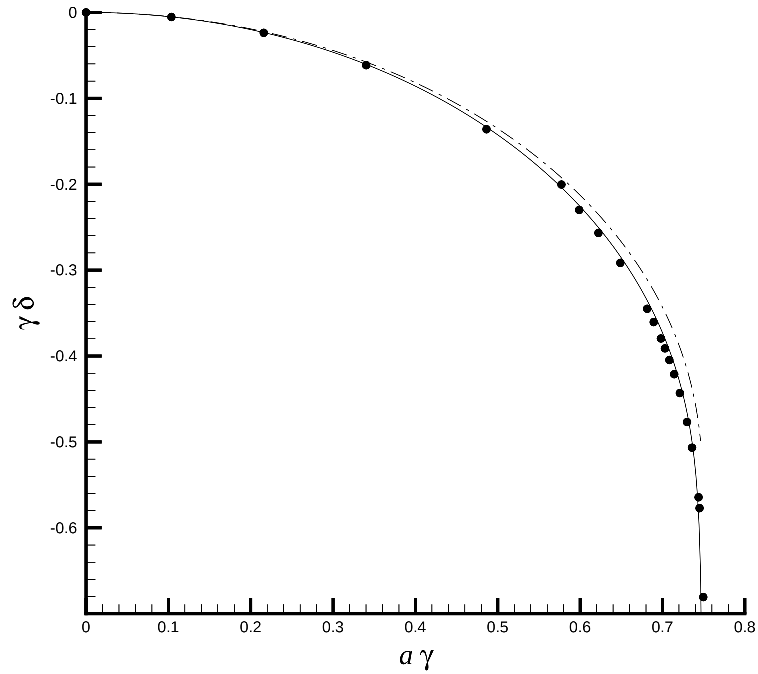


**FIGURE 12.4**
Comparison of the exact mean of motion $\delta$ of Example 12.2.1 with the approximate results when $\hbar_2 = -1$, $\tilde{\omega}_0 = (1 - 16a^2\gamma^2/9)^{1/4}$, and $\hbar = -\tilde{\omega}_0/2$. Symbols: exact result; dash-dotted line: first-order approximation (12.53); solid line: third-order approximation (12.55) .

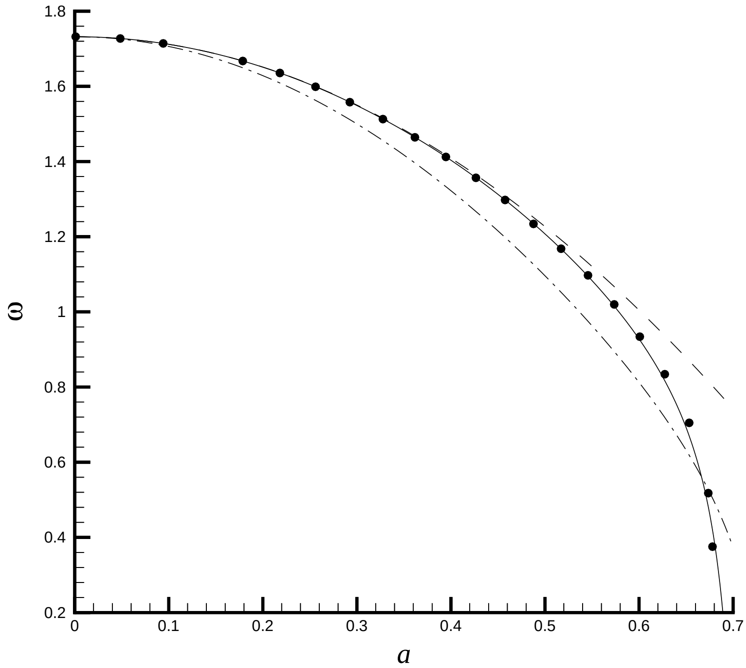


**FIGURE 12.5**

Comparison of the exact frequency of Example 12.2.2 with the first-order approximation when $\hbar_2 = 0$ and $\hbar = -1$. Symbols: exact result; dashed line: first-order perturbation approximation $\omega = \sqrt{3}(1 - 7a^2/6)$; dash-dotted line: initial approximation $\omega_0$ given by (12.62); solid line: first-order approximation (12.66).

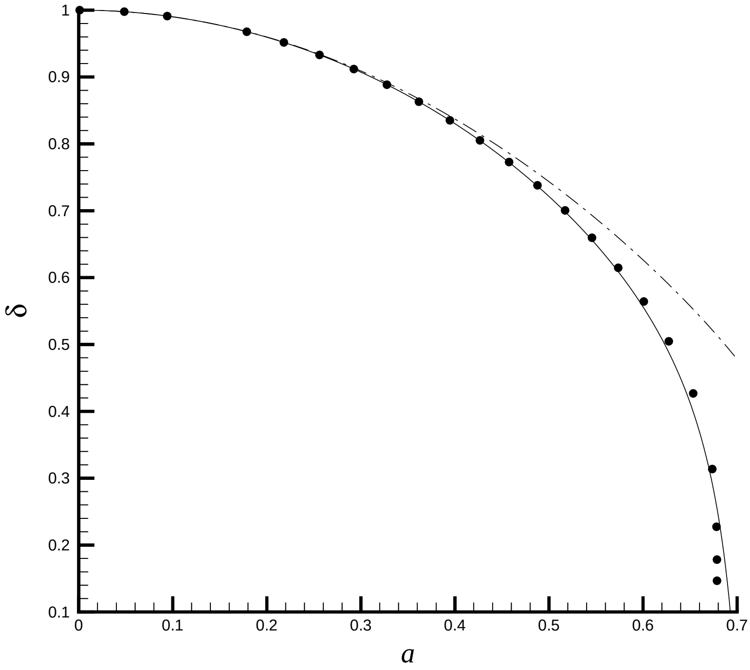


**FIGURE 12.6**
Comparison of the exact mean of motion $\delta$ of Example 12.2.2 with the first-order approximation when $\hbar_2 = 0$ and $\hbar = -3/4$. Symbols: exact result; dash-dotted line: initial approximation $\delta_0$ given by (12.63); solid line: first-order approximation (12.67).