11

Free oscillation systems with odd nonlinearity

Consider free oscillations of a conservative system with odd nonlinearity governed by

$$\ddot{U}(t) = f[U(t), \dot{U}(t), \ddot{U}(t)], \tag{11.1}$$

where t denotes the time, the dot denotes derivative with respect to t, and $f[U(t), \dot{U}(t), \ddot{U}(t)]$ is a known function of $U(t), \dot{U}(t)$, and $\ddot{U}(t)$, respectively. Unlike perturbation techniques, it is unnecessary to assume the existence of any small/large parameters in Equation (11.1). This equation is very general and describes many problems in science and engineering.

Physically speaking, free oscillation of a conservative system is a periodic motion. Let ω and a denote the frequency and amplitude of the oscillation, respectively. Physically, the frequency ω can be regarded as a time scale. For a linear system, the frequency is independent of the amplitude. However, for a nonlinear system, it is important to know the relationship between the frequency and amplitude. In a nonlinear conservative system, the amplitude a is physically determined by initial conditions and is related to the total kinetic energy. Without the loss of any generality, we may consider free oscillations with amplitude a under the initial conditions

$$\dot{U}(0) = 0, U(0) = a. \tag{11.2}$$

11.1 Homotopy analysis solution

11.1.1 Zero-order deformation equation

Obviously, free oscillations of a conservative system with odd nonlinearity can be expressed by the base functions

$$\{\cos(m\omega t) \mid m = 1, 2, 3, \cdots\}. \tag{11.3}$$

Under the transformation $\tau = \omega t$ and $U(t) = u(\tau)$, Equation (11.1) becomes

$$\omega^2 u''(\tau) = f[u(\tau), \omega u'(\tau), \omega^2 u''(\tau)], \tag{11.4}$$

subject to the initial conditions

$$u(\tau) = a, \quad u'(\tau) = 0, \quad \text{when } \tau = 0, \tag{11.5}$$

where the prime denotes derivative with respect to τ. From (11.3), $u(\tau)$ can be expressed by the base functions

$$\{\cos(m\tau) \mid m = 1, 2, 3, \cdots\} \tag{11.6}$$

in the form

$$u(\tau) = \sum_{k=1}^{+\infty} c_k \cos(k\tau), \tag{11.7}$$

where c_k is a coefficient. This provides us with the so-called *rule of solution expression.*

Let ω_0 denote the initial guess of the frequency ω. Obviously, under the *rule of solution expression* denoted by (11.7) and using the initial condition (11.5), it is easy to choose

$$u_0(\tau) = a \cos \tau \tag{11.8}$$

as the initial guess of $u(\tau)$, where a is the amplitude of oscillation. Under the *rule of solution expression* denoted by (11.7), we choose the auxiliary linear operator

$$\mathcal{L}[\Phi(\tau; q)] = \omega_0^2 \left[\frac{\partial^2 \Phi(\tau; q)}{\partial \tau^2} + \Phi(\tau; q) \right], \tag{11.9}$$

with the property

$$\mathcal{L}\left(C_1 \sin \tau + C_2 \cos \tau\right) = 0. \tag{11.10}$$

From Equation (11.4), we define the nonlinear operator

$$\begin{aligned} \mathcal{N}\left[\Phi(\tau; q), \Omega(q)\right] = {} & \Omega^2(q) \, \frac{\partial^2 \Phi(\tau; q)}{\partial \tau^2} \\ & - f\left[\Phi(\tau; q), \Omega(q) \, \frac{\partial \Phi(\tau; q)}{\partial \tau}, \Omega^2(q) \, \frac{\partial^2 \Phi(\tau; q)}{\partial \tau^2}\right], \end{aligned} \tag{11.11}$$

where $\Phi(\tau; q)$ is a function of τ and q, $\Omega(q)$ is a function of q. Let $\hbar$ denote a nonzero auxiliary parameter and $H(\tau)$ a nonzero auxiliary function, respectively. We then construct the zero-order deformation equation

$$(1 - q) \, \mathcal{L}\left[\Phi(\tau; q) - u_0(\tau)\right] = q \, \hbar \, H(\tau) \, \mathcal{N}[\Phi(\tau; q), \Omega(q)], \tag{11.12}$$

subject to the initial conditions

$$\Phi(0; q) = a, \qquad \left. \frac{\partial \Phi(\tau; q)}{\partial \tau} \right|_{\tau=0} = 0. \tag{11.13}$$

When $q = 0$, it is clear that Equations (11.12) and (11.13) have the solution

$$\Phi(\tau; 0) = u_0(\tau), \qquad \Omega(0) = \omega_0. \tag{11.14}$$

When $q = 1$, since $\hbar \neq 0$ and $H(\tau) \neq 0$, Equations (11.12) and (11.13) are equivalent to Equations (11.4) and (11.5), respectively, provided

$$\Phi(\tau; 1) = u(\tau), \qquad \Omega(1) = \omega. \tag{11.15}$$

Therefore, as q increases from 0 to 1, $\Phi(\tau;q)$ deforms from the initial guess $u_0(\tau) = a\cos\tau$ to the exact solution $u(\tau)$, and $\Omega(q)$ varies from the initial guess ω_0 to the exact frequency ω, respectively.

Using (11.14) and Taylor's theorem, $\Phi(\tau;q)$ and $\Omega(q)$ can be expanded in the power series of q as follows:

$$\Phi(\tau;q) = u_0(\tau) + \sum_{m=1}^{+\infty} u_m(\tau)\, q^m, \tag{11.16}$$

$$\Omega(q) = \omega_0 + \sum_{m=1}^{+\infty} \omega_m\, q^m, \tag{11.17}$$

where

$$u_m(\tau) = \frac{1}{m!}\left.\frac{\partial^m \Phi(\tau;q)}{\partial q^m}\right|_{q=0}, \qquad \omega_m = \frac{1}{m!}\left.\frac{\partial^m \Omega(q)}{\partial q^m}\right|_{q=0}. \tag{11.18}$$

Note that the zero-order deformation equation (11.12) contains the auxiliary parameter $\hbar$ and the auxiliary function $H(\tau)$. Thus, $\Phi(\tau;q)$ and $\Omega(q)$ are also dependent upon them. Assuming that $\hbar$ and $H(\tau)$ are properly chosen so that the above series converge at $q = 1$, we have, using (11.15), the solution series

$$u(\tau) = u_0(\tau) + \sum_{m=1}^{+\infty} u_m(\tau), \tag{11.19}$$

$$\omega = \omega_0 + \sum_{m=1}^{+\infty} \omega_m. \tag{11.20}$$

11.1.2 High-order deformation equation

For brevity, define the vectors

$$\vec{u}_n = \{u_0(\tau), u_1(\tau), \cdots, u_n(\tau)\}, \qquad \vec{\omega}_n = \{\omega_0, \omega_1, \cdots, \omega_n\}.$$

Differentiating the zero-order deformation equations (11.12) and (11.13) m times with respect to q, then setting $q = 0$, and finally dividing it by $m!$, we gain the so-called high-order deformation equation

$$\mathcal{L}\left[u_m(\tau) - \chi_m u_{m-1}(\tau)\right] = \hbar\, H(\tau)\, R_m(\vec{u}_{m-1}, \vec{\omega}_{m-1}), \tag{11.21}$$

subject to the initial conditions

$$u_m(0) = u'_m(0) = 0, \tag{11.22}$$

where χ_m is defined by (2.42) and

$$R_m(\vec{u}_{m-1}, \vec{\omega}_{m-1}) = \frac{1}{(m-1)!}\left.\frac{d^{m-1}\mathcal{N}[\Phi(\tau;q), \Omega(q)]}{dq^{m-1}}\right|_{q=0}. \tag{11.23}$$

Note that there are two unknowns: $u_m(\tau)$ and ω_{m-1}. However, we have only Equations (11.21) and (11.22) for $u_m(\tau)$. Thus, the problem is not closed and an additional algebraic equation is needed to determine ω_{m-1}. Under the *rule of solution expression* denoted by (11.7) and due to the odd nonlinearity of the conservative system, $R_m(\vec{u}_{m-1}, \vec{\omega}_{m-1})$ can be expressed by

$$R_m(\vec{u}_{m-1}, \vec{\omega}_{m-1}) = \sum_{n=0}^{\varphi(m)} b_{m,n}(\vec{\omega}_{m-1}) \cos[(2n+1)\tau], \tag{11.24}$$

where $b_{m,n}(\vec{\omega}_{m-1})$ is a coefficient dependent of $\vec{\omega}_{m-1}$, and the integer $\varphi(m)$ depends upon m and the form of Equation (11.1). In order to comply with the *rule of solution expression* denoted by (11.7), $H(\tau)$ should be in the form

$$H(\tau) = \cos(2\kappa\tau), \qquad \kappa = 0, 1, 2, 3, \cdots.$$

For simplicity, we choose $\kappa = 0$, corresponding to

$$H(\tau) = 1. \tag{11.25}$$

According to the property (11.10) of $\mathcal{L}$, the solution of Equation (11.21) involves the so-called secular term $\tau \ \cos\tau$ if

$$R_m(\vec{u}_{m-1}, \vec{\omega}_{m-1})$$

contains the term $\cos\tau$. This disobeys *the rule of solution expression* denoted by (11.7). Thus, the coefficient $b_{m,0}$ in (11.24) must be enforced to be zero. This provides us with the additional algebraic equation

$$b_{m,0}(\vec{\omega}_{m-1}) = 0, \tag{11.26}$$

which yields ω_{m-1}. This equation is often nonlinear when $m = 1$ for ω_0 but is always linear otherwise. Thereafter, it is easy to gain the solution of Equation (11.21):

$$\begin{aligned} u_m(\tau) = \chi_m u_{m-1}(\tau) + \frac{\hbar}{\omega_0^2} \sum_{n=2}^{\varphi(m)} \frac{b_{m,n}(\vec{\omega}_{m-1})}{(1-n^2)} \ \cos(n\tau) \\ + C_1 \sin\tau + C_2 \cos\tau, \end{aligned} \tag{11.27}$$

where C_1 and C_2 are two coefficients. Using (11.22), we obtain $C_1 = 0$. To ensure that the oscillation amplitude equals to a, we use

$$u_m(0) - u_m(\pi) = 0, \qquad m = 1, 2, 3, \cdots, \tag{11.28}$$

which determines C_2, thus producing ω_{m-1} and $u_m(\tau)$ successively. At the Mth-order of approximation,

$$u(\tau) \approx \sum_{m=0}^{M} u_m(\tau), \tag{11.29}$$

$$\omega \approx \sum_{m=0}^{M} \omega_m. \tag{11.30}$$

The above approach is very general and valid even for free oscillations of conservative systems with odd nonlinearity in more general cases, governed by

$$F\left[U(t), \dot{U}(t), \ddot{U}(t), signU(t), sign\dot{U}(t), sign\ddot{U}(t)\right] = 0, \tag{11.31}$$

where

$$sign(x) = \begin{cases} 1, & \text{when } x > 0, \\ -1, & \text{when } x < 0. \end{cases} \tag{11.32}$$

With transformation $\tau = \omega\, t$ and $U(t) = u(\tau)$,

$$F\left[u(\tau), \omega u'(\tau), \omega^2 u''(\tau), sign(u), sign(u'), sign(u'')\right] = 0. \tag{11.33}$$

Let a $(a > 0)$ denote the amplitude and $u_0(\tau) = a\cos\tau$ the initial approximation of oscillation. For free oscillations of conservative systems with odd nonlinearity,

$$sign(u) = sign(u_0) = sign(\cos\tau), \tag{11.34}$$

and similarly,

$$sign(u') = -sign(\sin\tau), sign(u'') = -sign(\cos\tau). \tag{11.35}$$

Thus, Equation (11.33) is equivalent to

$$F\left[u(\tau), \omega u'(\tau), \omega^2 u''(\tau), sign(\cos\tau), -sign(\sin\tau), -sign(\cos\tau)\right] = 0.$$

Using

$$sign(\cos\tau) = \frac{4}{\pi}\sum_{k=0}^{+\infty}\frac{(-1)^k}{2k+1}\cos[(2k+1)\tau], \tag{11.36}$$

$$sign(\sin\tau) = \frac{4}{\pi}\sum_{k=0}^{+\infty}\frac{1}{2k+1}\sin[(2k+1)\tau], \tag{11.37}$$

we write

$$\begin{aligned} &f[u(\tau), \omega u'(\tau), \omega^2 u''(\tau)] \\ &= F\left[u(\tau), \omega u'(\tau), \omega^2 u''(\tau), sign(\cos\tau), -sign(\sin\tau), -sign(\cos\tau)\right]. \end{aligned}$$

Similarly, we are able to solve free oscillations of conservative systems with odd nonlinearity, governed by Equation (11.31).

Note that

$$|x| = x\ sign(x).$$

Equation (11.31) is therefore equivalent to the equation

$$G\left[U(t), \dot{U}(t), \ddot{U}(t), |U(t)|, |\dot{U}(t)|, |\ddot{U}(t)|\right] = 0, \tag{11.38}$$

where G is a function of $U(t), \dot{U}(t), \ddot{U}(t), |U(t)|, |\dot{U}(t)|$, and $|\ddot{U}(t)|$.

11.2 Illustrative examples

11.2.1 Example 11.2.1

Consider free oscillations of a conservation system with odd nonlinearity governed by

$$\ddot{U}(t) + U(t) = \epsilon\, U(t)\, \dot{U}^2(t). \tag{11.39}$$

Under the transformation $\tau = \omega t$ and $U(t) = u(\tau)$,

$$\omega^2 u''(\tau) + u(\tau) = \epsilon\, \omega^2\, u(\tau) u'^2(\tau). \tag{11.40}$$

All other related formulae are the same as those given in §11.1. From (11.23) and (11.39),

$$\begin{aligned} & R_m(\vec{u}_{m-1}, \vec{\omega}_{m-1}) \\ = & \sum_{n=0}^{m-1} \left(\sum_{j=0}^{n} \omega_j \omega_{n-j} \right) u''_{m-1-n} + u_{m-1} \\ - & \epsilon \sum_{n=0}^{m-1} \left(\sum_{i=0}^{n} u_{n-i} \sum_{r=0}^{i} \omega_r \omega_{i-r} \right) \left(\sum_{j=0}^{m-1-n} u'_j u'_{m-1-n-j} \right). \end{aligned} \tag{11.41}$$

When $m = 1$ we obtain from (11.26) the algebraic equation

$$a - a\omega_0^2 - \frac{1}{4} a^3 \epsilon \omega_0^2 = 0, \tag{11.42}$$

which gives

$$\omega_0 = \frac{1}{\sqrt{1 + \frac{1}{4}\epsilon a^2}}. \tag{11.43}$$

The frequency ω at the first and second order of approximation is given by

$$\omega \approx \omega_0 + \frac{\hbar\, (\epsilon a^2)\, [2 + (\epsilon a^2 - 2)\omega_0^2]}{32(4 + \epsilon a^2)\omega_0}$$

and

$$\begin{aligned} \omega \approx \omega_0 &+ \frac{\hbar\, (\epsilon a^2)\, [2 + (\epsilon a^2 - 2)\omega_0^2]}{16(4 + \epsilon a^2)\omega_0} \\ &+ \frac{\hbar^2 (\epsilon a^2)}{6144(4 + \epsilon a^2)^2 \omega_0^3} \left[39\omega_0^4 (\epsilon a^2)^3 + 4\omega_0^2 (43\omega_0^2 + 17)(\epsilon a^2)^2 \right. \\ &\left. + 4(97\omega_0^4 + 98\omega_0^2 - 3)(\epsilon a^2) - 192\, (9\omega_0^4 - 10\omega_0^2 + 1) \right], \end{aligned}$$

respectively. These approximations contain the auxiliary parameter $\hbar$. When $\hbar = -1$, the series of frequency is convergent only in the region $0 \le \epsilon a^2 < 5$, as shown in Figure 11.1. Note that the convergence region becomes larger when $\hbar$ is chosen closer to 0, as shown in Figure 11.1. Therefore, $\hbar$ should be defined as a function of ϵa^2, whose absolute value should decrease as ϵa^2 increases. When $\hbar = -\omega_0^2 = -(1+\epsilon a^2/4)^{-1}$ the series for the frequency converges quickly in the whole region $0 \le \epsilon a^2 < +\infty$, as shown in Figure 11.1. Selecting

$$\hbar = -(1+\epsilon a^2/4)^{-1},$$

we gain the first-order approximation

$$\omega \approx \frac{256 + 128\epsilon a^2 + 13(\epsilon a^2)^2}{8(4+\epsilon a^2)^{5/2}}, \tag{11.44}$$

and the second-order approximation

$$\omega \approx \frac{393216 + 393216\epsilon a^2 + 142848(\epsilon a^2)^2 + 21248(\epsilon a^2)^3 + 1181(\epsilon a^2)^4}{768(4+\epsilon a^2)^{9/2}}. \tag{11.45}$$

These two approximations agree with the numerical results in the whole region:

$$0 \le \epsilon a^2 < +\infty,$$

as shown in Figure 11.1. This example illustrates that the auxiliary parameter $\hbar$ provides a convenient way to adjust the convergence region and rate of solution series.

11.2.2 Example 11.2.2

Consider free oscillations governed by

$$\ddot{U}(t) + U(t) + \epsilon\, U^3(t) = 0. \tag{11.46}$$

The exact frequency is

$$\omega = \frac{\pi\sqrt{1+\epsilon a^2/2}}{2K(\mu)}, \tag{11.47}$$

where

$$\mu = -\frac{\epsilon a^2}{2+\epsilon a^2}$$

and $K(\mu)$ is the complete elliptic integral of the first kind.

Under the transformation $\tau = \omega t$ and $U(t) = u(\tau)$, Equation (11.46) becomes

$$\omega^2 u''(\tau) + u(\tau) + \epsilon\, u^3(\tau) = 0. \tag{11.48}$$

All related formulae are the same as those given in §11.1. From (11.23) and (11.46),

$$R_m = \sum_{n=0}^{m-1}\left(\sum_{j=0}^{n}\omega_j\omega_{n-j}\right)u''_{m-1-n} + u_{m-1}$$

$$+\ \epsilon \sum_{n=0}^{m-1} \left(\sum_{j=0}^{n} u_j u_{n-j} \right) u_{m-1-n}. \quad (11.49)$$

When $m = 1$ we obtain from (11.26) the algebraic equation

$$a + \frac{3}{4}\epsilon a^3 - a\omega_0^2 = 0, \quad (11.50)$$

which yields

$$\omega_0 = \sqrt{1 + \frac{3}{4}\epsilon a^2}. \quad (11.51)$$

The frequency ω at the first and second order of approximation is given by

$$\omega \approx \omega_0 + \frac{\hbar(\epsilon a^2)}{128\omega_0^3}\left[2(1-\omega_0^2) + 3\ \epsilon a^2\right], \quad (11.52)$$

and

$$\begin{aligned} \omega \approx \omega_0 + \frac{\hbar(\epsilon a^2)}{32768\omega_0^7} & \left\{1024(\omega_0^4 - \omega_0^6) + 1536\omega_0^4(\epsilon a^2) \right. \\ & - \hbar\left[(576\omega_0^6 - 640\omega_0^4 + 64\omega_0^2) - (940\omega_0^4 - 168\omega_0^2 - 4)(\epsilon a^2)\right. \\ & \left.\left. +\ (84\omega_0^2 + 12)(\epsilon a^2)^2 + 9(\epsilon a^2)^3\right]\right\}, \end{aligned} \quad (11.53)$$

respectively. Note that the series of frequency contains the auxiliary parameter $\hbar$. When $-1 \le \hbar < 0$ the series of frequency converges in the whole region $0 \le \epsilon a^2 < +\infty$. Choosing $\hbar = -1$, we gain the first-order approximation

$$\omega \approx \frac{256 + 384\epsilon a^2 + 141\epsilon^2 a^4}{32(4 + 3\epsilon a^2)^{3/2}} \quad (11.54)$$

and the second-order approximation

$$\omega \approx \frac{131072 + 393216\epsilon a^2 + 440832\epsilon^2 a^4 + 218880\epsilon^3 a^6 + 40599\epsilon^4 a^8}{1024(4 + 3\epsilon a^2)^{7/2}}. \quad (11.55)$$

The maximum error of the first- and second-order approximation is only 0.09% and 0.07% in the whole region $0 \le \epsilon a^2 < +\infty$, respectively! The first-order approximation given by the proposed approach agrees with the exact result, as shown in Figure 11.2.

11.2.3 Example 11.2.3

Consider free oscillations of conservative system with odd nonlinearity, governed by

$$\ddot{U}(t) + U(t) + \epsilon U(t)|U(t)| = 0. \quad (11.56)$$

With transformation $\tau = \omega t$ and $U(t) = u(\tau)$,

$$\omega^2 u''(\tau) + u(\tau) + \epsilon u^2(\tau)\ sign[u(\tau)] = 0, \tag{11.57}$$

which is equivalent to the equation

$$\omega^2 u''(\tau) + u(\tau) + \epsilon u^2(\tau)\ sign[\cos\tau] = 0. \tag{11.58}$$

All related formulae are the same as those given in §11.1. From (11.23) and (11.46),

$$R_m = \sum_{n=0}^{m-1}\left(\sum_{j=0}^{n}\omega_j\omega_{n-j}\right)u''_{m-1-n}(\tau) + u_{m-1}(\tau)$$
$$+ \epsilon\ sign(\cos\tau)\sum_{n=0}^{m-1}u_n(\tau)u_{m-1-n}(\tau). \tag{11.59}$$

When $m = 1$ we obtain from (11.26) the algebraic equation

$$a + \frac{8\epsilon a^2}{3\pi} - a\omega_0^2 = 0, \tag{11.60}$$

which gives

$$\omega_0 = \sqrt{1 + \frac{8\epsilon a}{3\pi}}. \tag{11.61}$$

Note that the solution series for frequency contains the auxiliary parameter $\hbar$. We gain the frequency at the first several orders of approximation and find that the series of frequency is convergent when $-2 \le \hbar < 0$. When $\hbar = -1$, the first order of approximation

$$\omega \approx \sqrt{1 + \frac{8\epsilon a}{3\pi}} - \frac{20,1789,3901,1695}{406,4428,1993,5152}\left(\frac{\epsilon a}{\pi}\right)^2\left(1 + \frac{8\epsilon a}{3\pi}\right)^{-3/2} \tag{11.62}$$

agrees with the numerical results in the whole region $0 \le \epsilon\ a < +\infty$, as shown in Figure 11.3.

11.3 The control of convergence region

As mentioned before, $\hbar = -1$ corresponds to the traditional method of constructing a homotopy. In Example 11.2.2 and Example 11.2.3, using $\hbar = -1$ we obtain accurate approximations valid in the whole regions $0 < \epsilon a^2 < +\infty$ and $0 \le \epsilon a < +\infty$, respectively. In Example 11.2.2 and Example 11.2.3, using

(11.9) as the auxiliary linear operator $\mathcal{L}$, we gain accurate approximations by setting $\hbar = -1$, as shown in Figures 11.2 and 11.3. However, in Example 11.2.1, the series of frequency converges in a fairly small region $0 \leq \epsilon a^2 < 5$ when $\hbar = -1$. Thus, we had to choose $\hbar = -(1 + \epsilon a^2/4)^{-1}$ to adjust its convergence region to ensure that it is valid in the whole region $0 \leq \epsilon a^2 < +\infty$, as shown in Figure 11.1.

Note that the auxiliary linear operator $\mathcal{L}$ defined by (11.9) contains the term ω_0^2. If we replace (11.9) with

$$\mathcal{L}[\Phi(\tau;q)] = \frac{\partial^2 \Phi(\tau;q)}{\partial \tau^2} + \Phi(\tau;q), \tag{11.63}$$

we gain the frequency of Example 11.2.2 at the first order of approximation:

$$\omega \approx \omega_0 + \frac{\hbar(\epsilon a^2)}{128\omega_0}\left[2(1-\omega_0^2) + 3\,\epsilon a^2\right] \tag{11.64}$$

and at the second order of approximation:

$$\begin{aligned}\omega \approx \omega_0 + \frac{\hbar(\epsilon a^2)}{32768\omega_0^5}&\left\{1024(\omega_0^4 - \omega_0^6) + 1536\omega_0^4(\epsilon a^2)\right.\\ &-\hbar\,\omega_0^2\left[(576\omega_0^6 - 640\omega_0^4 + 64\omega_0^2) - (940\omega_0^4 - 168\omega_0^2 - 4)(\epsilon a^2)\right.\\ &\left.\left.+\ (84\omega_0^2 + 12)(\epsilon a^2)^2 + 9(\epsilon a^2)^3\right]\right\},\end{aligned} \tag{11.65}$$

respectively, where ω_0 is defined by (11.51). Unfortunately, when $\hbar = -1$, the above approximations are valid in restricted regions much smaller than those of (11.54) and (11.55) given by the auxiliary linear operator (11.9) and the same value of $\hbar$, as shown in Figure 11.2. In this case we must choose

$$\hbar = -\omega_0^{-2} = -\left(1 + \frac{3}{4}\epsilon a^2\right)^{-1}$$

to adjust the convergence region so that the series of frequency is convergent in the whole region $0 \leq \epsilon a^2 < +\infty$. Using the above expression of $\hbar$, we gain the same results as in (11.54) and (11.55), respectively.

Similarly, when $\hbar = -1$ and using the auxiliary linear operator defined by (11.63), the corresponding series of frequency of Example 11.2.3 also converges in a restricted region much smaller than that of (11.62) given by the auxiliary linear operator (11.9) and the same value of $\hbar$, as shown in Figure 11.3. In this case we must select

$$\hbar = -\left(1 + \frac{8\epsilon a}{3\pi}\right)^{-1}$$

to gain the accurate approximations valid in the whole region $0 \leq \epsilon a < +\infty$. Using the above expression of $\hbar$, we gain exactly the same result as (11.62).

However, using the auxiliary linear operator $\mathcal{L}$ defined by (11.63), we gain the frequency ω of Example 11.2.1 at the first order of approximation:

$$\omega \approx \omega_0 + \frac{\hbar\,\omega_0\,(\epsilon a^2)\,[2+(\epsilon a^2-2)\omega_0^2]}{32(4+\epsilon a^2)}$$

and at the second order of approximation:

$$\begin{aligned}\omega \approx \omega_0 &+ \frac{\hbar\,\omega_0\,(\epsilon a^2)\,[2+(\epsilon a^2-2)\omega_0^2]}{16(4+\epsilon a^2)}\\ &+ \frac{\hbar^2\omega_0(\epsilon a^2)}{6144(4+\epsilon a^2)^2}\left[39\omega_0^4(\epsilon a^2)^3 + 4\omega_0^2(43\omega_0^2+17)(\epsilon a^2)^2\right.\\ &\quad\left.+4(97\omega_0^4+98\omega_0^2-3)(\epsilon a^2) - 192\,(9\omega_0^4-10\omega_0^2+1)\right],\end{aligned}$$

respectively. They are exactly the same as (11.44) and (11.45) when $\hbar = -1$.

These examples illustrate that, for given auxiliary linear operator and auxiliary function, the auxiliary parameter $\hbar$ provides a convenient way to control the convergence region and rate of solution series. The auxiliary parameter $\hbar$ plays an important role in the homotopy analysis method.

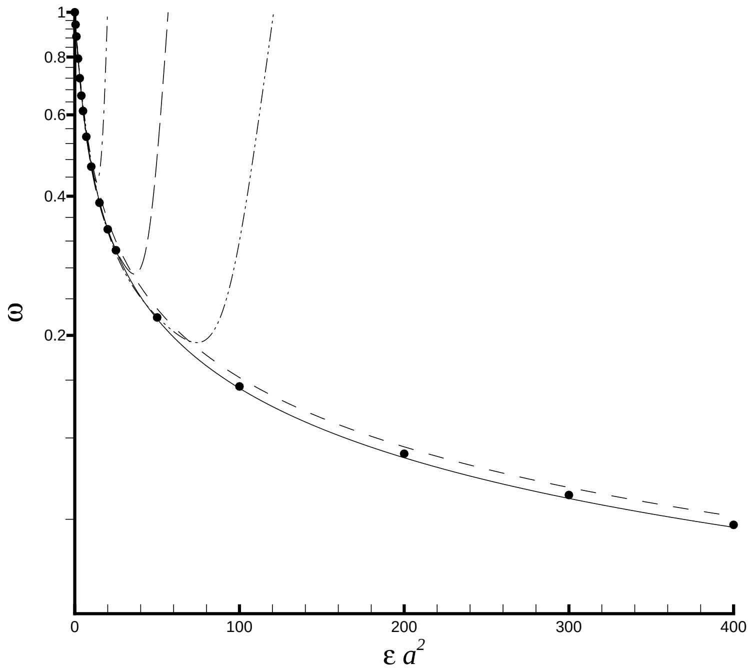

FIGURE 11.1
Comparison of the exact ω with the approximate results of Example 11.2.1 given by the auxiliary linear operator defined by (11.9). Symbols: exact result; dashed line: first-order approximation (11.44) when $\hbar = -(1+\epsilon a^2/4)^{-1}$; solid line: second-order approximation (11.45) when $\hbar = -(1+\epsilon a^2/4)^{-1}$; dash-dotted line: sixth-order approximation when $\hbar = -1/2$; long-dashed line: sixth-order approximation when $\hbar = -1/5$; dash-dot-dotted line: sixth-order approximation when $\hbar = -1/10$.

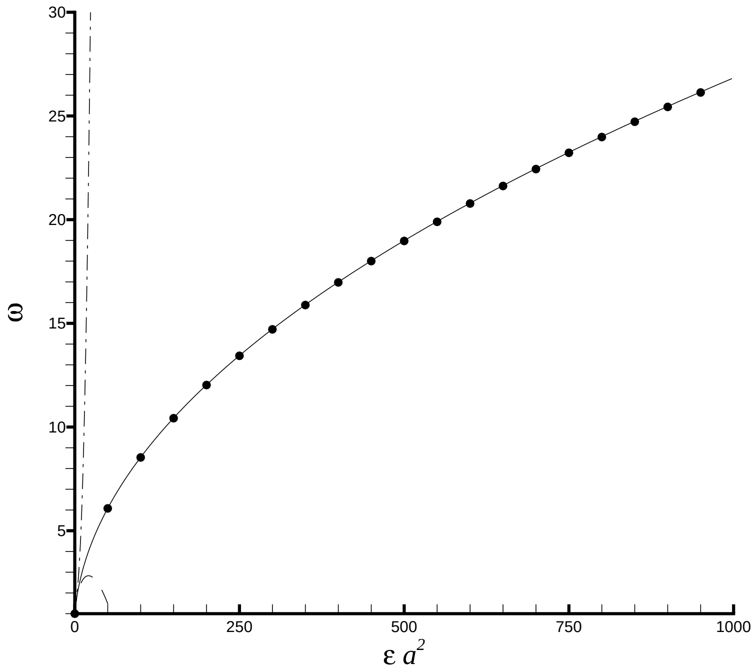

FIGURE 11.2
Comparison of the exact ω with the approximate results of Example 11.2.2 when $\hbar = -1$ by different auxiliary linear operators. Symbols: exact result; solid line: first-order approximation (11.54) given by the auxiliary operator defined by (11.9); dashed line: first-order approximation (11.64) given by the auxiliary linear operator defined by (11.63); dash-dotted line: second-order approximation (11.65) given by the auxiliary linear operator defined by (11.63).

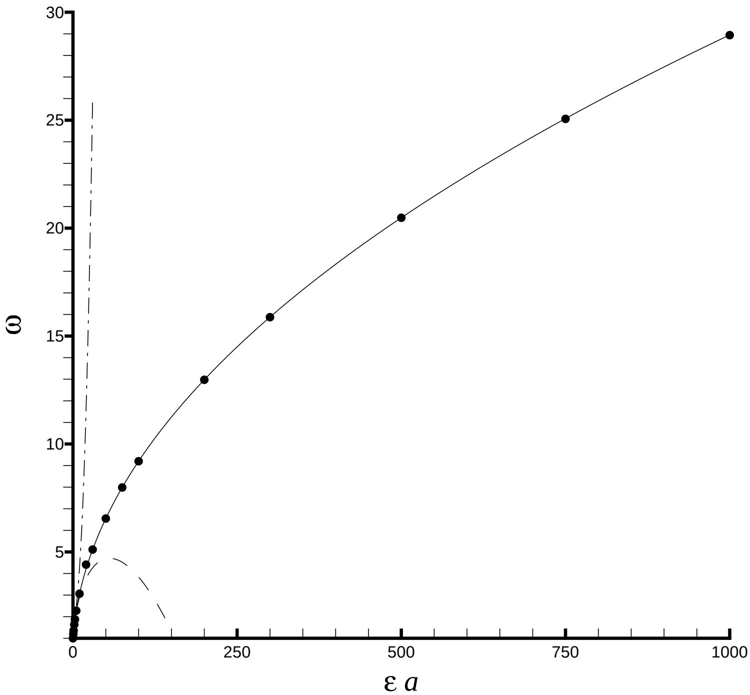

FIGURE 11.3
Comparison of the exact ω with the approximate results of Example 11.2.3 when $\hbar = -1$ by different auxiliary linear operators. Symbols: exact result; solid line: first-order approximation (11.62) given by the auxiliary linear operator defined by (11.9); dashed line: first-order approximation given by the auxiliary linear operator defined by (11.63) of $\mathcal{L}$; dash-dotted line: second-order approximation given by the auxiliary linear operator defined by (11.63).