8

Nonlinear eigenvalue problem

It is often necessary to know eigenvalues and eigenfunctions of a nonlinear problem. In this chapter we illustrate how to obtain eigenvalues and eigenfunctions of a given nonlinear problem by means of the homotopy analysis method.

For example, let us consider an eigenvalue problem governed by

$$u''(x) + \lambda\, u(x) + \epsilon\, u^3(x) = 0, \tag{8.1}$$

subject to the boundary conditions

$$u(0) = u(1) = 0, \tag{8.2}$$

where the prime denotes differentiation with respect to x, ϵ is a parameter. Our object is to find such an eigenvalue λ_n and a normalized eigenfunction $u_n(x)$ such that

$$u_n''(x) + \lambda_n\, u_n(x) + \epsilon\, u_n^3(x) = 0, \tag{8.3}$$

subject to the boundary conditions

$$u_n(0) = u_n(1) = 0 \tag{8.4}$$

and the normalization condition

$$\int_0^1 u_n^2(x)dx = 1, \tag{8.5}$$

where the subscript $n \geq 1$ is an integer. Nayfeh [12] described a perturbation approach to the same problem and gave the first-order perturbation approximation

$$u_n(x) = \sqrt{2}\sin(n\pi x) - \frac{\epsilon\sqrt{2}}{16n^2\pi^2}\sin(3n\pi x) + O(\epsilon^2), \tag{8.6}$$

$$\lambda_n = n^2\pi^2 - \frac{3}{2}\epsilon + O(\epsilon^2), \tag{8.7}$$

valid for small ϵ.

8.1 Homotopy analysis solution

8.1.1 Zero-order deformation equation

From the boundary conditions (8.4) and considering the nonlinearity of Equation (8.3), it is straightforward that the eigenfunction $u_n(x)$ can be expressed by the set of base functions

$$\{\sin[(2k+1)n\pi x] \ | \ n \geq 1, k = 0, 1, 2, 3, \cdots\} \tag{8.8}$$

in the form

$$u_n(x) = \sum_{k=0}^{+\infty} a_{n,k} \ \sin[(2k+1)n\pi x], \tag{8.9}$$

where $a_{n,k}$ is a coefficient. This provides us with the so-called *rule of solution expression.*

Under the *rule of solution expression* denoted by (8.9) and from (8.4) and (8.5), it is straightforward to choose

$$u_{n,0}(x) = \sqrt{2} \ \sin(n\pi x) \tag{8.10}$$

as an initial guess of $u_n(x)$. Furthermore, under the *rule of solution expression* denoted by (8.9) and from Equation (8.3), we choose an auxiliary linear operator

$$\mathcal{L}\Phi = \frac{\partial^2 \Phi}{\partial x^2} + (n\pi)^2 \Phi \tag{8.11}$$

with the property

$$\mathcal{L}\left[C_1 \sin(n\pi x) + C_2 \cos(n\pi x)\right] = 0, \tag{8.12}$$

where C_1 and C_2 are constant coefficients. From Equation (8.3), we define the nonlinear operator

$$\mathcal{N}\left[\Phi(x;q), \Lambda(q)\right] = \frac{\partial^2 \Phi(x;q)}{\partial x^2} + \Lambda(q) \ \Phi(x;q) + \epsilon \ \Phi^3(x;q), \tag{8.13}$$

where $q \in [0,1]$ is an embedding parameter, $\Phi(x;q)$ is a function of x and q, and $\Lambda(q)$ is a function of q, corresponding to $u_n(x)$ and λ_n, respectively. Let $\hbar \neq 0$ denote a nonzero auxiliary parameter and $H(x) \neq 0$ a nonzero auxiliary function. Then, we construct the so-called zero-order deformation equation

$$(1-q) \ \mathcal{L}\left[\Phi(x;q) - u_{n,0}(x)\right] = q \ \hbar \ H(x) \ \mathcal{N}[\Phi(x;q), \Lambda(q)], \tag{8.14}$$

subject to the boundary conditions

$$\Phi(0;q) = \Phi(1;q) = 0. \tag{8.15}$$

When $q = 0$, it is straightforward to show that the zero-order deformation equations (8.14) and (8.15) have the solution

$$\Phi(x; 0) = u_{n,0}(x). \tag{8.16}$$

When $q = 1$, they are equivalent to the original equations (8.3) and (8.4), respectively, provided

$$\Phi(x; 1) = u_n(x), \quad \Lambda(1) = \lambda_n. \tag{8.17}$$

Thus, as the embedding parameter q increases from 0 to 1, $\Phi(x; q)$ varies from the initial guess $u_{n,0}(x)$ to the exact eigenfunction $u_n(x)$, so does $\Lambda(q)$ from the initial guess $\lambda_{n,0}$ to the exact eigenvalue λ_n, respectively. Note that the zero-order deformation equations contain the auxiliary parameter $\hbar$ and the auxiliary function $H(x)$, therefore $\Phi(x; q)$ and $\Lambda(q)$ are dependent of $\hbar$ and $H(x)$. Assume that $\hbar$ and $H(x)$ are properly chosen so that the zero-order deformation equations (8.14) and (8.15) have solutions for all $q \in [0, 1]$, that the terms

$$u_{n,k}(x) = \frac{1}{k!} \frac{\partial^k \Phi(x; q)}{\partial q^k}\bigg|_{q=0}, \qquad \lambda_{n,k} = \frac{1}{k!} \frac{\partial^k \Lambda(q)}{\partial q^k}\bigg|_{q=0} \tag{8.18}$$

exist for $k \geq 1$, and that the Taylor series

$$\Phi(x; q) = \sum_{k=0}^{+\infty} u_{n,k}(x)\, q^k, \tag{8.19}$$

$$\Lambda(q) = \sum_{k=0}^{+\infty} \lambda_{n,k}\, q^k \tag{8.20}$$

are convergent at $q = 1$. Then, using (8.16) and (8.17), we have

$$u_n(x) = u_{n,0}(x) + \sum_{k=1}^{+\infty} u_{n,k}(x), \tag{8.21}$$

$$\lambda_n = \lambda_{n,0} + \sum_{k=1}^{+\infty} \lambda_{n,k}, \tag{8.22}$$

which provide us with the relationships between the exact eigenfunction, eigenvalue, and their initial guesses by $u_{n,k}(x)$ and $\lambda_{n,k}$, respectively.

8.1.2 High-order deformation equation

For brevity, define the vector

$$\vec{u}_{n,m} = \{u_{n,0}(x), u_{n,1}(x), u_{n,2}(x), \cdots, u_{n,m}(x)\}$$

and

$$\vec{\lambda}_{n,m} = \{\lambda_{n,0}, \lambda_{n,1}, \lambda_{n,2}, \cdots, \lambda_{n,m}\}.$$

Differentiating the zero-order deformation equations (8.14) and (8.15) k times with respect to q and then dividing by $k!$ and finally setting $q = 0$, we have the high-order deformation equation

$$\mathcal{L}\left[u_{n,k}(x) - \chi_k\, u_{n,k-1}(x)\right] = \hbar\, H(x)\, R_{n,k}(\vec{u}_{n,k-1}, \vec{\lambda}_{n,k-1}), \tag{8.23}$$

subject to the boundary conditions

$$u_{n,k}(0) = u_{n,k}(1) = 0, \tag{8.24}$$

where χ_k is defined by (2.42) and

$$\begin{aligned} & R_{n,k}(\vec{u}_{n,k-1}, \vec{\lambda}_{n,k-1}) \\ &= u''_{n,k-1}(x) + \sum_{m=0}^{k-1} \lambda_m u_{n,k-1-m}(x) \\ &+ \epsilon \sum_{m=0}^{k-1} u_{n,k-1-m}(x) \sum_{j=0}^{m} u_{n,j}(x) u_{n,m-j}(x). \end{aligned} \tag{8.25}$$

Note that, for any a given integer $n \geq 1$, both $u_{n,k}(x)$ and $\lambda_{n,k-1}$ are unknown for $k \geq 1$, but we have only one differential equation (8.23) for $u_{n,k}(x)$. Thus, the problem is not closed and an additional algebraic equation is needed to determine $\lambda_{n,k-1}$. Note that we have great freedom to choose the auxiliary function $H(x)$. Under the *rule of solution expression* denoted by (8.9), $R_{n,k}(\vec{u}_{n,k-1}, \vec{\lambda}_{n,k-1})$ can be expressed in the form

$$R_{n,k}(\vec{u}_{n,k-1}, \vec{\lambda}_{n,k-1}) = \sum_{m=0}^{M_{n,k}} d_{n,m} \sin[(2m+1)n\pi x],$$

where $d_{n,m}$ is a coefficient, and $M_{n,k}$ is an integer dependent on both n and k. Under the *rule of solution expression* denoted by (8.9) and from (8.11) and (8.23), $H(x)$ can be in the form

$$H(x) = \sin^2[(2m-1)\, n\pi x], \qquad (m \geq 1) \tag{8.26}$$

or

$$H(x) = \sin[(2m)n\pi x], \qquad (m \geq 1) \tag{8.27}$$

or

$$H(x) = \cos^2[(2m-1)\, n\pi x], \qquad (m \geq 1) \tag{8.28}$$

or

$$H(x) = \cos[(2m)n\pi x], \qquad (m \geq 1) \tag{8.29}$$

or even simply

$$H(x) = 1, \tag{8.30}$$

where $m \geq 1$ is an integer. So, it holds

$$H(x)\ R_{n,k}(\vec{u}_{n,k-1}, \vec{\lambda}_{n,k-1}) = \sum_{m=0}^{\mu_{n,k}} b_m^{n,k}(\vec{\lambda}_{k-1})\ \ \sin[(2m+1)n\pi x], \tag{8.31}$$

where $b_m^{n,k}(\vec{\lambda}_{k-1})$ is a coefficient, and $\mu_{n,k}$ is an integer determined by $H(x)$ and the values of n and k. Thus, there exists the term

$$\hbar\, b_0^{n,k}(\vec{\lambda}_{k-1})\ \ \sin(n\pi x)$$

on the right-hand side of the high-order deformation equation (8.23). If

$$b_0^{n,k}(\vec{\lambda}_{k-1}) \neq 0,$$

due to the property (8.12), the solution $u_{n,k}(x)$ contains the term

$$x\ \sin(n\pi x),$$

which disobeys the *rule of solution expression* denoted by (8.9). To avoid this, we had to enforce

$$b_0^{n,k}(\vec{\lambda}_{k-1}) = 0, \tag{8.32}$$

which provides us with an additional equation to determine $\lambda_{n,k-1}$. In this way, the problem is closed.

After solving the above algebraic equation to gain $\lambda_{n,k-1}$, it is easy to get the solution

$$u_{n,k}(x) = \chi_k\ u_{n,k-1}(x) + \sum_{m=1}^{\mu_{n,k}} \frac{\hbar\, b_m^{n,k}(\vec{\lambda}_{k-1})}{n^2\pi^2\,[1-(2m+1)^2]} \sin[(2m+1)n\pi x]$$
$$+ C_1 \sin(n\pi x) + C_2 \cos(n\pi x), \tag{8.33}$$

where C_1 and C_2 are coefficients. Using the *rule of solution expression* denoted by (8.9), we have

$$C_2 = 0.$$

Note that the above solution automatically satisfies the boundary conditions (8.24) so that the coefficient C_1 cannot be determined. However, from the normalization condition (8.5), we have

$$\int_0^1 \left(\sum_{m=0}^{k} u_{n,m}(x) \right)^2 dx = 1, \tag{8.34}$$

which gives an algebraic equation

$$C_1^2 + 4\alpha C_1 + 2\beta = 1, \tag{8.35}$$

where

$$\alpha = \int_0^1 w_{n,k}(x)\sin(n\pi x)dx, \qquad \beta = \int_0^1 w_{n,k}^2(x)dx \tag{8.36}$$

and

$$w_{n,k}(x) = \sum_{j=0}^{k-1} u_{n,j}(x) + \chi_k\, u_{n,k-1}(x) + \sum_{m=1}^{\mu_{n,k}} \frac{\hbar\, b_m^{n,k}(\vec{\lambda}_{k-1})}{n^2\pi^2\,[1-(2m+1)^2]} \sin[(2m+1)n\pi x]. \tag{8.37}$$

Solving Equation (8.35), we have two solutions

$$C_1 = -2\alpha + \sqrt{4\alpha^2 - 2\beta} \tag{8.38}$$

and

$$C_1 = -2\alpha - \sqrt{4\alpha^2 - 2\beta}, \tag{8.39}$$

which correspond to two different eigenfunctions, respectively. In this way, we can gain $\lambda_{n,0}, u_{n,1}(x)$, $\lambda_{n,1}, u_{n,2}(x)$, and so on, successively.

For any given n, the mth-order approximation of the eigenfunction and eigenvalue are given by

$$u_n(x) \approx u_{n,0} + \sum_{k=1}^{m} u_{n,k}(x), \tag{8.40}$$

$$\lambda_n \approx \lambda_{n,0} + \sum_{k=1}^{m} \lambda_{n,k}, \tag{8.41}$$

respectively.

8.1.3 Convergence theorem

THEOREM 8.1
If the series

$$u_{n,0}(x) + \sum_{k=1}^{+\infty} u_{n,k}(x)$$

and

$$\lambda_{n,0} + \sum_{k=1}^{+\infty} \lambda_{n,k}$$

are convergent, where $u_{n,k}(x)$ is governed by Equations (8.23), (8.24), and (8.34) under the definitions (8.11), (8.25), and (2.42), they must be the eigenfunction and eigenvalue of Equations (8.1) and (8.2), respectively.

Proof: If the series of the eigenfunction is convergent, it is necessary that

$$\lim_{m \to +\infty} u_{n,m}(x) = 0.$$

Then, using (8.11), (8.23), and (2.42), we have

$$\begin{aligned}
&\hbar \, H(x) \sum_{k=1}^{+\infty} R_k(\vec{u}_{n,k-1}, \vec{\lambda}_{n,k-1}) \\
&= \lim_{m \to +\infty} \sum_{k=1}^{m} \mathcal{L}[u_{n,k}(x) - \chi_k u_{n,k-1}(x)] \\
&= \mathcal{L}\left\{ \lim_{m \to +\infty} \sum_{k=1}^{m} [u_{n,k}(x) - \chi_k u_{n,k-1}(x)] \right\} \\
&= \mathcal{L}\left[\lim_{m \to +\infty} u_{n,m}(x) \right] \\
&= 0,
\end{aligned}$$

which gives, since $\hbar \neq 0$ and $H(x) \neq 0$,

$$\sum_{k=1}^{+\infty} R_k(\vec{u}_{n,k-1}, \vec{\lambda}_{n,k-1}) = 0.$$

Substituting (8.25) into the above expression and simplifying it, we obtain, due to the convergence of the series of the eigenfunction and eigenvalue, that

$$\frac{d^2}{dx^2} \left[\sum_{k=0}^{+\infty} u_{n,k}(x) \right] + \left(\sum_{m=0}^{+\infty} \lambda_{n,m} \right) \left[\sum_{k=0}^{+\infty} u_{n,k}(x) \right] + \epsilon \left[\sum_{k=0}^{+\infty} u_{n,k}(x) \right]^3 = 0.$$

From (8.10) and (8.24) it holds that

$$\sum_{k=0}^{+\infty} u_{n,k}(0) = \sum_{k=0}^{+\infty} u_{n,k}(1) = 0.$$

Furthermore, from (8.34), the normalization condition (8.5) is satisfied. Thus, as long as the two series are convergent, they must be the eigenfunction $u_n(x)$ and the eigenvalue λ_n of the nonlinear problem governed by Equations (8.3), (8.4), and (8.5). This ends the proof.

8.2 Result analysis

Note that the series (8.21) and (8.22) contain the auxiliary parameter $\hbar$ and the auxiliary function $H(x)$. In particular, for any given values of n and ϵ, the series (8.22) for the eigenvalue is a power series of $\hbar$ so that its convergence region and rate are dependent on $\hbar$. According to Theorem 8.1, we need only focus on the choice of the auxiliary parameter $\hbar$ and the auxiliary function $H(x)$ to ensure that the two series converge. Note that, under the *rule of solution expression* denoted by (8.9) and the *rule of coefficient ergodicity*, the auxiliary functions $H(x)$ can be chosen in many different forms such as those expressed by (8.26) to (8.30). For the sake of simplicity, we first consider the case of $H(x) = 1$. For any given values of n and ϵ, we can investigate the influence of $\hbar$ on the convergence region of the series (8.22) for the eigenvalue by plotting the so-called $\hbar$-curves (see page 26 and §3.5.1) of λ_n versus $\hbar$. For example, the $\hbar$-curves of the eigenvalue λ_1 when $\epsilon = 5, 25, -50$ are as shown in Figure 8.1. Using the $\hbar$-curves, we can easily find out the valid regions of $\hbar$ which ensure that the corresponding series (8.22) converge. From Figure 8.1, it is clear that the series (8.22) of λ_1 when $\epsilon = -50$ converges by means of $\hbar = -1/2$ or $\hbar = -2/5$. This is indeed true, as shown in Table 8.1, and the convergence rate can be accelerated by means of the homotopy-Padé technique (see page 38 and §3.5.2), as shown in Table 8.2. It is found that, as long as the series (8.22) for the eigenvalue is convergent, the series (8.21) for the corresponding eigenfunction also converges in the whole region $0 \le x \le 1$. For example, the approximations of the eigenfunction $u_1(x)$ when $\epsilon = -50$ are as shown in Figure 8.2. In this way, we can gain the convergent eigenvalue and eigenfunction for any given values of n and ϵ. For instance, some convergent analytic results of the eigenvalues are listed in Table 8.3 and some eigenfunctions are as shown in Figures 8.3 and 8.4.

It is found that the valid region of $\hbar$ becomes shorter as the nonlinearity of the problem is stronger, as shown in Figure 8.1. So, as the absolute value of ϵ increases, the value of $\hbar$ had to be chosen closer to zero from the below. It is found that, for $\epsilon < 0$, we can always gain convergent results by means of

$$\hbar = -\frac{1}{\sqrt{1+|\epsilon|}}. \tag{8.42}$$

Using this expression, we can investigate the eigenvalues and eigenfunctions when the nonlinearity becomes very strong. Some eigenvalues for negative ϵ far away from zero are listed in Table 8.4 and some eigenfunctions are as shown in Figures 8.5 to 8.7, respectively. According to these analytic results, it seems that

$$\lim_{\epsilon \to -\infty} \frac{\lambda_n}{\epsilon} = -1 \tag{8.43}$$

and

$$\lim_{\epsilon \to -\infty} u_n(x) = \begin{cases} 1, & 2k/n < x < (2k+1)/n, \\ -1, & (2k+1)/n < x < (2k+2)/n, \end{cases} \tag{8.44}$$

where $n \ge 2$, $k = 0, 1, 2, \cdots, [(n-1)/2]$ and $[x]$ denotes the integer part of x. Note that, when $\epsilon = -10000$, we had to choose a negative value of $\hbar$ with a small enough absolute value so as to ensure that the series (8.21) and (8.22) converge. This indicates once again that the auxiliary parameter $\hbar$ plays an important role in the homotopy analysis method.

It is found that, if (8.38) is used to calculate the coefficient C_1, the corresponding eigenfunction is positive in the region

$$0 < x < 1/n.$$

Let $u_n^+(x)$ denote such kind of eigenfunction. If (8.39) is used, the corresponding eigenfunction is negative in the same region, denoted by $u_n^-(x)$. It also is found that these two kinds of eigenfunctions are symmetrical about the x axis. Thus, for given ϵ and n, there exists a unique eigenvalue λ_n but two eigenfunctions $u_n^+(x)$ and $u_n^-(x)$ satisfying

$$u_n^-(x) = -u_n^+(x).$$

However, the series of the eigenfunction $u_n^-(x)$ converges more slowly than that of the eigenfunction $u_n^+(x)$. This is mainly because $u_n^+(x)$ is closer to the initial guess (8.10). It is found that, if we use the initial guess

$$u_{n,0}(x) = -\sqrt{2}\sin(n\pi x)$$

and employ the formula (8.39), it is easier to get convergent eigenfunctions $u_n^-(x)$. So, by means of the homotopy analysis method we can gain the multiple eigenfunctions of the considered nonlinear problem.

All of the above results are given by means of the auxiliary function $H(x) = 1$. It is found that the other four types of the auxiliary functions denoted by (8.26) to (8.29) can also give convergent results. Using different auxiliary functions, we gain the same eigenvalue and eigenfunction, however, the solution series given by the other four types of the auxiliary functions converge more slowly than those by $H(x) = 1$. It seems that $H(x) = 1$ might be the best auxiliary function for the considered problem, although we cannot prove it.

This example illustrates that the homotopy analysis method can be employed to gain all eigenvalues and eigenfunctions of nonlinear boundary-value problems with very strong nonlinearity.

TABLE 8.1
The analytic approximations of λ_1/π^2 when $\epsilon = -50$ by means of $H(x) = 1$.

order of approximation	$\hbar = -1/2$	$\hbar = -2/5$
5	7.5375343842	7.5399457051
10	7.5384488341	7.5384600578
15	7.5384471198	7.5384473078
20	7.5384471141	7.5384471261
25	7.5384471141	7.5384471146
30	7.5384471141	7.5384471141
35	7.5384471141	7.5384471141
40	7.5384471141	7.5384471141

TABLE 8.2
The $[m, m]$ homotopy-Padé approximant of λ_1/π^2 when $\epsilon = -50$ by means of $H(x) = 1$.

$[m, m]$	$\hbar = -1/2$	$\hbar = -2/5$
$[2, 2]$	7.5407539111	7.5410810211
$[4, 4]$	7.5384474321	7.5384485282
$[6, 6]$	7.5384473644	7.5384480394
$[8, 8]$	7.5384471141	7.5384471141
$[10, 10]$	7.5384471141	7.5384471141
$[12, 12]$	7.5384471141	7.5384471141
$[14, 14]$	7.5384471141	7.5384471141
$[16, 16]$	7.5384471141	7.5384471141
$[18, 18]$	7.5384471141	7.5384471141
$[20, 20]$	7.5384471141	7.5384471141

TABLE 8.3
The analytic value of $\lambda_n/(n\pi)^2$ by means of $\hbar = -1$ and $H(x) = 1$.

ϵ	$n = 1$	$n = 2$	$n = 3$
-25	4.43277	1.91746	1.41524
-20	3.78508	1.73857	1.33324
-15	3.12328	1.55758	1.25074
-10	2.44317	1.37430	1.16771
-5	1.73857	1.18852	1.08414
0	1	1	1
5	0.212582	0.808470	0.915264
10	-0.647567	0.613626	0.829909
15	-1.61838	0.415125	0.743906
20	-2.75608	0.212582	0.657228
25	-4.13061	0.005561	0.569843

TABLE 8.4
The analytic value of λ_n/ϵ by means of $H(x) = 1$.

ϵ	λ_1/ϵ	λ_2/ϵ	λ_3/ϵ
-200	-1.221	-1.488	-1.810
-400	-1.152	-1.325	-1.524
-600	-1.122	-1.272	-1.412
-1000	-1.093	-1.196	-1.310
-2000	-1.065	-1.135	-1.215
-5000	-1.041	-1.083	-1.133
-10000	-1.029	-1.059	-1.090

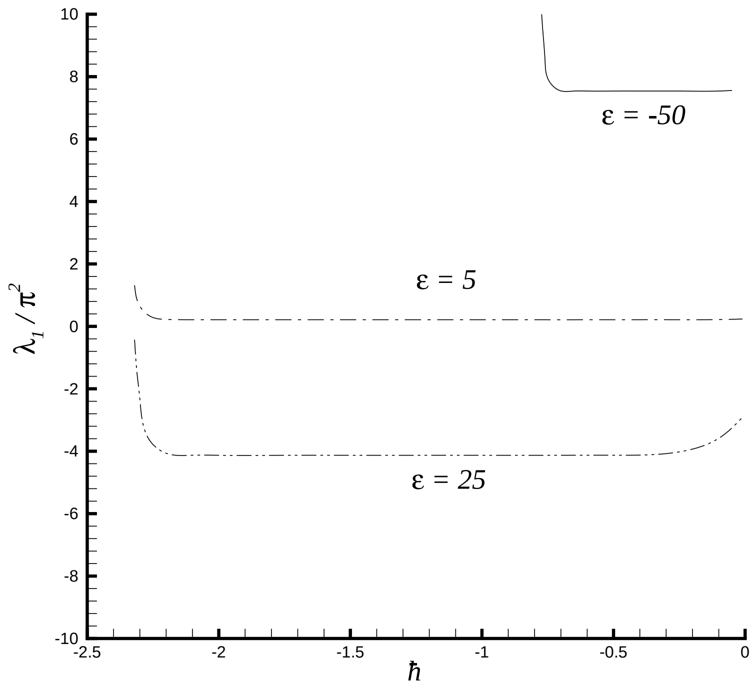

FIGURE 8.1
The $\hbar$-curves of λ_1/π^2 by means of $H(x) = 1$. Dash-dotted line: 20th-order approximation when $\epsilon = 5$; dash-dot-dotted line: 20th-order approximation when $\epsilon = 25$; solid line: 30th-order approximation when $\epsilon = -50$.

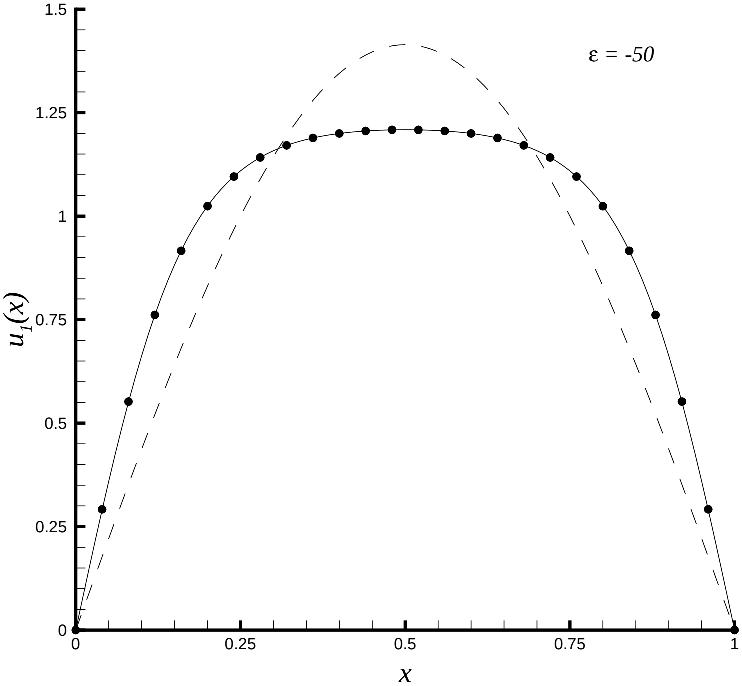

FIGURE 8.2
The analytic approximations of the eigenfunction $u_1(x)$ when $\epsilon = -50$ by means of $\hbar = -1/2$ and $H(x) = 1$. Dashed line: zero-order approximation; solid line: fifth-order approximation; symbols: 10th-order approximation.

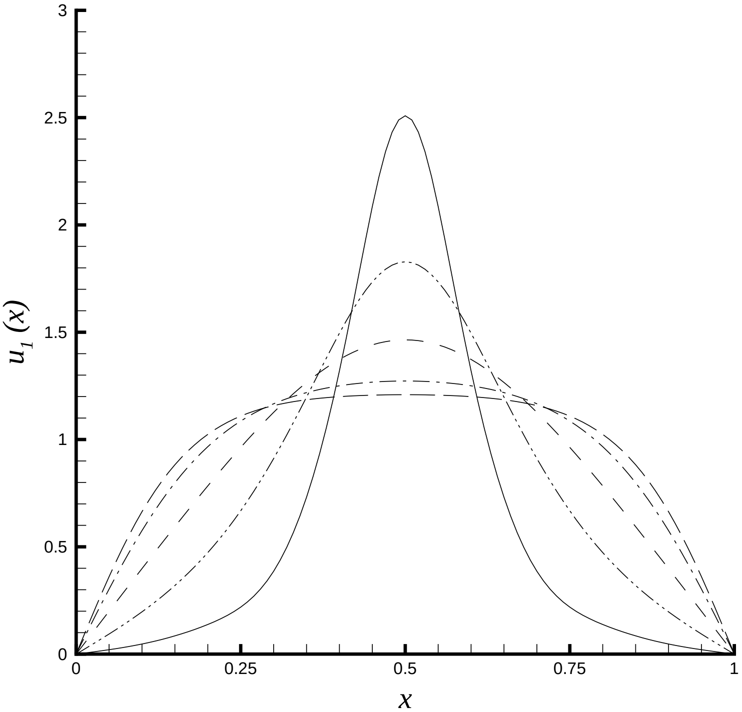

FIGURE 8.3
The convergent analytic results of the eigenfunction $u_1(x)$ by means of $H(x) = 1$. Solid line: 30th-order approximation when $\epsilon = 50$ and $\hbar = -1/2$; dash-dot-dotted line: 10th-order approximation when $\epsilon = 25$ and $\hbar = -1/2$; dashed line: fifth-order approximation when $\epsilon = 5$ and $\hbar = -1$; dash-dotted line: 20th-order approximation when $\epsilon = -25$ and $\hbar = -1/2$; long-dashed line: 20th-order approximation when $\epsilon = -50$ and $\hbar = -1/2$.

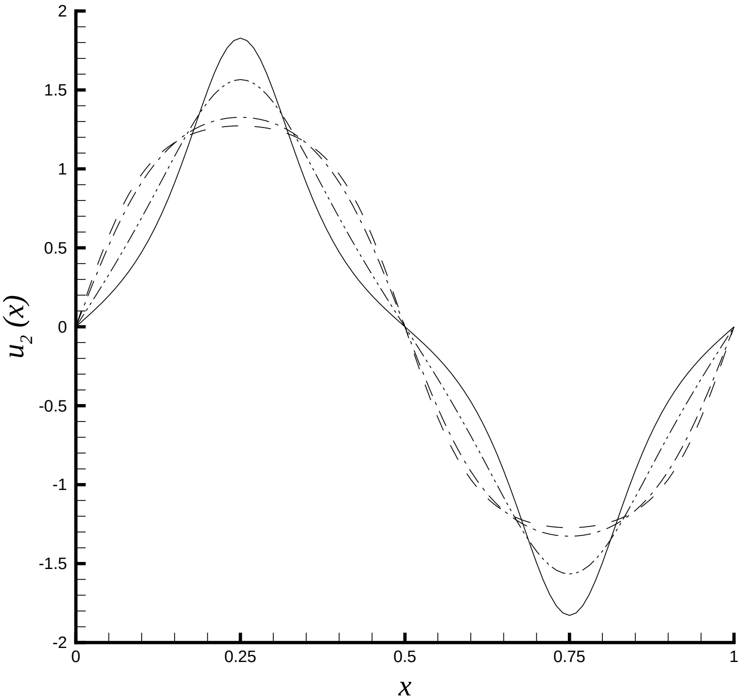

FIGURE 8.4

The convergent analytic results of the eigenfunction $u_2(x)$ by means of $\hbar = -1$ and $H(x) = 1$. Solid line: 10th-order approximation when $\epsilon = 100$; dash-dot-dotted line: fifth-order approximation when $\epsilon = 50$; dash-dotted line: 10th-order approximation when $\epsilon = -50$; dashed line: 20th-order approximation when $\epsilon = -100$.

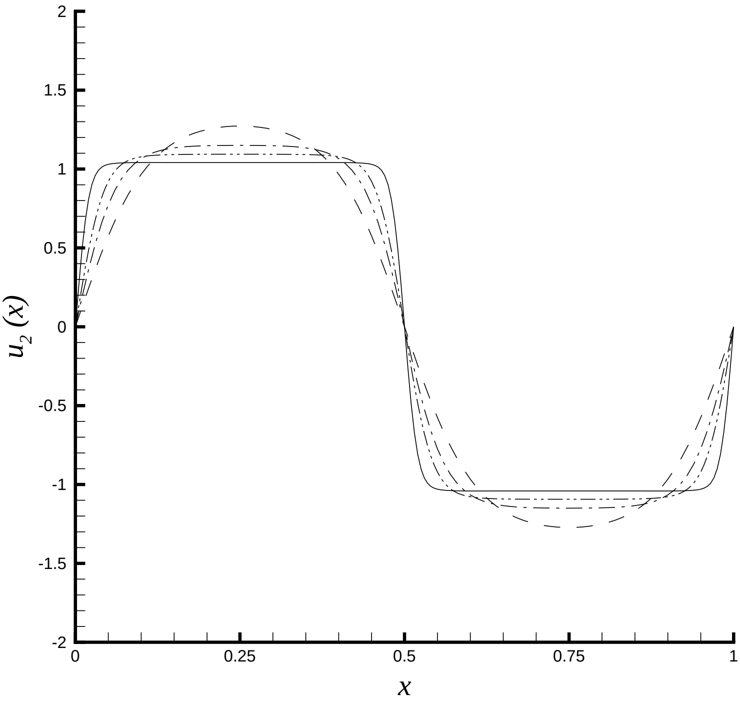

FIGURE 8.5
The convergent analytic result of the eigenfunction $u_2(x)$ by means of $H(x) = 1$. Solid line: 100th-order approximation when $\epsilon = -5000$ and $\hbar = -1/50$; dash-dot-dotted line: 20th-order approximation when $\epsilon = -1000$ and $\hbar = -1/10$; dash-dotted line: 20th-order approximation when $\epsilon = -400$ and $\hbar = -1/4$; dashed line: 20th-order approximation when $\epsilon = -100$ and $\hbar = -1$.

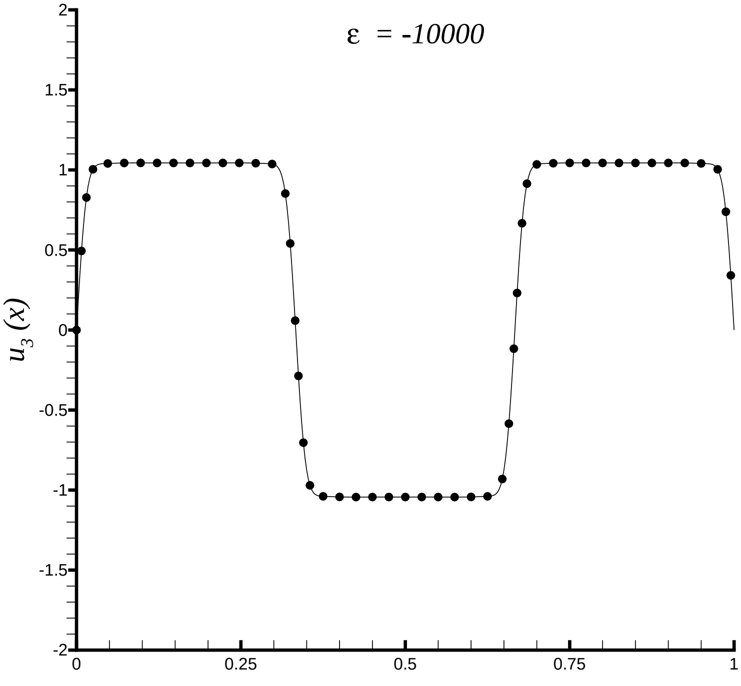

FIGURE 8.6
The analytic result of the eigenfunction $u_3(x)$ when $\epsilon = -10000$ by means of $H(x) = 1$ and $\hbar = -1/50$. Solid line: 70th-order approximation; symbols: 90th-order approximation

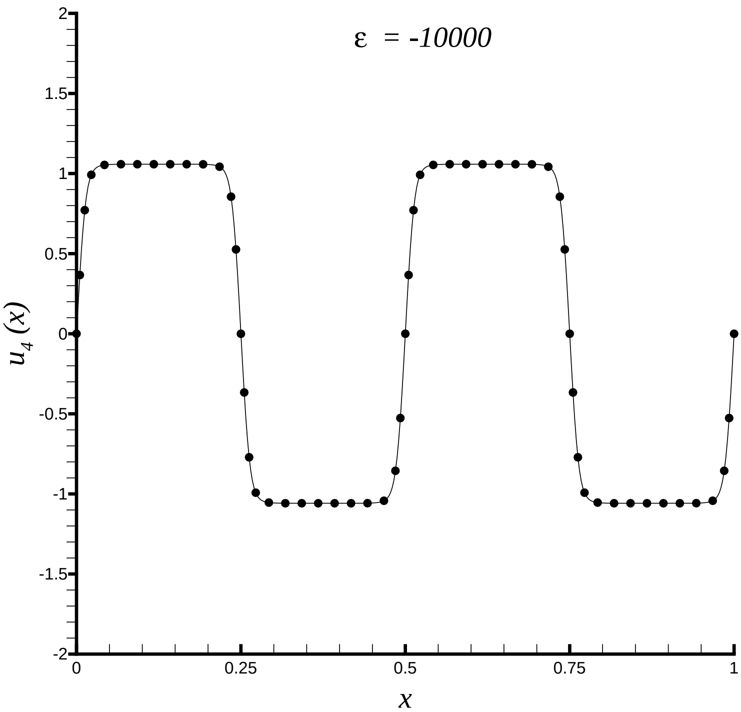

FIGURE 8.7
The analytic result of the eigenfunction $u_4(x)$ when $\epsilon = -10000$ by means of $H(x) = 1$ and $\hbar = -1/20$. Solid line: 40th-order approximation; symbols: 60th-order approximation