7

Multiple solutions of a nonlinear problem

It is well known that many nonlinear problems have multiple solutions. For example, let us consider again the so-called Duffing oscillator in space, governed by

$$v'' + \epsilon(v - v^3) = 0, \tag{7.1}$$

subject to the boundary conditions

$$v(0) = v(\pi) = 0, \tag{7.2}$$

where the prime denotes the derivation with respect to ξ. In Chapter 6, we use the homotopy analysis method to solve the same problem and correctly discover its critical condition $\epsilon = 1$ for the simple bifurcation and express its solution by such a set of base functions

$$\{\sin[(2m+1)\xi] \ \mid \ m \geq 0\}. \tag{7.3}$$

Notice that there exist an infinite number of sets of base functions denoted by

$$\{\sin[(2m+1)\kappa\, \xi] \ \mid \ m \geq 0, \kappa \geq 1\}, \tag{7.4}$$

where $\kappa \geq 1$ is a positive integer, which can be used to express a real function satisfying the boundary conditions (7.2). This implies that Equations (7.1) and (7.2) might have multiple solutions. This is indeed true. We show in this chapter that, using base functions denoted by (7.4), we can gain all multiple solutions of Equations (7.1) and (7.2) by means of the homotopy analysis method.

Without loss of any generality, define

$$A = v(\pi/2\kappa), \qquad v(\xi) = A\, u(\xi). \tag{7.5}$$

Then, Equation (7.1) becomes

$$u'' + \epsilon(u - A^2 u^3) = 0, \qquad u(0) = u(\pi) = 0. \tag{7.6}$$

Note that A is unknown in the above equation. From (7.5), it holds

$$u(\pi/2\kappa) = 1. \tag{7.7}$$

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7.1 Homotopy analysis solution

7.1.1 Zero-order deformation equation

Using the base functions (7.4) and the boundary conditions $u(0) = u(\pi) = 0$ and considering the nonlinearity of Equation (7.6), we express the solution $u(\xi)$ in the form

$$u(\xi) = \sum_{m=0}^{+\infty} c_m \sin[(2m+1)\kappa\xi], \tag{7.8}$$

where c_m is a coefficient. This provides us with the so-called *rule of solution expression*.

Under the *rule of solution expression* denoted by (7.8) and using (7.7), it is straightforward to choose

$$u_0(\xi) = \sin(\kappa\, \xi) \tag{7.9}$$

as an initial guess of $u(\xi)$, where $\kappa \geq 1$ is an integer. To obey the *rule of solution expression* denoted by (7.8), we choose an auxiliary linear operator

$$\mathcal{L}[\Phi(\xi;q)] = \frac{\partial^2 \Phi(\xi;q)}{\partial \xi^2} + \kappa^2\ \Phi(\xi;q) \tag{7.10}$$

such that

$$\mathcal{L}[C_1 \sin(\kappa\, \xi) + C_2 \cos(\kappa\, \xi)] = 0, \tag{7.11}$$

where C_1 and C_2 are coefficients. Furthermore, from Equation (7.6), we define the nonlinear operator

$$\mathcal{N}[\Phi(\xi;q), \alpha(q)] = \frac{\partial^2 \Phi(\xi;q)}{\partial \xi^2} + \epsilon\ \left[\Phi(\xi;q) - \alpha^2(q)\Phi^3(\xi;q)\right], \tag{7.12}$$

where $q \in [0,1]$ is the embedding parameter, $\Phi(\xi;q)$ is an unknown function of ξ and q, $\alpha(q)$ is an unknown function dependent upon q. Let $\hbar \neq 0$ denote an auxiliary parameter and $H(\xi) \neq 0$ an auxiliary function. We construct the so-called zero-order deformation equation

$$(1-q)\mathcal{L}[\Phi(\xi;q) - u_0(\xi)] = \hbar\ q\ H(\xi)\ \mathcal{N}[\Phi(\xi;q), \alpha(q)], \tag{7.13}$$

subject to the boundary conditions

$$\Phi(0;q) = \Phi(\pi;q) = 0. \tag{7.14}$$

When $q = 0$, the solution of Equations (7.13) and (7.14) is

$$\Phi(\xi;0) = u_0(\xi), \qquad \xi \in [0,\pi]. \tag{7.15}$$

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When $q = 1$, Equations (7.13) and (7.14) are equivalent to Equations (7.6), provided

$$\Phi(\xi;1) = u(\xi), \qquad \alpha(1) = A. \tag{7.16}$$

Thus, $\Phi(\xi;q)$ varies (or deforms) from the initial approximation $u_0(\xi) = \sin(\kappa\xi)$ to the exact solution $u(\xi)$ of Equations (7.6), as does $\alpha(q)$ from its initial approximation A_0 to the exact value $A = u(\pi/2\kappa)$. Note that the zero-order deformation equation (7.13) contains the auxiliary parameter $\hbar$ and the auxiliary function $H(\xi)$. Assume that $\hbar$ and $H(\xi)$ are properly chosen so that the zero-order deformation equations (7.13) and (7.14) have solutions for all $q \in [0,1]$ and that the terms

$$u_m(\xi) = \frac{1}{m!} \left. \frac{\partial^m \Phi(\xi;q)}{\partial q^m} \right|_{q=0}, \qquad A_m = \frac{1}{m!} \left. \frac{d^m \alpha(q)}{dq^m} \right|_{q=0} \tag{7.17}$$

exist for $m \geq 1$. Then, by Taylor's theorem and using (7.15), we can expand $\Phi(\xi;q)$ and $\alpha(q)$ in power series of q as follows

$$\Phi(\xi;q) = u_0(\xi) + \sum_{m=1}^{+\infty} u_m(\xi)\, q^m, \tag{7.18}$$

$$\alpha(q) = A_0 + \sum_{m=1}^{+\infty} A_m\, q^m. \tag{7.19}$$

Furthermore, assuming that $\hbar$ and $H(\xi)$ are so properly chosen that the power series (7.18) and (7.19) are convergent at $q = 1$, we have from (7.16) the solution series

$$u(\xi) = u_0(\xi) + \sum_{m=1}^{+\infty} u_m(\xi), \tag{7.20}$$

$$A = A_0 + \sum_{m=1}^{+\infty} A_m. \tag{7.21}$$

7.1.2 High-order deformation equation

For brevity, write

$$\vec{u}_k = \{u_0(\xi), u_1(\xi), u_2(\xi), \cdots, u_k(\xi)\}, \qquad \vec{A}_k = \{A_0, A_1, A_2, \cdots, A_k\}.$$

Differentiating the zero-order deformation equations (7.13) and (7.14) m times with respect to q and then dividing them by $m!$ and finally setting $q = 0$, we have the high-order deformation equation

$$\mathcal{L}\left[u_m(\xi) - \chi_m u_{m-1}(\xi)\right] = \hbar\, H(\xi)\, R_m(\vec{u}_{m-1}, \vec{A}_{m-1}), \tag{7.22}$$

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subject to the boundary conditions

$$u_m(0) = u_m(\pi) = 0, \tag{7.23}$$

where χ_m is defined by (2.42) and

$$R_m(\vec{u}_{m-1}, \vec{A}_{m-1}) = \frac{1}{(m-1)!} \left. \frac{\partial^{m-1} \mathcal{N}[\Phi(\xi;q), \alpha(q)]}{\partial q^{m-1}} \right|_{q=0}$$
$$= u''_{m-1}(\xi) + \epsilon\, u_{m-1}(\xi)$$
$$- \epsilon \sum_{n=0}^{m-1} \left(\sum_{i=0}^{n} A_i\, A_{n-i} \right) \left[\sum_{j=0}^{m-1-n} u_j(\xi) \sum_{r=0}^{m-1-n-j} u_r(\xi) u_{m-1-n-j-r}(\xi) \right]. \tag{7.24}$$

Note that both $u_m(\xi)$ and A_{m-1} are unknown, but we have only one differential equation for $u_m(\xi)$. So, the problem is not closed and an additional algebraic equation is needed to determine A_{m-1}. Assume that $H(\xi)$ is properly chosen so that the right-hand side term of the high-order deformation equation (7.22) can be expressed by

$$\hbar\, H(\xi)\, R_m(\vec{u}_{m-1}, \vec{A}_{m-1}) = \sum_{n=0}^{\mu_m} b_{m,n}(\vec{A}_{m-1})\ \sin[(2n+1)\kappa\, \xi], \tag{7.25}$$

where $b_{m,n}(\vec{A}_{m-1})$ is a coefficient and the positive integer μ_m depends upon $H(\xi)$ and m. According to the property (7.11) of $\mathcal{L}$, when $b_{m,0}(\vec{A}_{m-1}) \neq 0$, the solution of the mth-order deformation equation (7.22) contains the term

$$\xi\ \sin(\kappa\, \xi),$$

which disobeys the *rule of solution expression* denoted by (7.8). To avoid this, we had to enforce

$$b_{m,0}(\vec{A}_{m-1}) = 0, \tag{7.26}$$

which provides us with an additional algebraic equation for A_{m-1}. In this way, the problem is closed. Thereafter, it is easy to gain the solution of Equation (7.22), say,

$$u_m(\xi) = \chi_m\, u_{m-1}(\xi) + \sum_{n=1}^{\mu_m} \frac{b_{m,n}}{[1-(2n+1)^2\kappa^2]}\ \sin[(2n+1)\kappa\, \xi]$$
$$+ C_1\ \sin(\kappa\xi) + C_2\ \cos(\kappa\xi), \tag{7.27}$$

where C_1 and C_2 are coefficients. Under the *rule of solution expression* denoted by (7.8), C_2 must be zero. Note that the coefficient C_1 cannot be determined by the boundary conditions (7.23), which is automatically satisfied when $C_2 = 0$. However, from (7.7), it should hold

$$u_m(\pi/2\kappa) = 0, \tag{7.28}$$

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which uniquely determines C_1. In this way, we gain A_{m-1} and $u_m(\xi)$ successively.

At the Nth-order of approximation, we have

$$u(\xi) \approx u_0(\xi) + \sum_{m=1}^{N} u_m(\xi), \tag{7.29}$$

$$A \approx A_0 + \sum_{m=1}^{N-1} A_m. \tag{7.30}$$

7.1.3 Convergence theorem

THEOREM 7.1
If the solution series (7.20) and (7.21) are convergent, where $u_k(\xi)$ is governed by Equations (7.22) and (7.23) under the definitions (7.24) and (2.42), they must be the exact solution of Equations (7.6).

Proof: If the solution series (7.20) is convergent, it is necessary that

$$\lim_{m->+\infty} u_m(\xi) = 0, \qquad \xi \in [0, \pi].$$

From (7.10), (2.42), and (7.22) and using the above expression, we have

$$\begin{aligned}
&\hbar\, H(\xi) \sum_{k=1}^{+\infty} R_k(\vec{u}_{k-1}, \vec{A}_{k-1}) \\
&= \lim_{m\to+\infty} \sum_{k=1}^{m} \mathcal{L}[u_k(\xi) - \chi_k\, u_{k-1}(\xi)] \\
&= \mathcal{L}\left\{ \lim_{m\to+\infty} \sum_{k=1}^{m} [u_k(\xi) - \chi_k\, u_{k-1}(\xi)] \right\} \\
&= \mathcal{L}\left[\lim_{m\to+\infty} u_m(\xi) \right] \\
&= 0.
\end{aligned}$$

Since $\hbar \neq 0$ and $H(\xi) \neq 0$, the above expression gives

$$\sum_{k=1}^{+\infty} R_k(\vec{u}_{k-1}, \vec{A}_{k-1}) = 0.$$

Substituting (7.24) into the above expression and simplifying it, we have, due to the convergence of the series (7.20) and (7.21), that

$$\frac{d^2}{d\xi^2}\left[\sum_{k=0}^{+\infty} u_k(\xi)\right] + \epsilon \left\{ \left[\sum_{k=0}^{+\infty} u_k(\xi)\right] - \left(\sum_{m=0}^{+\infty} A_m\right)^2 \left[\sum_{k=0}^{+\infty} u_k(\xi)\right]^3 \right\} = 0.$$

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Using (7.9) and (7.23), it holds

$$\sum_{k=0}^{+\infty} u_k(0) = \sum_{k=0}^{+\infty} u_k(\pi) = 0.$$

Thus, as long as the solution series (7.20) and (7.21) are convergent, they must be the exact solution of Equations (7.6). This ends the proof.

7.2 Result analysis

According to Theorem 7.1, we need only to properly choose an auxiliary function $H(\xi)$ and an auxiliary parameter $\hbar$ to ensure that the solution series (7.20) and (7.21) converge. As pointed out in Chapter 6, the auxiliary function $H(\xi)$ can be chosen in many different forms without disobeying the *rule of coefficient ergodicity*. For the sake of simplicity, we choose here

$$H(\xi) = 1. \tag{7.31}$$

Then, using (7.9) and (7.24), we have

$$\begin{aligned} &\hbar\ H(\xi)\ R_1(\vec{u}_0, \vec{A}_0) \\ &= \hbar \left(\epsilon - \kappa^2 - \frac{3}{4}\epsilon\ A_0^2\right) \sin(\kappa\ \xi) + \frac{1}{4}\hbar\ \epsilon\ A_0^2 \sin(3\kappa\ \xi), \end{aligned} \tag{7.32}$$

which gives according to (7.25) that

$$b_{1,0} = \hbar \left(\epsilon - \kappa^2 - \frac{3}{4}\epsilon\ A_0^2\right), \qquad b_{1,1} = \frac{1}{4}\hbar\ \epsilon\ A_0^2.$$

Thus, from Equation (7.26), we have an algebraic equation

$$\epsilon - \kappa^2 - \frac{3}{4}\epsilon\ A_0^2 = 0, \tag{7.33}$$

which has the nonzero solution

$$A_0 = \pm\frac{2}{\sqrt{3}}\sqrt{1 - \frac{\kappa^2}{\epsilon}} \tag{7.34}$$

when $\epsilon > \kappa^2$. Thus, for any a positive integer $\kappa \geq 1$, the so-called bifurcation occurs when

$$\epsilon = \kappa^2. \tag{7.35}$$

This critical condition of bifurcations indicates that there exist multiple bifurcation points for large ϵ. Note that κ determines the set of base functions denoted by (7.4). So, for large ϵ, there exist multiple solutions.

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Without the loss of generality, we consider here the two cases of $\kappa = 2$ and $\kappa = 3$. Note that the convergence region and rate of the solution series (7.20) and (7.21) are determined by the auxiliary parameter $\hbar$. For a given ϵ and a positive integer κ, where $\epsilon > \kappa^2 \geq 1$, we can always find, by means of plotting the so-called $\hbar$-curves (see page 26 and §3.5.1) of A, a valid region of $\hbar$ to ensure that the solution series (7.21) converges. For example, the $\hbar$-curves of $A \sim \hbar$ when $\kappa = 2, \epsilon = 10, 25, 100$, and $\kappa = 3, \epsilon = 40, 90, 225$ are as shown in Figures 7.1 and 7.2, respectively. From these $\hbar$-curves, it is clear that the series (7.21) converges when $\epsilon = 10$ and $\kappa = 2, 3$ by means of $\hbar = -1$, or $\epsilon = 40, \kappa = 2$ and $\epsilon = 90, \kappa = 3$ by means of $\hbar = -1/2$, or $\epsilon = 100, \kappa = 2$ and $\epsilon = 225, \kappa = 3$ by means of $\hbar = -1/5$. This is indeed true, as shown in Tables 7.1 and 7.2.

From Figures 7.1 and 7.2, the so-called valid region of $\hbar$ for A decreases as ϵ increases for a given κ. So, the absolute value of $\hbar$ should decrease as ϵ increases. It is found that, for any a given ϵ and a given κ satisfying $\epsilon > \kappa^2 \geq 1$, the series (7.21) is always convergent in the region

$$\kappa^2 \leq \epsilon < +\infty,$$

when

$$\hbar = -\left(1 + \frac{\epsilon}{3\kappa^2}\right)^{-1}. \tag{7.36}$$

Besides, the corresponding 10th-order approximation

$$\begin{aligned} A \approx \pm \left(1 + \frac{\epsilon}{3\kappa^2}\right)^{-10} \sqrt{1 - \frac{\kappa^2}{\epsilon}} \; \Bigg(& 1.1803 + 3.9075\, \frac{\epsilon}{\kappa^2} + 5.8128\, \frac{\epsilon^2}{\kappa^4} \\ & + 5.1149\, \frac{\epsilon^3}{\kappa^6} + 2.9466\, \frac{\epsilon^4}{\kappa^8} + 1.1603\, \frac{\epsilon^5}{\kappa^1 0} + 0.31602\, \frac{\epsilon^6}{\kappa^{12}} \\ & + 5.8726 \times 10^{-2}\, \frac{\epsilon^7}{\kappa^{14}} + 7.1298 \times 10^{-3}\, \frac{\epsilon^8}{\kappa^{16}} + 5.1396 \times 10^{-4}\, \frac{\epsilon^9}{\kappa^{18}} \\ & + 1.7001 \times 10^{-5}\, \frac{\epsilon^{10}}{\kappa^{20}} \Bigg) \end{aligned} \tag{7.37}$$

agrees well in the whole region $\kappa^2 \leq \epsilon < +\infty$ with the exact analytic result given by the implicit formula

$$\epsilon = \frac{8\kappa^2}{\pi^2(2 - A^2)}\, K\left(\frac{A^2}{2 - A^2}\right), \tag{7.38}$$

where K denotes the complete elliptic integral of the first kind, as shown in Figure 7.3. In fact, Figure 7.3 provides us with a complete bifurcation diagram of the so-called Duffing oscillator in space problem.

Using the so-called homotopy-Padé technique (see page 38 and §3.5.2), we can greatly accelerate the convergence of the series (7.21), as shown in Tables 7.3 and 7.4. It is found that the $[m, m]$ homotopy-Padé approximant

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does not depend upon the auxiliary parameter $\hbar$. The $[4,4]$ homotopy-Padé approximant

$$A \approx 2\sqrt{3}\sqrt{1-\frac{\kappa^2}{\epsilon}}\ \frac{P(\epsilon)}{Q(\epsilon)} \tag{7.39}$$

gives an accurate approximation of A in the whole region $1 \le \epsilon/\kappa^2 < +\infty$, where

$$\begin{aligned}
P(\epsilon) &= 8665210296046039923 + 2500964782519057396\left(\frac{\epsilon}{\kappa^2}\right) \\
&\quad + 604034298653768562\left(\frac{\epsilon}{\kappa^2}\right)^2 + 62408285303687028\left(\frac{\epsilon}{\kappa^2}\right)^3 \\
&\quad + 3874319809940915\left(\frac{\epsilon}{\kappa^2}\right)^4, \\
Q(\epsilon) &= 2543093833757545089 + 7921677254280814588\left(\frac{\epsilon}{\kappa^2}\right) \\
&\quad + 1930521704826790758\left(\frac{\epsilon}{\kappa^2}\right)^2 + 213027971364041596\left(\frac{\epsilon}{\kappa^2}\right)^3 \\
&\quad + 13310678950379441\left(\frac{\epsilon}{\kappa^2}\right)^4.
\end{aligned}$$

It is found that, as long as the series (7.21) of A is convergent, the corresponding series (7.20) of $u(\xi)$ given by the same value of $\hbar$ converges in the whole region $\xi \in [0,\pi]$, as shown in Figures 7.4 and 7.5. Due to the odd nonlinearity of Equation (7.6), if $u(\xi)$ is a solution, $-u(\xi)$ must be also a solution. However, for brevity, we do not give this kind of solution in Figures 7.4 and 7.5. The nonlinear problem of the so-called Duffing oscillator in space has multiple solutions for large ϵ. For example, when $\epsilon = 10$, there exist two nonzero solutions corresponding to $\kappa = 1$, two nonzero solutions to $\kappa = 2$, and two nonzero solutions to $\kappa = 3$, respectively, so that there are six nonzero solutions. In general, for any given $\epsilon \ge 1$, the problem of the so-called Duffing oscillator in space has $2[\sqrt{\epsilon}]$ nonzero solutions, where $[x]$ denotes the integer part of x. Therefore, the larger the value of ϵ, the more the multiple solutions, as shown in Figure 7.3. As ϵ tends to infinity, there exists an infinite number of solutions. Therefore, the nonlinear equation (7.1) with boundary conditions (7.2) contains rather rich mathematical structure and a complicated bifurcation diagram.

The *rule of solution expression* plays an important role in finding these multiple solutions. This example clearly indicates that, by means of different base functions, we can employ the homotopy analysis method to gain all multiple solutions of some nonlinear problems. Indeed, the so-called *rule of solution expression* of the homotopy analysis method provides us with a new viewpoint and a different starting point to investigate nonlinear problems.

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TABLE 7.1
The analytic approximations of A when $\epsilon = 10$ and $\kappa = 2, 3$ by means of $\hbar = -1$ and $H(\xi) = 1$.

Order of approximation	$\kappa = 2$	$\kappa = 3$
2	0.8694142054	0.3643175731
4	0.8696932532	0.3643100899
6	0.8696857656	0.3643100194
8	0.8696860265	0.3643100187
10	0.8696860164	0.3643100187
12	0.8696860168	0.3643100187
14	0.8696860168	0.3643100187
16	0.8696860168	0.3643100187
18	0.8696860168	0.3643100187
20	0.8696860168	0.3643100187

TABLE 7.2
The analytic approximations of A when $\epsilon = 40, \kappa = 2$ and $\epsilon = 90, \kappa = 3$ by means of $\hbar = -1/2$ and $H(\xi) = 1$.

Order of approximation	$\epsilon = 40, \kappa = 2$	$\epsilon = 90, \kappa = 3$
2	0.98070	0.98171
4	0.99912	0.99854
6	0.99613	0.99639
8	0.99634	0.99625
10	0.99656	0.99658
12	0.99635	0.99635
14	0.99649	0.99648
16	0.99641	0.99642
18	0.99645	0.99644
20	0.99643	0.99644
22	0.99644	0.99644
24	0.99644	0.99644
26	0.99644	0.99644
28	0.99644	0.99644
30	0.99644	0.99644

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TABLE 7.3
The $[m, m]$ homotopy-Padé approximant of A when $\epsilon = 10$ and $\kappa = 2, 3$ by means of $H(\xi) = 1$.

$[m, m]$	$\kappa = 2$	$\kappa = 3$
$[1, 1]$	0.8694029457	0.3643104636
$[2, 2]$	0.8696902377	0.3643100178
$[3, 3]$	0.8696859569	0.3643100187
$[4, 4]$	0.8696860176	0.3643100187
$[5, 5]$	0.8696860168	0.3643100187
$[6, 6]$	0.8696860168	0.3643100187
$[7, 7]$	0.8696860168	0.3643100187
$[8, 8]$	0.8696860168	0.3643100187
$[9, 9]$	0.8696860168	0.3643100187
$[10, 10]$	0.8696860168	0.3643100187

TABLE 7.4
The $[m, m]$ homotopy-Padé approximant of A when $\epsilon = 40, \kappa = 2$ and $\epsilon = 90, \kappa = 3$ by means of $H(\xi) = 1$.

$[m, m]$	$\epsilon = 40, \kappa = 2$	$\epsilon = 90, \kappa = 3$
$[1, 1]$	0.9747449855	0.9753179745
$[2, 2]$	0.9988803766	0.9988250895
$[3, 3]$	0.9960551840	0.9960761350
$[4, 4]$	0.9964957766	0.9964921829
$[5, 5]$	0.9964304420	0.9964305571
$[6, 6]$	0.9964370766	0.9964368336
$[7, 7]$	0.9964352860	0.9964352709
$[8, 8]$	0.9964353707	0.9964353614
$[9, 9]$	0.9964353314	0.9964353314
$[10, 10]$	0.9964353352	0.9964353355
$[11, 11]$	0.9964353351	0.9964353363
$[12, 12]$	0.9964353362	0.9964353363
$[13, 13]$	0.9964353363	0.9964353363
$[14, 14]$	0.9964353363	0.9964353363
$[15, 15]$	0.9964353363	0.9964353363

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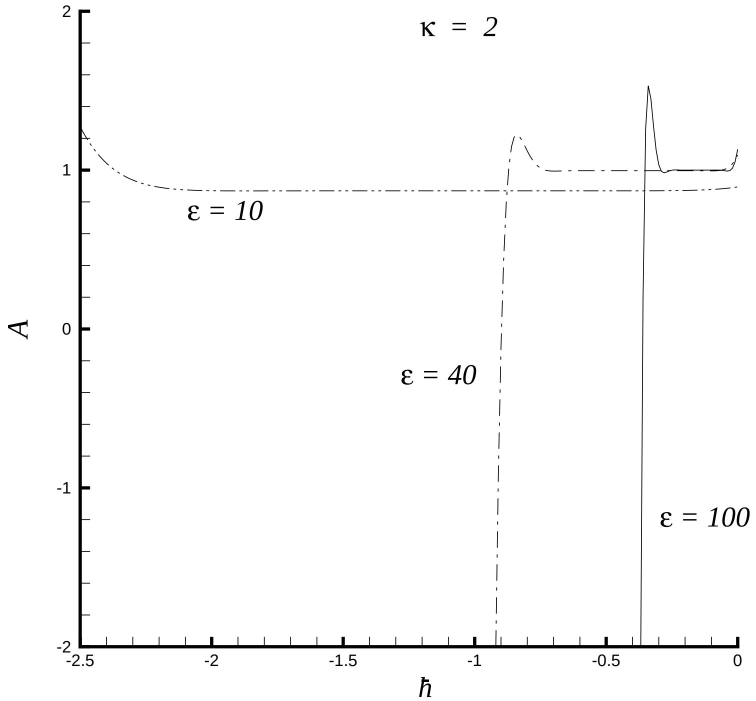

FIGURE 7.1

The $\hbar$-curves of A when $\kappa = 2$ and $\epsilon = 10, 40, 100$ by means of $H(\xi) = 1$. Dash-dot-dotted line: 20th-order approximation of A when $\epsilon = 10$; dash-dotted line: 20th-order approximation of A when $\epsilon = 40$; solid line: 20th-order approximation of A when $\epsilon = 100$.

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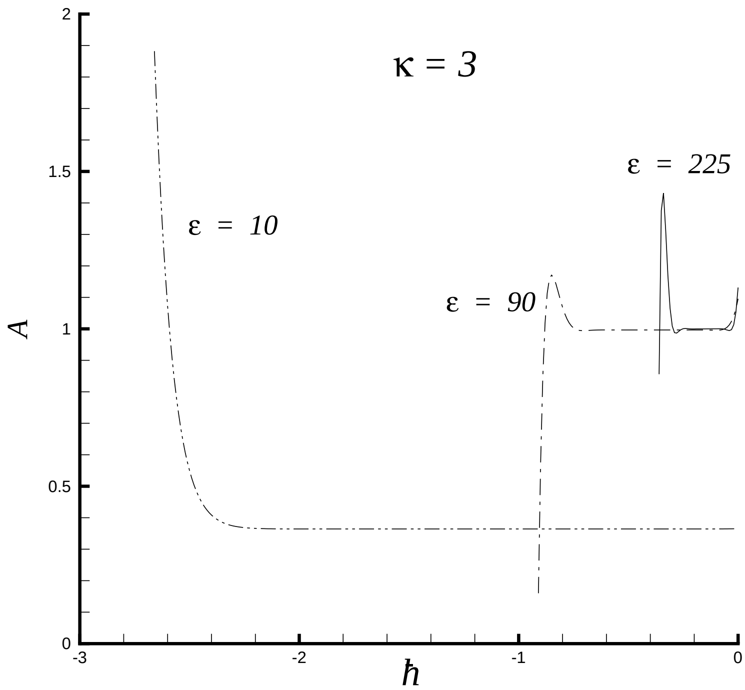

FIGURE 7.2
The $\hbar$-curves of A when $\kappa = 3$ and $\epsilon = 10, 90, 225$ by means of $H(\xi) = 1$. Dash-dot-dotted line: 20th-order approximation of A when $\epsilon = 10$; dash-dotted line: 20th-order approximation of A when $\epsilon = 90$; solid line: 20th-order approximation of A when $\epsilon = 225$.

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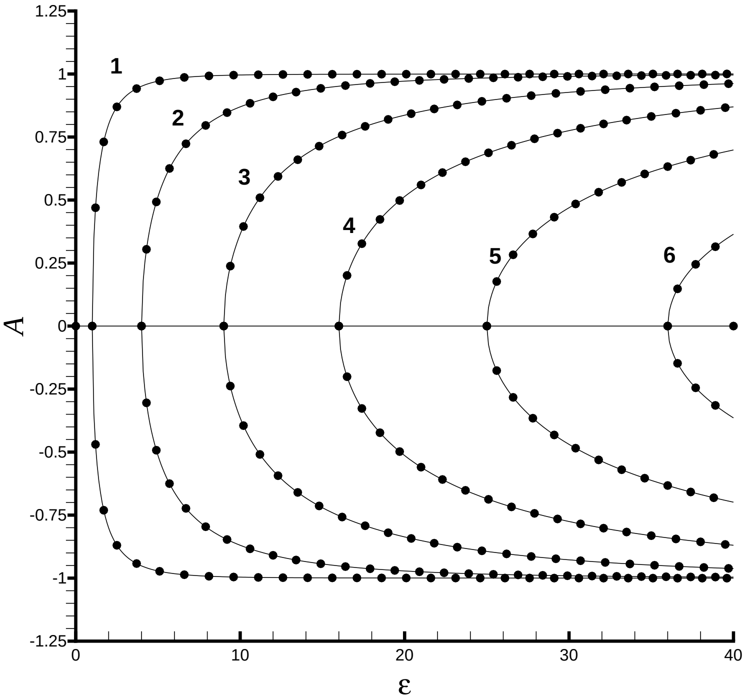

FIGURE 7.3
The comparison of the 10th-order approximation (7.37) of A with the exact implicit solution (7.38). Symbols: exact result; curve 1: $\kappa = 1$; curve 2: $\kappa = 2$; curve 3: $\kappa = 3$; curve 4: $\kappa = 4$; curve 5: $\kappa = 5$; curve 6: $\kappa = 6$.

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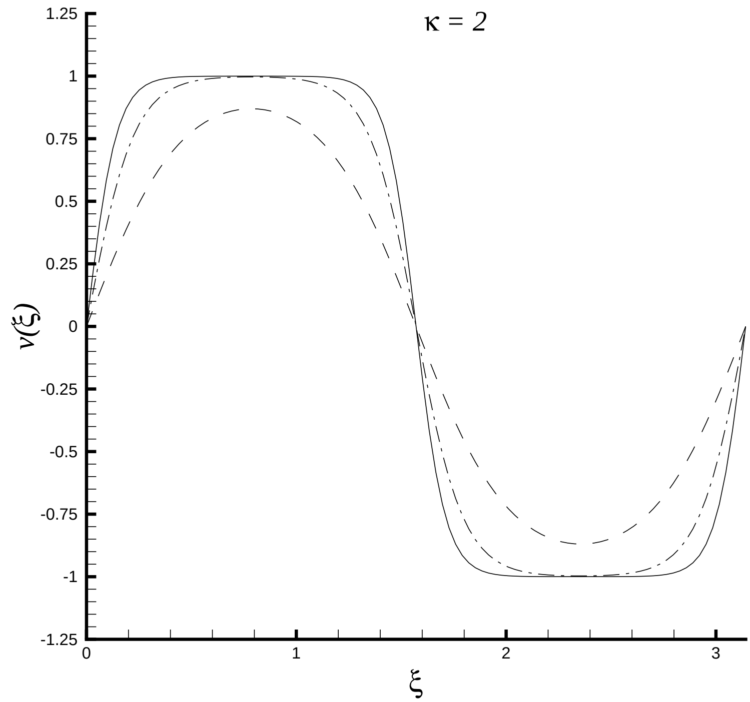

FIGURE 7.4
The convergent analytic result of $v(\xi) = Au(\xi)$ when $\kappa = 2$ and $\epsilon = 10, 40, 100$ by means of $H(\xi) = 1$. Dashed line: fifth-order approximation of $v(\xi)$ when $\epsilon = 10$ by means of $\hbar = -1$; dash-dotted line: 10th-order approximation of $v(\xi)$ when $\epsilon = 40$ by means of $\hbar = -1/2$; solid line: 20th-order approximation of $v(\xi)$ when $\epsilon = 100$ by means of $\hbar = -1/5$.

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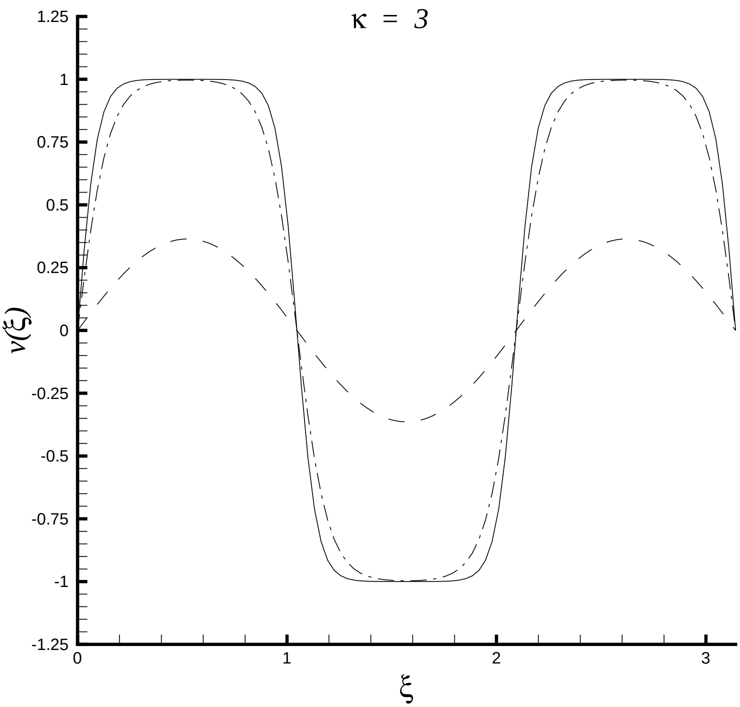

FIGURE 7.5

The convergent analytic result of $v(\xi) = Au(\xi)$ when $\kappa = 3$ and $\epsilon = 10, 90, 225$ by means of $H(\xi) = 1$. Dashed line: fifth-order approximation of $v(\xi)$ when $\epsilon = 10$ by means of $\hbar = -1$; dash-dotted line: 10th-order approximation of $v(\xi)$ when $\epsilon = 90$ by means of $\hbar = -1/2$; solid line: 20th-order approximation of $v(\xi)$ when $\epsilon = 225$ by means of $\hbar = -1/5$.

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