PART II

APPLICATIONS

Great straightness seems bent;
Great skill seems awkward;
Great eloquence seems tongue-tied.

Lao Tzu, an ancient Chinese philosopher

6

Simple bifurcation of a nonlinear problem

Consider a nonlinear problem of the so-called Duffing oscillator in space (see Kahn and Zarmi [11], page 198) governed by

$$w''(x) + w(x) - w^3(x) = 0, \qquad w(0) = w(L) = 0, \tag{6.1}$$

where x is a spatial variable, $w(x)$ is a real function of x defined in the region $0 \leq x \leq L$, and the prime denotes the derivation. Obviously,

$$w(x) = 0$$

satisfies all of the above equations and thus is one of its solutions. However, for some values of L, there exist nonzero solutions so that the so-called simple bifurcation occurs.

Under the transformation

$$x = \left(\frac{L}{\pi}\right) \xi, \qquad \epsilon = \left(\frac{L}{\pi}\right)^2, \qquad v(\xi) = w(x), \tag{6.2}$$

Equation (6.1) becomes

$$v'' + \epsilon(v - v^3) = 0, \qquad v(0) = v(\pi) = 0, \tag{6.3}$$

where the prime denotes the derivation with respect to ξ.

For any $\epsilon \geq 0$, the above equation has the solution $v(\xi) = 0$. The so-called bifurcation occurs when a nonzero solution of Equation (6.3) exists for some values of ϵ. Thus, we focus on the nonzero solution of Equation (6.3) and the critical condition of its existence. Obviously, if $v(\xi)$ is a nonzero solution of Equation (6.3), then $-v(\xi)$ must be its solution as well. Without loss of any generality, define

$$A = v(\pi/2), \qquad v(\xi) = A\, u(\xi). \tag{6.4}$$

Substituting the above expressions into Equation (6.3), we have

$$u'' + \epsilon(u - A^2 u^3) = 0, \qquad u(0) = u(\pi) = 0. \tag{6.5}$$

Note that A is unknown in the above equation and it holds from (6.4) that

$$u(\pi/2) = 1. \tag{6.6}$$

According to Kahn and Zarmi [11], the exact relation between A and L is given by

$$L = 2 \int_0^A \frac{dz}{\sqrt{A^2 - z^2 - (A^4 - z^4)/2}},$$

which gives the exact solution

$$\frac{L}{\pi} = \frac{2}{\pi \sqrt{1 - A^2/2}} K\left(\frac{A^2}{2 - A^2}\right), \tag{6.7}$$

where $K(\zeta)$ is the complete elliptic integral of the first kind. According to the above exact solution, $\epsilon = (L/\pi)^2$ tends to infinity as $|A|$ approaches to 1. By means of the method of normal forms, Kahn and Zarmi [11] gave the perturbation solution

$$A \approx \pm 2\sqrt{\frac{\epsilon - 1}{3}}, \qquad \epsilon \geq 1, \tag{6.8}$$

which breaks down for large ϵ. In this chapter we employ the homotopy analysis method to solve the nonlinear boundary-value problem with simple bifurcations.

6.1 Homotopy analysis solution

6.1.1 Zero-order deformation equation

Using the boundary conditions $u(0) = u(\pi) = 0$ and considering the nonlinearity of Equation (6.5), it is straightforward to express the solution $u(\xi)$ by a set of base functions

$$\{\sin[(2m+1)\xi] \mid m \geq 0\} \tag{6.9}$$

such that

$$u(\xi) = \sum_{m=0}^{+\infty} c_m \sin[(2m+1)\xi], \tag{6.10}$$

where c_m is a coefficient. This provides us with the so-called *rule of solution expression.*

Under the *rule of solution expression* denoted by (6.10) and using (6.6), it is straightforward to choose

$$u_0(\xi) = \sin(\xi) \tag{6.11}$$

as the initial guess of $u(\xi)$. Under the *rule of solution expression* denoted by (6.10), we choose an auxiliary linear operator

$$\mathcal{L}[\Phi(\xi;q)] = \frac{\partial^2\Phi(\xi;q)}{\partial\xi^2} + \Phi(\xi;q) \tag{6.12}$$

with the property

$$\mathcal{L}[C_1 \sin\xi + C_2 \cos\xi] = 0, \tag{6.13}$$

where C_1 and C_2 are coefficients. From Equation (6.5), we define a nonlinear operator

$$\mathcal{N}[\Phi(\xi;q), \alpha(q)] = \frac{\partial^2\Phi(\xi;q)}{\partial\xi^2} + \epsilon\ \left[\Phi(\xi;q) - \alpha^2(q)\Phi^3(\xi;q)\right], \tag{6.14}$$

where $q \in [0,1]$ is the embedding parameter and $\alpha(q)$ is an unknown function dependent upon q. Let $\hbar \neq 0$ denote an auxiliary parameter and $H(\xi)$ an auxiliary function. We construct the so-called zero-order deformation equation

$$(1-q)\ \mathcal{L}[\Phi(\xi;q) - u_0(\xi)] = \hbar\ q\ H(\xi)\ \mathcal{N}[\Phi(\xi;q), \alpha(q)], \tag{6.15}$$

subject to the boundary conditions

$$\Phi(0;q) = \Phi(\pi;q) = 0. \tag{6.16}$$

Obviously, when $q = 0$, the solution of Equations (6.15) and (6.16) is

$$\Phi(\xi;0) = u_0(\xi). \tag{6.17}$$

When $q = 1$, Equations (6.15) and (6.16) are exactly the same as the original equations (6.5), provided

$$\Phi(\xi;1) = u(\xi), \qquad \alpha(1) = A. \tag{6.18}$$

Thus, $\Phi(\xi;q)$ varies (or deforms) from the initial approximation $u_0(\xi) = \sin\xi$ to the exact solution $u(\xi)$ of Equations (6.5), as does $\alpha(q)$ from its initial approximation A_0 to the exact value $A = u(\pi/2)$.

Note that the zero-order deformation equation (6.15) contains the auxiliary parameter $\hbar$ and the auxiliary function $H(\xi)$. Assume that $\hbar$ and $H(\xi)$ are properly chosen so that the zero-order deformation equations (6.15) and (6.16) have solutions for all $q \in [0,1]$, and that there exist the derivatives

$$u_m(\xi) = \frac{1}{m!}\ \frac{\partial^m\Phi(\xi;q)}{\partial q^m}\bigg|_{q=0}, \qquad A_m = \frac{1}{m!}\ \frac{d^m\alpha(q)}{dq^m}\bigg|_{q=0}. \tag{6.19}$$

Then, using Taylor's theorem and Equation (6.17), we can expand $\Phi(\xi;q)$ and $\alpha(q)$ in power series of q as follows

$$\Phi(\xi;q) = u_0(\xi) + \sum_{m=1}^{+\infty} u_m(\xi)\ q^m, \tag{6.20}$$

$$\alpha(q) = A_0 + \sum_{m=1}^{+\infty} A_m\ q^m, \tag{6.21}$$

respectively. Furthermore, assuming that $\hbar$ and $H(\xi)$ are so properly chosen that the power series (6.20) and (6.21) are convergent at $q = 1$, we have using (6.18) the solution series

$$u(\xi) = u_0(\xi) + \sum_{m=1}^{+\infty} u_m(\xi), \tag{6.22}$$

$$A = A_0 + \sum_{m=1}^{+\infty} A_m. \tag{6.23}$$

6.1.2 High-order deformation equation

For simplicity, define the vectors

$$\vec{u}_k = \{u_0(\xi), u_1(\xi), u_2(\xi), \cdots, u_k(\xi)\}, \qquad \vec{A}_k = \{A_0, A_1, A_2, \cdots, A_k\}.$$

Differentiating the zero-order deformation equations (6.15) and (6.16) m times with respect to the embedding parameter q and then dividing them by $m!$ and finally setting $q = 0$, we have the high-order deformation equation

$$\mathcal{L}\left[u_m(\xi) - \chi_m u_{m-1}(\xi)\right] = \hbar\, H(\xi)\, R_m(\vec{u}_{m-1}, \vec{A}_{m-1}), \tag{6.24}$$

subject to the boundary conditions

$$u_m(0) = u_m(\pi) = 0, \tag{6.25}$$

where χ_m is defined by (2.42) and

$$\begin{aligned} R_m(\vec{u}_{m-1}, \vec{A}_{m-1}) &= \frac{1}{(m-1)!} \left. \frac{\partial^{m-1} \mathcal{N}[\Phi(\xi;q), \alpha(q)]}{\partial q^{m-1}} \right|_{q=0} \\ &= u''_{m-1}(\xi) + \epsilon\, u_{m-1}(\xi) \\ &- \epsilon \sum_{n=0}^{m-1} \left(\sum_{i=0}^{n} A_i\, A_{n-i} \right) \left[\sum_{j=0}^{m-1-n} u_j(\xi) \sum_{r=0}^{m-1-n-j} u_r(\xi) u_{m-1-n-j-r}(\xi) \right]. \end{aligned} \tag{6.26}$$

Note that there are two unknowns $u_m(\xi)$ and A_{m-1}, but we have only one differential equation for $u_m(\xi)$. The problem is therefore not closed and an additional algebraic equation is needed to determine A_{m-1}. Considering the *rule of solution expression* denoted by (6.10) and the property (6.13) of the auxiliary linear operator $\mathcal{L}$, $H(\xi)$ should be properly chosen so that the right-hand side term of the high-order deformation equation (6.24) can be expressed by

$$\hbar\, H(\xi)\, R_m(\vec{u}_{m-1}, \vec{A}_{m-1}) = \sum_{n=0}^{\mu_m} b_{m,n}(\vec{A}_{m-1})\, \sin[(2n+1)\xi], \tag{6.27}$$

where $b_{m,n}(\vec{A}_{m-1})$ is a coefficient and the positive integer μ_m depends upon m and the auxiliary function $H(\xi)$. Then, according to the property (6.13), if $b_{m,0}(\vec{A}_{m-1}) \neq 0$, the solution of the mth-order deformation equation (6.24) contains the term

$$\xi \ \sin \xi,$$

which disobeys the *rule of solution expression* denoted by (6.10). To avoid this, we had to enforce

$$b_{m,0}(\vec{A}_{m-1}) = 0, \tag{6.28}$$

which provides us with an additional algebraic equation for A_{m-1}. In this way, the problem is closed. Thereafter, it is easy to gain the solution of Equation (6.24)

$$\begin{aligned} u_m(\xi) = \chi_m \ u_{m-1}(\xi) - \sum_{n=1}^{\mu_m} \frac{b_{m,n}}{4n(n+1)} \ \sin[(2n+1)\xi] \\ + C_1 \ \sin \xi + C_2 \ \cos \xi, \end{aligned} \tag{6.29}$$

where C_1 and C_2 are coefficients. Under the *rule of solution expression* denoted by (6.10), C_2 must be zero. However, the coefficient C_1 cannot be determined by the boundary conditions (6.25), which is automatically satisfied when $C_2 = 0$. But, from Equation (6.6), it holds that

$$u_m(\pi/2) = 0, \tag{6.30}$$

which uniquely determines the value of C_1. In this way, we gain A_{m-1} and $u_m(\xi)$ successively. At the Nth-order of approximation, we have

$$u(\xi) \approx u_0(\xi) + \sum_{m=1}^{N} u_m(\xi), \tag{6.31}$$

$$A \approx A_0 + \sum_{m=1}^{N-1} A_m. \tag{6.32}$$

6.1.3 Convergence theorem

THEOREM 6.1

If the solution series (6.22) and (6.23) are convergent, where $u_k(\xi)$ is governed by Equations (6.24) and (6.25) under the definitions (6.26) and (2.42), they must be the exact solution of Equations (6.5).

Proof: If the solution series (6.22) is convergent, it is necessary that

$$\lim_{m->+\infty} u_m(\xi) = 0, \qquad \xi \in [0, \pi].$$

Using (6.12) and (2.42) and from (6.24), we have

$$\begin{aligned}
&\hbar\, H(\xi) \sum_{k=1}^{+\infty} R_k(\vec{u}_{k-1}, \vec{A}_{k-1}) \\
&= \lim_{m\to+\infty} \sum_{k=1}^{m} \mathcal{L}[u_k(\xi) - \chi_k\, u_{k-1}(\xi)] \\
&= \mathcal{L}\left\{ \lim_{m\to+\infty} \sum_{k=1}^{m} [u_k(\xi) - \chi_k\, u_{k-1}(\xi)] \right\} \\
&= \mathcal{L}\left[\lim_{m\to+\infty} u_m(\xi) \right] \\
&= 0,
\end{aligned}$$

which gives, since $\hbar \neq 0$ and $H(\xi) \neq 0$,

$$\sum_{k=1}^{+\infty} R_k(\vec{u}_{k-1}, \vec{A}_{k-1}) = 0.$$

Substituting (6.26) into the above expression and simplifying it, we have, due to the convergence of the series (6.22) and (6.23), that

$$\frac{d^2}{d\xi^2}\left[\sum_{k=0}^{+\infty} u_k(\xi)\right] + \epsilon \left\{ \left[\sum_{k=0}^{+\infty} u_k(\xi)\right] - \left(\sum_{m=0}^{+\infty} A_m\right)^2 \left[\sum_{k=0}^{+\infty} u_k(\xi)\right]^3 \right\} = 0.$$

From (6.11) and (6.25), it holds that

$$\sum_{k=0}^{+\infty} u_k(0) = \sum_{k=0}^{+\infty} u_k(\pi) = 0.$$

Thus, as long as the solution series (6.22) and (6.23) are convergent, they must be the exact solution of Equations (6.5). This ends the proof.

6.2 Result analysis

According to Theorem 6.1, we only need choose a proper auxiliary parameter $\hbar$ and a auxiliary function $H(\xi)$ to ensure that the solution series (6.22) and (6.23) are convergent. Note that, under the *rule of solution expression* denoted by (6.10) and without disobeying the *rule of coefficient ergodicity*, different auxiliary functions such as

$$H(\xi) = 1,\ H(\xi) = \sin^2(\xi),\ H(\xi) = \cos^2(\xi),\ H(\xi) = \cos(2\xi)$$

and so on can be used. However, for the sake of simplicity, we choose here

$$H(\xi) = 1. \tag{6.33}$$

In this case, using (6.11) and (6.26), we have

$$\hbar\, H(\xi)\; R_1(\vec{u}_0, \vec{A}_0) = \hbar\, \left(\epsilon - 1 - \frac{3}{4}\epsilon\, A_0^2\right) \sin\xi + \frac{1}{4}\,\hbar\,\epsilon\, A_0^2 \sin(3\xi), \tag{6.34}$$

which gives from Equation (6.27) that

$$b_{1,0} = \hbar\, \left(\epsilon - 1 - \frac{3}{4}\epsilon\, A_0^2\right), \qquad b_{1,1} = \frac{1}{4}\,\hbar\,\epsilon\, A_0^2.$$

Thus, when $m = 1$, we have from Equation (6.28) an additional algebraic equation

$$\epsilon - 1 - \frac{3}{4}\epsilon\, A_0^2 = 0. \tag{6.35}$$

Since $\epsilon = (L/\pi)^2 \geq 0$, the above equation has no solution when $\epsilon < 1$. Thus, there does not exist a nonzero solution when $0 \leq \epsilon \leq 1$. However, when $\epsilon > 1$, Equation (6.35) has the solution

$$A_0 = \pm\, \frac{2}{\sqrt{3}}\sqrt{1 - \frac{1}{\epsilon}}, \qquad \epsilon > 1. \tag{6.36}$$

Therefore, the so-called simple bifurcation occurs at $\epsilon = 1$. The homotopy analysis method correctly provides us with the critical condition of the simple bifurcation of the considered nonlinear problem.

It should be emphasized that the homotopy analysis method provides us with two families of solution expressions in the auxiliary parameter $\hbar$, and $\hbar$ influences the convergence of the solution series (6.22) and (6.23). In particular, the series (6.23) of A is a power series of $\hbar$. To investigate the influence of $\hbar$ on the solution series (6.23), we plot the so-called $\hbar$-curve (see page 26 and §3.5.1) of A for any a given ϵ. For example, the $\hbar$-curves $A \sim \hbar$ when $\epsilon = 10$ and $\epsilon = 25$ are as shown in Figure 6.1, which clearly indicate the corresponding valid regions of $\hbar$. From Figure 6.1, the solution series (6.23) when $\epsilon = 10$ converges if $-3/4 \leq \hbar < 0$. When $\epsilon = 25$, it converges if $-1/4 \leq \hbar < 0$. So, by means of Theorem 6.1 and using the $\hbar$-curves, it is very clear that when $\epsilon = 10$ and $\epsilon = 25$ the solution series (6.23) is convergent to the exact value if we choose $\hbar$ in the corresponding valid region of $\hbar$, i.e., $-3/4 \leq \hbar < 0$ or $-1/4 \leq \hbar < 0$, respectively. For example, the approximations of A when $\epsilon = 10, \hbar = -1/2$ and $\epsilon = 25, \hbar = -1/5$ are listed in Table 6.1. It is found that, in general, as long as the solution series (6.23) is convergent, the corresponding solution series (6.22) of $u(\xi)$ given by the same auxiliary parameter $\hbar$ also converges in the whole region $0 \leq \xi \leq \pi$. For example, the 10th-order approximation of $u(\xi)$ when $\epsilon = 10, \hbar = -1/2$ and the 30th-order approximation of $u(\xi)$ when $\epsilon = 25, \hbar = -1/5$ agree well with the exact solution of $u(\xi)$, respectively, as shown in Figure 6.2. So, using the $\hbar$-curves, it is convenient to find out the valid region of $\hbar$ to ensure that the solution series (6.22) and (6.23) converge.

It is found that the mth-order approximation of A can be expressed by

$$A \approx \pm \sqrt{3(1-\epsilon^{-1})} \sum_{k=0}^{m} \beta_{m,k}(\hbar)\, \epsilon^k, \tag{6.37}$$

where $\beta_{m,k}$ is a coefficient dependent upon $\hbar$. From Figure 6.1, it is clear that as ϵ enlarges the corresponding valid region of $\hbar$ becomes smaller. It is found that the convergence region of the solution series of A is governed by $\hbar$, as shown in Figure 6.3. Clearly, the closer the value of $\hbar$ is to zero from below ($\hbar < 0$), the larger the convergence region of A becomes. This implies that $\hbar$ should be a function of ϵ, whose absolution value should decrease as ϵ increases. It is found that, when

$$\hbar = -\frac{1}{1+\epsilon/3}, \tag{6.38}$$

the 10th-order of approximation of A, i.e.,

$$\begin{aligned} A \approx \pm \frac{1}{(1+\epsilon/3)^{10}} \sqrt{1-\frac{1}{\epsilon}}\, \big(&1.1803 + 3.9075\,\epsilon + 5.8128\,\epsilon^2 + 5.1149\,\epsilon^3 \\ &+ 2.9466\,\epsilon^4 + 1.1603\,\epsilon^5 + 0.31602\,\epsilon^6 + 5.8726\times 10^{-2}\,\epsilon^7 \\ &+ 7.1298\times 10^{-3}\,\epsilon^8 + 5.1396\times 10^{-4}\,\epsilon^9 + 1.7001\times 10^{-5}\,\epsilon^{10}\big), \end{aligned} \tag{6.39}$$

agrees well with the exact result in the *whole* region $1 \le \epsilon < +\infty$, as shown in Figure 6.3. The 10th-order approximation (6.39) of A gives

$$\lim_{\epsilon \to +\infty} |A| = 1.0039,$$

corresponding to a relative error of 0.39%. Using (6.38), even the third-order approximation of A, i.e.

$$A \approx \pm \frac{(7015\,\epsilon^3 + 70251\,\epsilon^2 + 220917\,\epsilon + 226105)}{4096\sqrt{3}(\epsilon+3)^3} \sqrt{1-\frac{1}{\epsilon}} \tag{6.40}$$

agrees well with the exact result, as shown in Figure 6.4. So, it is the auxiliary parameter $\hbar$ which provides us with a convenient way to control and adjust the convergence region and rate of solutions series. Thus, the auxiliary parameter $\hbar$ indeed plays an important role within the frame of the homotopy analysis method.

It should be emphasized that the *rule of solution expression* denoted by (6.10) also plays an important role within the frame of the homotopy analysis method. It is under the *rule of solution expression* that the initial approximation (6.11) and the auxiliary linear operator (6.12) are chosen. Furthermore, it is under the *rule of solution expression* that Equation (6.28) is given to avoid the appearance of the term $\xi\ \sin\xi$ and to close the problem. Note that for the problem considered in this chapter, the term $\xi\ \sin\xi$ is not a traditional secular term because it is possible that $u(\xi)$ can be expressed by such a set of base functions

$$\{\xi^m \sin(n\xi), \xi^m \cos(n\xi) \mid m \geq 0, n \geq 1\}.$$

This is the reason why the auxiliary function $H(\xi)$ is not unique for the considered problem. However, it seems more efficient to use the set of base functions denoted by (6.9) to approximate $u(\xi)$.

Note that our approximations are much better than the perturbation result (6.8), as shown in Figures 6.3 and 6.4. Also note that even the third-order approximation (6.40) of A agrees well with the exact result in the whole region $1 \leq \epsilon < +\infty$ and correctly gives the bifurcation point $\epsilon = 1$ and two breaches of nonzero solutions.

Using the so-called homotopy-Padé technique (see page 38 and §3.5.2), the convergence of the solution series of A is greatly accelerated, as shown in Table 6.2 when $\epsilon = 10$ and $\epsilon = 25$. It is found that the $[m, m]$ homotopy-Padé approximant of A does not depend upon the auxiliary parameter $\hbar$. The $[4, 4]$ homotopy-Padé approximant

$$A \approx 2\sqrt{3}\sqrt{1 - \frac{1}{\epsilon}}\ \frac{P(\epsilon)}{Q(\epsilon)} \tag{6.41}$$

agrees well with the exact result in the whole region $1 \leq \epsilon < +\infty$, where

$$\begin{aligned} P(\epsilon) &= 8665210296046039923 + 2500964782519057396\,\epsilon \\ &\quad + 604034298653768562\,\epsilon^2 + 62408285303687028\,\epsilon^3 \\ &\quad + 3874319809940915\,\epsilon^4, \\ Q(\epsilon) &= 25430938337575455089 + 7921677254280814588\,\epsilon \\ &\quad + 1930521704826790758\,\epsilon^2 + 213027971364041596\,\epsilon^3 \\ &\quad + 13310678950379441\,\epsilon^4. \end{aligned}$$

This example illustrates that the homotopy analysis method is valid for nonlinear problems with bifurcations.

TABLE 6.1
The 30th-order analytic approximations of A when $\epsilon = 10, \hbar = -1/2$ and $\epsilon = 25, \hbar = -1/5$ by means of $H(\xi) = 1$.

Order of approximation	$\epsilon = 10, \hbar = -1/2$	$\epsilon = 25, \hbar = -1/5$
5	0.99833	1.01046
10	0.99588	1.00313
15	0.99624	1.00117
20	0.99644	1.00049
25	0.99644	1.00017
30	0.99644	1.00000

TABLE 6.2
The $[m, m]$ homotopy-Padé approximations of A when $\epsilon = 10$ and $\epsilon = 25$ by means of $H(\xi) = 1$.

$[m, m]$	$\epsilon = 10$	$\epsilon = 25$
$[2, 2]$	0.99914	1.01167
$[4, 4]$	0.99651	1.00113
$[6, 6]$	0.99644	1.00012
$[8, 8]$	0.99644	0.99996
$[10, 10]$	0.99644	0.99994
$[12, 12]$	0.99644	0.99994
$[15, 15]$	0.99644	0.99994

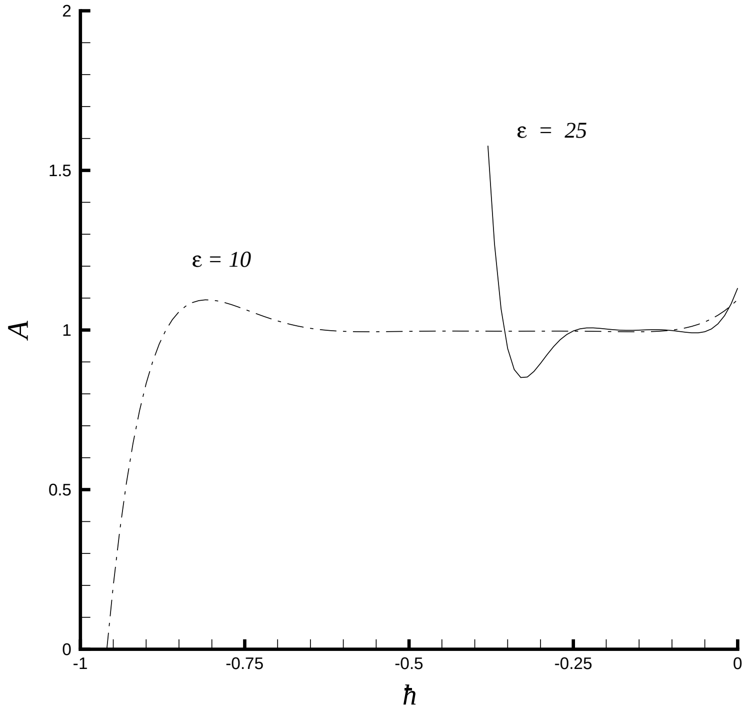

FIGURE 6.1
The $\hbar$-curves of A when $\epsilon = 10, 25$ by means of $H(\xi) = 1$. Dash-dotted line: 10th-order approximation of A when $\epsilon = 10$; solid line: 10th-order approximation of A when $\epsilon = 25$.

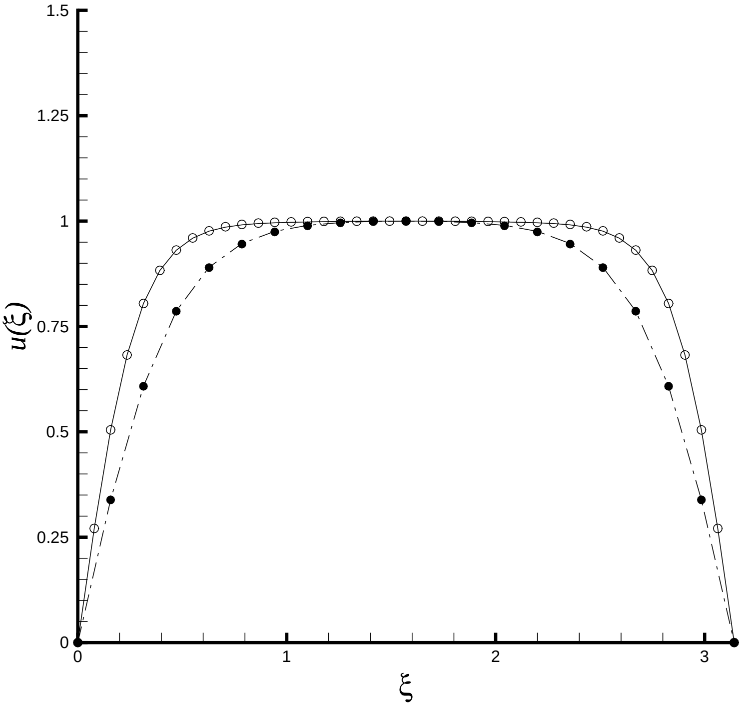

FIGURE 6.2
The analytic approximations of $u(\xi)$ by means of $H(\xi) = 1$. Dash-dotted line: 10th-order approximation when $\epsilon = 10$ and $\hbar = -1/2$; filled cycles: 20th-order approximation when $\epsilon = 10$ and $\hbar = -1/2$; solid line: 20th-order approximation of when $\epsilon = 25$ and $\hbar = -1/5$; open cycles: 30th-order approximation when $\epsilon = 25$ and $\hbar = -1/5$.

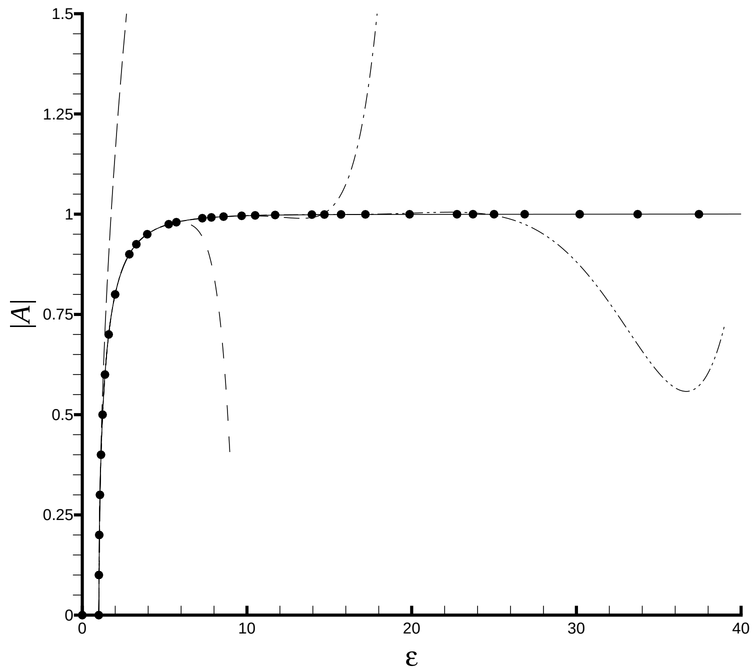

FIGURE 6.3
Comparison of the exact result (6.7) with the 10th-order approximation (6.37) of A when $H(\xi) = 1$. Symbols: exact result given by (6.7); long-dashed line: perturbation result (6.8); dashed line: approximation (6.37) when $\hbar = -1$; dash-dotted line: approximation (6.37) when $\hbar = -1/2$; dash-dot-dotted line: approximation (6.37) when $\hbar = -1/4$; solid line: approximation (6.39) when $\hbar = -1/(1 + \epsilon/3)$.

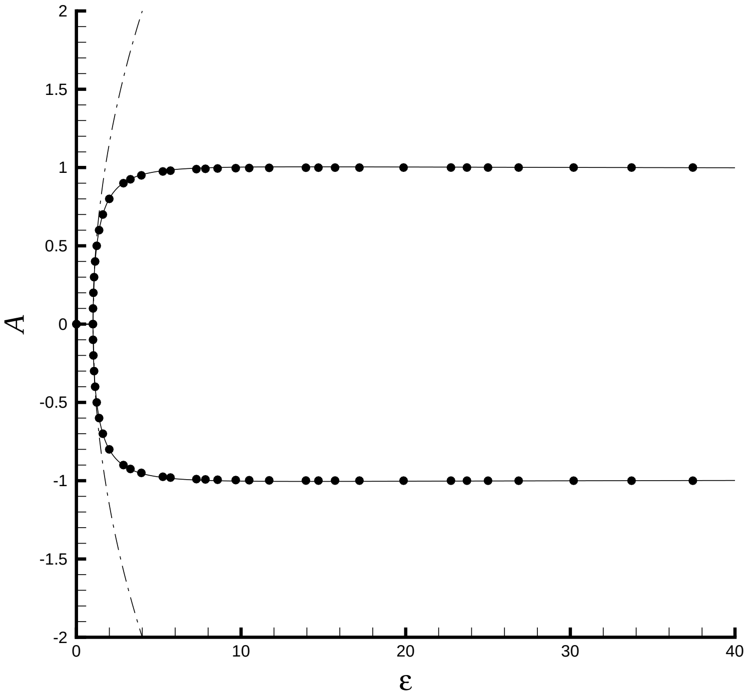

FIGURE 6.4
Comparison of the exact result (6.7) with the third-order analytic approximation (6.40) of A when $\hbar = -1/(1+\epsilon/3)$ and $H(\xi) = 1$. Symbols: exact result given by (6.7); dash-dotted line: perturbation result (6.8); solid line: analytic approximation (6.40).