4

Relations to some previous analytic methods

In this chapter we reveal the relationships between the homtopy analysis method and other nonperturbation techniques such as Adomian's decomposition method, Lyapunov's artificial small parameter method, and the δ-expansion method. We show that these methods can be unified by the homotopy analysis method.

4.1 Relation to Adomian's decomposition method

Adomian's decomposition method [23, 24, 25] is a well-known, easy-to-use analytic tool for nonlinear problems and has been widely applied in science and engineering [63, 64, 65, 66, 67, 68, 69, 70, 71, 72, 73, 74, 75, 76, 77, 78, 79]. In Chapter 2 we show by an example that the solution expression (2.17) given by Adomian's decomposition method is just a special one of the solution expressions (2.57) given by the homotopy analysis method. In this section we prove that the homotopy analysis method logically contains Adomian's decomposition method in general.

To simply describe the basic ideas of Adomian's decomposition method, let us consider a nonlinear problem governed by

$$\mathcal{N}\left[u(\mathbf{r},t)\right] = f(\mathbf{r},t), \tag{4.1}$$

where $\mathcal{N}$ is a nonlinear operator, u is a dependent variable, $f(\mathbf{r},t)$ is a known function, and $\mathbf{r}$ and t denote the spatial and temporal variables, respectively. Assume that the nonlinear operator $\mathcal{N}$ can be divided into

$$\mathcal{N} = \mathcal{L}_0 + \mathcal{N}_0, \tag{4.2}$$

where $\mathcal{L}_0$ and $\mathcal{N}_0$ are linear and nonlinear operators, respectively. Under this assumption the original nonlinear equation becomes

$$\mathcal{L}_0[u(\mathbf{r},t)] + \mathcal{N}_0[u(\mathbf{r},t)] = f(\mathbf{r},t). \tag{4.3}$$

By means of Adomian's decomposition method we express $u(\mathbf{r},t)$ in such a series

$$u(\mathbf{r},t) = u_0(\mathbf{r},t) + \sum_{n=1}^{+\infty} u_n(\mathbf{r},t), \tag{4.4}$$

where

$$u_0(\mathbf{r},t) = \mathcal{L}_0^{-1}\,[f(\mathbf{r},t)] \tag{4.5}$$

and

$$u_n(\mathbf{r},t) = -\mathcal{L}_0^{-1}\,[A_{n-1}(\mathbf{r},t)], \qquad n \geq 1, \tag{4.6}$$

in which $\mathcal{L}_0^{-1}$ is the inverse operator of $\mathcal{L}_0$, and $A_n(\mathbf{r},t)$ is the so-called Adomian polynomial defined by (see Cherruault [66] and Babolian et al. [75])

$$A_n(\mathbf{r},t) = \frac{1}{n!}\,\left[\frac{d^n}{dq^n}\mathcal{N}_0\left(u_0(\mathbf{r},t) + \sum_{n=1}^{+\infty} u_n(\mathbf{r},t)q^n\right)\right]\Bigg|_{q=0}. \tag{4.7}$$

Unlike Adomian's decomposition method, the homotopy analysis method is valid even without the assumption denoted by (4.2). Let $\mathcal{L}$ denote an auxiliary linear operator, $u_0(\mathbf{r},t)$ an initial approximation that is unnecessary to be given by (4.5), $\hbar$ a nonzero auxiliary parameter, $H(\mathbf{r},t)$ a nonzero auxiliary function, and $q \in [0,1]$ an imbedding parameter, respectively. By means of the homotopy analysis method, we construct the so-called zero-order deformation equation

$$(1-q)\,\mathcal{L}\,\left[\Phi(\mathbf{r},t;q) - u_0(\mathbf{r},t)\right] = \hbar\,q\,H(\mathbf{r},t)\,\left\{\mathcal{N}\left[\Phi(\mathbf{r},t;q)\right] - f(\mathbf{r},t)\right\}, \tag{4.8}$$

where $\Phi(\mathbf{r},t;q)$ is a unknown dependent variable. It clearly holds

$$\Phi(\mathbf{r},t;0) = u_0(\mathbf{r},t) \tag{4.9}$$

and

$$\Phi(\mathbf{r},t;1) = u(\mathbf{r},t) \tag{4.10}$$

when $q = 0$ and $q = 1$, respectively. Thus, the unknown function $\Phi(\mathbf{r},t;q)$ governed by Equation (4.8) deforms from the initial approximation $u_0(\mathbf{r},t)$ to the exact solution $u(\mathbf{r},t)$ of the original equation (4.1) as the embedding parameter q increases from 0 to 1. By Taylor's theorem and using (4.9) we expand $\Phi(\mathbf{r},t;q)$ in a power series of q in the form

$$\Phi(\mathbf{r},t;q) = u_0(\mathbf{r},t) + \sum_{n=1}^{+\infty} u_n(\mathbf{r},t)\,q^n, \tag{4.11}$$

where

$$u_n(\mathbf{r},t) = \frac{1}{n!}\,\frac{d^n\Phi(\mathbf{r},t;q)}{dq^n}\bigg|_{q=0}. \tag{4.12}$$

The zero-order deformation equation (4.8) contains the initial approximation $u_0(\mathbf{r},t)$, the auxiliary linear operator $\mathcal{L}$, the auxiliary parameter $\hbar$, the auxiliary function $H(\mathbf{r},t)$, and more importantly, we have great freedom to choose them. Assuming that all of them are properly chosen so that the series (4.11) converges at $q = 1$, we have, using (4.10), the solution series

$$u(\mathbf{r},t) = u_0(\mathbf{r},t) + \sum_{n=1}^{+\infty} u_n(\mathbf{r},t). \tag{4.13}$$

Note that in form this expression is the same as (4.4).

Differentiating the zero-order deformation equation (4.8) n times with respect to q and then dividing it by $n!$ and finally setting $q = 0$, we have the first-order deformation equation (when $n = 1$)

$$\mathcal{L}\left[u_1(\mathbf{r},t)\right] = \hbar\, H(\mathbf{r},t)\ \left\{\mathcal{N}\left[u_0(\mathbf{r},t)\right] - f(\mathbf{r},t)\right\} \tag{4.14}$$

and the nth-order deformation equation (when $n \geq 2$)

$$\mathcal{L}\left[u_n(\mathbf{r},t) - u_{n-1}(\mathbf{r},t)\right] = \hbar\, H(\mathbf{r},t)\ R_n(\mathbf{r},t), \tag{4.15}$$

where

$$R_n(\mathbf{r},t) = \frac{1}{(n-1)!}\ \left.\frac{d^{n-1}\mathcal{N}\left[\Phi(\mathbf{r},t;q)\right]}{dq^{n-1}}\right|_{q=0}. \tag{4.16}$$

We then can prove that Adomian's decomposition method is just a special case of the homotopy analysis method under the assumption (4.2). Because we have great freedom to choose the auxiliary linear operator $\mathcal{L}$ and the initial guess $u_0(\mathbf{r},t)$, we certainly can choose

$$\mathcal{L} = \mathcal{L}_0, \qquad u_0(\mathbf{r},t) = \mathcal{L}_0^{-1}\left[f(\mathbf{r},t)\right]. \tag{4.17}$$

Setting

$$\hbar = -1, \qquad H(\mathbf{r},t) = 1 \tag{4.18}$$

and substituting (4.2) and (4.17) into Equations (4.14) and (4.15), we have

$$\mathcal{L}_0\left[u_1(\mathbf{r},t)\right] = f(\mathbf{r},t) - \mathcal{L}_0\left[u_0(\mathbf{r},t)\right] - \mathcal{N}_0\left[u_0(\mathbf{r},t)\right] \tag{4.19}$$

and

$$\begin{aligned} &\mathcal{L}_0\left[u_n(\mathbf{r},t)\right] \\ &= \mathcal{L}_0\left[u_{n-1}(\mathbf{r},t)\right] - \frac{1}{(n-1)!}\ \left.\frac{d^{n-1}\mathcal{L}_0\left[\Phi(\mathbf{r},t;q)\right]}{dq^{n-1}}\right|_{q=0} \\ &\quad - \frac{1}{(n-1)!}\ \left.\frac{d^{n-1}\mathcal{N}_0\left[\Phi(\mathbf{r},t;q)\right]}{dq^{n-1}}\right|_{q=0}, \qquad n \geq 2, \end{aligned} \tag{4.20}$$

respectively. From (4.17), it holds

$$f(\mathbf{r},t) - \mathcal{L}_0\left[u_0(\mathbf{r},t)\right] = 0$$

so that Equation (4.19) becomes, by the definition (4.7),

$$\mathcal{L}_0\left[u_1(\mathbf{r},t)\right] = -A_0(\mathbf{r},t), \tag{4.21}$$

where $A_0(\mathbf{r}, t)$ is an Adomian polynomial. According to definition (4.12), it holds

$$\begin{aligned}
&\mathcal{L}_0\left[u_{n-1}(\mathbf{r}, t)\right] - \frac{1}{(n-1)!} \left.\frac{d^{n-1}\mathcal{L}_0\left[\Phi(\mathbf{r}, t; q)\right]}{dq^{n-1}}\right|_{q=0} \\
&= \mathcal{L}_0\left[u_{n-1}(\mathbf{r}, t)\right] - \mathcal{L}_0\left[\frac{1}{(n-1)!} \left.\frac{d^{n-1}\Phi(\mathbf{r}, t; q)}{dq^{n-1}}\right|_{q=0}\right] \\
&= \mathcal{L}_0\left[u_{n-1}(\mathbf{r}, t)\right] - \mathcal{L}_0\left[u_{n-1}(\mathbf{r}, t)\right] \\
&= 0. \qquad (4.22)
\end{aligned}$$

Thus, Equation (4.20) becomes

$$\mathcal{L}_0\left[u_n(\mathbf{r}, t)\right] = -\frac{1}{(n-1)!} \left.\frac{d^{n-1}\mathcal{N}_0\left[\Phi(\mathbf{r}, t; q)\right]}{dq^{n-1}}\right|_{q=0}. \qquad (4.23)$$

Substituting (4.11) of $\Phi(\mathbf{r}, t; q)$ into the above expression, we have, according to the definition (4.7) of the Adomian polynomial,

$$\begin{aligned}
&\mathcal{L}_0\left[u_n(\mathbf{r}, t)\right] \\
&= -\frac{1}{(n-1)!}\left[\frac{d^{n-1}}{dq^{n-1}}\mathcal{N}_0\left(u_0(\mathbf{r}, t) + \sum_{n=1}^{+\infty} u_n(\mathbf{r}, t)\, q^n\right)\right]\Bigg|_{q=0} \\
&= -A_{n-1}(\mathbf{r}, t). \qquad (4.24)
\end{aligned}$$

So, the solution of Equation (4.21) and Equation (4.24) can be uniformly expressed by

$$u_n(\mathbf{r}, t) = -\mathcal{L}_0^{-1}\left[A_{n-1}(\mathbf{r}, t)\right], \qquad n \geq 1, \qquad (4.25)$$

which is exactly the same as the solution (4.6) given by Adomian's decomposition method. Therefore, Adomian's decomposition method is just a special case of the homotopy analysis method under the assumption (4.2) when

$$u_0(\mathbf{r}, t) = \mathcal{L}_0^{-1}\left[f(\mathbf{r}, t)\right], \;\; \mathcal{L} = \mathcal{L}_0, \;\; H(\mathbf{r}, t) = 1, \;\; \hbar = -1.$$

Some points should be emphasized here. First, we have great freedom to choose the initial guess $u_0(\mathbf{r}, t)$, the auxiliary linear operator $\mathcal{L}$, and the auxiliary function $H(\mathbf{r}, t)$ different from the above expressions so that the solution of high-order deformation equations (4.14) and (4.15) can be expressed by better base functions than those employed by Adomian's decomposition method that often uses polynomials. Second, it is unnecessary for us to assume that the nonlinear operator $\mathcal{N}$ should be divided into the form (4.2). Finally but most importantly, solutions given by the homotopy analysis method contain the auxiliary parameter $\hbar$, which provides us with a simply way to adjust and control convergence region and rate of solution series. Therefore, the homotopy analysis method is more general than Adomian's decomposition method.

4.2 Relation to artificial small parameter method

In 1892 Lyapunov [21] proposed the so-called artificial small parameter method. In Chapter 2 we illustrate that the solution expression (2.15) given by Lyapunov's artificial small parameter method is just a special one of the solution expressions (2.57) given by the homotopy analysis method. In this section we prove that Lyapunov's artificial small parameter method is in essence equivalent to Adomian's decomposition method and therefore is also a special case of the homotopy analysis method.

To simply describe the basic ideas of Lyapunov's artificial small parameter method, let us consider a nonlinear equation

$$\mathcal{N}\left[u(\mathbf{r},t)\right] = f(\mathbf{r},t), \tag{4.26}$$

where $\mathcal{N}$ is a nonlinear operator, u is a dependent variable, $f(\mathbf{r},t)$ is a known function, and $\mathbf{r}$ and t denote the spatial and temporal variables, respectively. Assume that the nonlinear operator $\mathcal{N}$ can be divided into

$$\mathcal{N} = \mathcal{L}_0 + \mathcal{N}_0, \tag{4.27}$$

where $\mathcal{L}_0$ and $\mathcal{N}_0$ are linear and nonlinear operators, respectively. Using the above expression and introducing the artificial small parameter ϵ, the original equation (4.26) becomes

$$\mathcal{L}_0\ [\phi(\mathbf{r},t;\epsilon)] + \epsilon\, \mathcal{N}_0\ [\phi(\mathbf{r},t;\epsilon)] = f(\mathbf{r},t), \tag{4.28}$$

where $\phi(\mathbf{r},t;\epsilon)$ is an unknown function. When $\epsilon = 1$, the above equation is clearly the same as Equation (4.26) so that

$$\phi(\mathbf{r},t;1) = u(\mathbf{r},t). \tag{4.29}$$

Expanding $\phi(\mathbf{r},t;\epsilon)$ in a power series of the artificial small parameter ϵ, we have

$$\phi(\mathbf{r},t;\epsilon) = u_0(\mathbf{r},t) + \sum_{n=1}^{+\infty} u_n(\mathbf{r},t)\ \epsilon^n. \tag{4.30}$$

Setting $\epsilon = 1$ in the above expression we have, using (4.29),

$$u(\mathbf{r},t) = u_0(\mathbf{r},t) + \sum_{n=1}^{+\infty} u_n(\mathbf{r},t), \tag{4.31}$$

which in form is exactly the same as the solution expression (4.4) given by Adomian's decomposition method.

Substituting (4.30) into Equation (4.28), we have

$$\mathcal{L}_0[u_0(\mathbf{r},t)] - f(\mathbf{r},t) + \sum_{n=1}^{+\infty} \epsilon^n \, \mathcal{L}_0 \, [u_n(\mathbf{r},t)]$$
$$+ \, \epsilon \, \mathcal{N}_0 \, \left[u_0(\mathbf{r},t) + \sum_{n=1}^{+\infty} u_n(\mathbf{r},t) \, \epsilon^n \right] = 0. \tag{4.32}$$

Write

$$\mathcal{N}_0 \, \left[u_0(\mathbf{r},t) + \sum_{n=1}^{+\infty} u_n(\mathbf{r},t) \, \epsilon^n \right] = \sum_{n=0}^{+\infty} w_n(\mathbf{r},t) \, \epsilon^n.$$

Differentiating both sides of the above expression m times with respect to the artificial small parameter ϵ and then setting $\epsilon = 0$, we have

$$\left\{ \frac{\partial^m}{\partial \epsilon^m} \mathcal{N}_0 \left[u_0(\mathbf{r},t) + \sum_{n=1}^{+\infty} u_n(\mathbf{r},t) \, \epsilon^n \right] \right\} \Bigg|_{\epsilon=0} = m! \; w_m(\mathbf{r},t),$$

which gives, using the definition (4.7), that

$$w_m(\mathbf{r},t) = \frac{1}{m!} \left\{ \frac{\partial^m}{\partial \epsilon^m} \mathcal{N}_0 \left[u_0(\mathbf{r},t) + \sum_{n=1}^{+\infty} u_n(\mathbf{r},t) \, \epsilon^n \right] \right\} \Bigg|_{\epsilon=0} = A_m(\mathbf{r},t),$$

where $A_m(\mathbf{r},t)$ is the so-called Adomian polynomial. So, substituting

$$\mathcal{N}_0 \, \left[u_0(\mathbf{r},t) + \sum_{n=1}^{+\infty} u_n(\mathbf{r},t) \, \epsilon^n \right] = \sum_{n=0}^{+\infty} A_n(\mathbf{r},t) \, \epsilon^n$$

into Equation (4.32), we have

$$\{\mathcal{L}_0[u_0(\mathbf{r},t)] - f(\mathbf{r},t)\} + \sum_{n=1}^{+\infty} \epsilon^n \; \{\mathcal{L}_0 \, [u_n(\mathbf{r},t)] + A_{n-1}(\mathbf{r},t)\} = 0,$$

which gives

$$\mathcal{L}_0[u_0(\mathbf{r},t)] - f(\mathbf{r},t) = 0$$

and

$$\mathcal{L}_0 \, [u_n(\mathbf{r},t)] + A_{n-1}(\mathbf{r},t) = 0, \qquad n \geq 1.$$

Solving the above equations successively, we have

$$u_0(\mathbf{r},t) = \mathcal{L}_0^{-1} \, [f(\mathbf{r},t)]$$

and

$$u_n(\mathbf{r},t) = -\mathcal{L}_0^{-1} \, [A_{n-1}(\mathbf{r},t)], \qquad n \geq 1,$$

which are exactly the same as the solutions (4.5) and (4.6) given by Adomian's decomposition method, respectively. So, Adomian's decomposition method is in essence equivalent to the artificial small parameter method.

In §4.1 we prove that Adomian's decomposition method is just a special case of the homotopy analysis method. Therefore, Lyapunov's artificial small parameter method is also a special case of the homotopy analysis method under the assumption

$$\mathcal{N} = \mathcal{L}_0 + \mathcal{N}_0$$

when

$$\hbar = -1, \quad H(\mathbf{r}, t) = 1, \quad \mathcal{L} = \mathcal{L}_0, \quad u_0(\mathbf{r}, t) = \mathcal{L}_0^{-1}[f(\mathbf{r}, t)].$$

This is easy to understand if we regard the so-called artificial small parameter ϵ as the embedding parameter and Equation (4.28) as a special zero-order deformation equation.

4.3 Relation to δ-expansion method

In Chapter 2 only the solution expression (2.21) given by the δ-expansion method is not among the four families of solution expressions given by means of the homotopy analysis method. However, using the generalized zero-order deformation equation (3.34) in §3.6, we can show that the δ-expansion method is also a special case of the homotopy analysis method. To illustrate this point, let us consider the same example in Chapter 2, i.e.,

$$\dot{V}(t) + V^2(t) = 1, \qquad V(0) = 0. \tag{4.33}$$

To solve this problem by means of the homotopy analysis method, we choose an auxiliary linear operator

$$\mathcal{L}\Phi = \frac{\partial \Phi}{\partial t} + \Phi - 1 \tag{4.34}$$

and an initial approximation $V_0(t)$ satisfying

$$\mathcal{L}[V_0(t)] = 0, \qquad V_0(0) = 0,$$

which gives

$$V_0(t) = 1 - \exp(-t). \tag{4.35}$$

From Equation (4.33), we define the nonlinear operator

$$\mathcal{N}\left[\Phi(t;q), q\right] = \frac{\partial \Phi(t;q)}{\partial t} + [\Phi(t;q)]^{q+1} - 1. \tag{4.36}$$

Define the auxiliary operator

$$\Pi\left[\Phi(t;q),q\right]=(1-q)\left\{[\Phi(t;q)]^{q+1}-\Phi(t;q)\right\} \tag{4.37}$$

which equals zero when $q = 0$ and $q = 1$. Let $\hbar, \hbar_2$ denote the auxiliary parameters, and $H(t), H_2(t)$ the auxiliary functions, respectively. According to (3.34), we construct the zero-order deformation equation

$$\begin{aligned}(1-q)\mathcal{L}[\Phi(t;q)-V_0(t)] &= q\,\hbar\,H(t)\,\mathcal{N}\left[\Phi(t;q),q\right]\\ &\quad+\hbar_2\,H_2(t)\,\Pi\left[\Phi(t;q),q\right],\end{aligned} \tag{4.38}$$

subject to the initial condition

$$\Phi(0;q)=0. \tag{4.39}$$

When $q = 0$, it is straightforward that

$$\Phi(t;0)=V_0(t)=1-\exp(-t). \tag{4.40}$$

When $q = 1$, Equation (4.38) is equivalent to the original equation (4.33), provided

$$\Phi(t;1)=V(t). \tag{4.41}$$

Expand $\Phi(t;q)$ in a power series

$$\Phi(t;q)=\Phi(t;0)+\sum_{n=1}^{+\infty}V_n(t)\,q^n, \tag{4.42}$$

where

$$V_n(t)=\left.\frac{1}{n!}\frac{\partial^n\Phi(t;q)}{\partial q^n}\right|_{q=0}. \tag{4.43}$$

Assuming that the series (4.42) is convergent at $q = 1$, we have using Equations (4.40) and (4.41)

$$V(t)=V_0(t)+\sum_{m=1}^{+\infty}V_m(t). \tag{4.44}$$

The governing equation of $V_m(t)$ is deduced by means of the definition (4.43). Differentiating the zero-order deformation equation (4.38) m times with respect to the embedding parameter q and then dividing by $m!$ and finally setting $q = 0$, we have the high-order deformation equation

$$\mathcal{L}_0[V_m(t)-\chi_m\,V_{m-1}(t)]=\hbar\,H(t)\,R_m(t)+\hbar_2\,H_2(t)\,\Delta_m(t), \tag{4.45}$$

subject to the initial condition

$$V_m(0)=0, \tag{4.46}$$

where χ_m is defined by (2.42) and

$$R_m(t) = \frac{1}{(m-1)!} \left. \frac{\partial^{m-1} \mathcal{N}\left[\Phi(t;q), q\right]}{\partial q^{m-1}} \right|_{q=0}, \tag{4.47}$$

$$\Delta_m(t) = \frac{1}{m!} \left. \frac{\partial^m \Pi\left[\Phi(t;q), q\right]}{\partial q^m} \right|_{q=0} \tag{4.48}$$

under the definition

$$\mathcal{L}_0 \Phi = \frac{\partial \Phi}{\partial t} + \Phi. \tag{4.49}$$

Substituting Equations (4.36) and (4.37) into Equations (4.47) and (4.48), respectively, we have

$$\begin{aligned} R_1(t) &= \dot{V}_0(t) + V_0(t) - 1, \\ R_2(t) &= \dot{V}_1(t) + V_1(t) + V_0(t) \ln V_0(t), \\ &\vdots \end{aligned}$$

and

$$\begin{aligned} \Delta_1(t) &= V_0(t) \ln V_0(t), \\ \Delta_2(t) &= -V_0(t) \ln V_0(t) + V_1(t)\left[1 + \ln V_0(t)\right] + \frac{1}{2} V_0(t) \ln^2 V_0(t), \\ &\vdots \end{aligned}$$

In the special case

$$\hbar = \hbar_2 = -1, \qquad H(t) = H_2(t) = 1, \tag{4.50}$$

we have the high-order deformation equations

$$\begin{aligned} \dot{V}_1 + V_1 &= -V_0 \ln V_0 - R_1(t), \qquad V_1(0) = 0, \\ \dot{V}_2 + V_2 &= -V_1(1 + \ln V_0) - \frac{1}{2} V_0 \ln^2 V_0 - R_2(t), \qquad V_2(0) = 0, \\ &\vdots \end{aligned}$$

Solving the above high-order deformation equations successively, we obtain

$$\begin{aligned} V_1(t) &= \exp(-t)\left[t - \frac{\pi^2}{6} + P_2^L(e^{-t})\right] - (1 - e^{-t}) \ln(1 - e^{-t}), \\ &\vdots \end{aligned}$$

where

$$P_n^L(z) = \sum_{k=1}^{+\infty} \frac{z^k}{k^n}$$

is the nth polylogarithm function of z. So, the first-order approximation is

$$V(t) \approx 1 + \exp(-t)\left[t - \frac{\pi^2}{6} - 1 + P_2^L(e^{-t})\right] - (1 - e^{-t})\ln(1 - e^{-t}), \quad (4.51)$$

which is exactly the same as the approximation (2.21) given by the δ-expansion method in Chapter 2. It should be emphasized that the solution expression given by Equations (4.45) and (4.46) contains the two auxiliary parameters $\hbar$ and $\hbar_2$ and thus is more general than the solution expression (2.21) given by the δ-expansion method. In fact, from (4.35) it holds $R_1(t) = 0$. Furthermore, using the first-order deformation equation, we have $R_2(t) = 0$. Thus, the high-order deformation equations are exactly the same as those given by the δ-expansion method in Chapter 2. Substituting

$$\hbar = \hbar_2 = -1, \qquad H(t) = H_2(t) = 1$$

into the zero-order deformation equation (4.38) we have

$$\frac{\partial \Phi(t;q)}{\partial t} + [\Phi(t;q)]^{1+q} = 1, \quad (4.52)$$

which is the same as the equation

$$\dot{V}(t) + V^{1+\delta}(t) = 1$$

in Chapter 2 used by the δ-expansion method, if δ and $V(t)$ are replaced by q and $\Phi(t;q)$, respectively. In general, we can regard δ as an embedding parameter and the corresponding equation as a special zero-order deformation equation. Therefore, the δ-expansion method is only a special case of the homotopy analysis method.

4.4 Unification of nonperturbation methods

As shown above, Adomian's decomposition method, Lyapunov's artificial small parameter method, and the δ-expansion method are only special cases of the homotopy analysis method. Therefore, these three nonperturbation methods can be unified in the frame of the homotopy analysis method. A unified theory is often believed to be closer to the truth. This, from another side, further indicates the validity of the homotopy analysis method.