2

Illustrative description

In this chapter we use a simple nonlinear ordinary differential equation as an example to introduce the basic ideas of the homotopy analysis method.

2.1 An illustrative example

Consider a free sphere dropping in the air from a static state. Let $\tilde{t}$ denote the time, $U(\tilde{t})$ the velocity of the sphere, m the mass, and g the acceleration of gravity. Assume that the air resistance on the sphere is $a\,U^2(\tilde{t})$, where a is a constant. Then, due to Newton's second law, it holds

$$m\frac{dU(\tilde{t})}{d\tilde{t}} = mg - aU^2(\tilde{t}), \tag{2.1}$$

subject to the initial condition

$$U(0) = 0. \tag{2.2}$$

Physically speaking, the speed of a freely dropping sphere is increased due to the gravity until a steady velocity U_∞ is reached. So, even not knowing the solution $U(\tilde{t})$ in detail, we can gain the limit velocity U_∞ directly from (2.1), i.e.,

$$U_\infty = \sqrt{\frac{mg}{a}}. \tag{2.3}$$

Using U_∞ and U_∞/g as the characteristic velocity and time, respectively, and writing

$$\tilde{t} = \left(\frac{U_\infty}{g}\right) t, \qquad U(\tilde{t}) = U_\infty V(t), \tag{2.4}$$

we have the dimensionless equation

$$\dot{V}(t) + V^2(t) = 1, \qquad t \geq 0, \tag{2.5}$$

subject to the initial condition

$$V(0) = 0, \tag{2.6}$$

where t denotes the dimensionless time and the dot denotes the derivative with respect to t. Obviously, as $t \to +\infty$, i.e., $\tilde{t} \to \infty$ and $U(\tilde{t}) \to U_\infty$, we have from (2.4) that

$$\lim_{t \to +\infty} V(t) = 1, \tag{2.7}$$

even without solving Equations (2.5) and (2.6).

The exact solution of Equations (2.5) and (2.6) is

$$V(t) = \tanh(t), \tag{2.8}$$

useful for the comparisons of different approximations.

2.2 Solution given by some previous analytic techniques

For the sake of comparison, we first apply some well-known previous analytic techniques to solve the illustrative nonlinear problem.

2.2.1 Perturbation method

To give perturbation approximation, we assume that the dimensionless time t is a small variable (called perturbation quantity) and then express $V(t)$ in a power series

$$V(t) = \alpha_0 + \alpha_1 t + \alpha_2 t^2 + \alpha_3 t^3 + \cdots. \tag{2.9}$$

Using the initial condition (2.6) we gain $\alpha_0 = 0$. Then, substituting the above expression into Equation (2.5), we obtain

$$\sum_{k=0}^{+\infty} \left[(k+1)\,\alpha_{k+1} + \sum_{j=0}^{k} \alpha_j \alpha_{k-j} \right] t^k = 1,$$

which holds for any $t \geq 0$, provided

$$\alpha_1 = 1, \tag{2.10}$$

$$\alpha_{k+1} = -\frac{1}{k+1} \sum_{j=0}^{k} \alpha_j \alpha_{k-j}, \qquad k \geq 1. \tag{2.11}$$

We therefore have the perturbation solution

$$V_{pert}(t) = t - \frac{1}{3} t^3 + \frac{2}{15} t^5 - \frac{17}{315} t^7 + \cdots = \sum_{n=0}^{+\infty} \alpha_{2n+1}\, t^{2n+1}, \tag{2.12}$$

which converges in a rather small region $0 \leq t < \rho_0$, where $\rho_0 \approx 3/2$, as shown in Figure 2.1. Note that the convergence region and rate of the perturbation solution (2.12) are uniquely determined.

2.2.2 Lyapunov's artificial small parameter method

By Lyapunov's artificial small parameter method we first replace Equation (2.5) by the equation

$$\dot{V}(t) + \varepsilon\, V^2(t) = 1 \tag{2.13}$$

and then write

$$V(t) = V_0(t) + \varepsilon\, V_1(t) + \varepsilon^2\, V_2(t) + \cdots, \tag{2.14}$$

where ε is an artificial small parameter. Substituting Equation (2.14) into Equations (2.13) and (2.6) and then balancing the coefficients of power series of ε, we obtain the equations

$$\begin{aligned} &\dot{V}_0(t) = 1, \qquad V_0(0) = 0,\\ &\dot{V}_1(t) + V_0^2(t) = 0, \qquad V_1(0) = 0,\\ &\vdots \end{aligned}$$

which give

$$V_0(t) = t, \quad V_1(t) = -\frac{t^3}{3}, \quad V_2(t) = \frac{2t^5}{15}, \cdots,$$

successively. Finally, setting $\varepsilon = 1$ in Equation (2.14), we have

$$V(t) = t - \frac{1}{3}t^3 + \frac{2}{15}t^5 - \frac{17}{315}t^7 + \cdots = \sum_{n=0}^{+\infty} \alpha_{2n+1}\, t^{2n+1}, \tag{2.15}$$

which is exactly the same as the perturbation solution (2.12) and thus is valid in a rather restricted region of t, as shown in Figure 2.1. Note that the convergence region and rate of the solution (2.15) given by Lyapunov's artificial small parameter method are also uniquely determined.

2.2.3 Adomian's decomposition method

By Adomian's decomposition method we first replace Equations (2.5) and (2.6) by

$$V(t) = t - \int_0^t V^2(t) dt. \tag{2.16}$$

The solution is given by

$$V(t) = V_0(t) + \sum_{k=1}^{+\infty} V_k(t),$$

where

$$\begin{aligned} V_0(t) &= t,\\ V_k(t) &= -\int_0^t A_{k-1}(t)\, dt, \qquad k \geq 1 \end{aligned}$$

and

$$A_k(t) = \sum_{n=0}^{k} V_n(t)\, V_{k-n}(t)$$

is the so-called Adomian polynomial. We gain successively

$$V_1(t) = -\frac{t^3}{3}, \qquad V_2(t) = \frac{2t^5}{15}, \qquad V_3(t) = -\frac{17}{315}t^7, \quad \cdots$$

such that

$$V(t) = t - \frac{1}{3}t^3 + \frac{2}{15}t^5 - \frac{17}{315}t^7 + \cdots = \sum_{n=0}^{+\infty} \alpha_{2n+1}\, t^{2n+1}, \tag{2.17}$$

which is exactly the same as the perturbation result (2.12), and thus is also valid in a rather small region, as shown in Figure 2.1. Note that the convergence region and rate of the solution (2.17) given by Adomian's decomposition method are also uniquely determined.

2.2.4 The δ-expansion method

By the δ-expansion method we first replace Equation (2.5) by

$$\dot{V}(t) + V^{1+\delta}(t) = 1, \tag{2.18}$$

where δ is a real number. Write

$$V(t) = V_0(t) + \sum_{n=1}^{+\infty} V_n(t)\, \delta^n. \tag{2.19}$$

Then we expand $V^{1+\delta}(t)$ in the power series of δ as follows:

$$\begin{aligned} V^{1+\delta} &= V_0 + [V_1 + V_0 \ln V_0]\,\delta \\ &\quad + \left[(V_1(1+\ln V_0) + \frac{1}{2}V_0 \ln^2 V_0 + V_2\right]\delta^2 + \cdots. \end{aligned} \tag{2.20}$$

Substituting Equations (2.19) and (2.20) into Equation (2.18) and balancing the power series of δ, we have the following equations:

$$\begin{aligned} &\dot{V}_0 + V_0 = 1, \qquad V_0(0) = 0, \\ &\dot{V}_1 + V_1 = -V_0 \ln V_0, \qquad V_1(0) = 0, \\ &\dot{V}_2 + V_2 = -V_1(1+\ln V_0) - \frac{1}{2}V_0 \ln^2 V_0, \qquad V_2(0) = 0, \\ &\dot{V}_3 + V_3 = -V_2(1+\ln V_0) - V_1\left(1 + \frac{1}{2}\ln V_0\right)\ln V_0 \\ &\qquad\qquad -\frac{1}{6}V_0 \ln^3 V_0 - \frac{V_1^2}{2V_0}, \qquad V_3(0) = 0, \\ &\qquad\vdots \end{aligned}$$

Solving the above linear equations successively, we obtain

$$V_0(t) = 1 - \exp(-t),$$
$$V_1(t) = \exp(-t)\left[t - \frac{\pi^2}{6} + P_2^L(e^{-t})\right] - (1 - e^{-t})\ln(1 - e^{-t}),$$
$$\vdots$$

where

$$P_n^L(z) = \sum_{k=1}^{+\infty} \frac{z^k}{k^n}$$

is the nth polylogarithm function of z. The first-order approximation is

$$V(t) \approx 1 + \exp(-t)\left[t - \frac{\pi^2}{6} - 1 + P_2^L(e^{-t})\right] - (1 - e^{-t})\ln(1 - e^{-t}). \quad (2.21)$$

The first several approximations of this kind of solution seem to be valid in the whole region $0 \le t < +\infty$. However, due to the appearance of the special function $P_n^L(z)$, it becomes more and more difficult to get higher order approximations.

It is interesting that the solutions given by the perturbation method, Lyapunov's artificial small parameter method, and Adomian's decomposition method are the *same* for the illustrative problem. However, this solution is valid in a rather small region $0 \le t < 3/2$. This illustrates that, like perturbation approximations, solutions given by previous nonperturbation techniques such as Lyapunov's artificial small parameter method and Adomian's decomposition method might break down as physical parameters or variables increase and nonlinearity goes stronger. This fact also implies that there might exist some relationships between them. It should be emphasized that the convergence region and rate of solutions given by *all* of these previous analytic methods are uniquely determined, and neither perturbation techniques nor previous nonperturbation methods such as Lyapunov's artificial small parameter method, Adomian's decomposition method, and the δ-expansion method can provide us with a convenient way to *control* and *adjust* convergence region and rate of solution series. Furthermore, the important property (2.7) of $V(t)$ at infinity, which we obtain without solving the problem, seems useless for all of these previous analytic techniques. This example illustrates that, in essence, neither perturbation techniques nor the previous nonperturbation methods can provide us with any ways to utilize such kinds of valuable information to approximate a given nonlinear problem more efficiently. Finally, note that the artificial parameters ϵ and δ appear in the different places in Equations (2.13) and (2.18), respectively, but the solution (2.21) given by the δ-expansion method is valid in the whole region $0 \le t < +\infty$ and thus is much better than the solution (2.15) given by Lyapunov's artificial small parameter method. As mentioned in Chapter 1, Equations (2.13) and (2.18) can be regarded as a kind of homotopy, if we define ϵ and δ as the embedding parameter. This example illustrates that, in order to approximate a given nonlinear problem more efficiently, it seems important to properly construct a homotopy. However, up to now, we have not had any fundamental rules to direct us.

2.3 Homotopy analysis solution

In this section the basic ideas of the homotopy analysis method are introduced by the same illustrative problem mentioned above.

2.3.1 Zero-order deformation equation

Let $V_0(t)$ denote an initial guess of $V(t)$, which satisfies the initial condition (2.6), i.e.,

$$V_0(0) = 0. \tag{2.22}$$

Let $q \in [0, 1]$ denote the so-called embedding parameter. The homotopy analysis method is based on a kind of continuous mapping $V(t) \to \Phi(t; q)$ such that, as the embedding parameter q increases from 0 to 1, $\Phi(t; q)$ varies from the initial guess $V_0(t)$ to the exact solution $V(t)$. To ensure this, choose such an auxiliary linear operator as

$$\mathcal{L}[\Phi(t; q)] = \gamma_1(t) \frac{\partial \Phi(t; q)}{\partial t} + \gamma_2(t)\ \Phi(t; q), \tag{2.23}$$

where $\gamma_1(t) \neq 0$ and $\gamma_2(t)$ are real functions to be determined later. From Equation (2.5), we define the nonlinear operator

$$\mathcal{N}[\Phi(t; q)] = \frac{\partial \Phi(t; q)}{\partial t} + \Phi^2(t; q) - 1. \tag{2.24}$$

Let $\hbar \neq 0$ and $H(t) \neq 0$ denote the so-called auxiliary parameter and auxiliary function, respectively. Using the embedding parameter $q \in [0, 1]$, we construct a family of equations

$$(1 - q)\ \mathcal{L}\left[\Phi(t; q) - V_0(t)\right] = \hbar\ q\ H(t)\ \mathcal{N}[\Phi(t; q)], \tag{2.25}$$

subject to the initial condition

$$\Phi(0; q) = 0. \tag{2.26}$$

It should be emphasized that we have great freedom to choose the auxiliary parameter $\hbar$, the auxiliary function $H(t)$, the initial approximation $V_0(t)$, and the auxiliary linear operator $\mathcal{L}$. It is such freedom that plays important roles and establishes the cornerstone of the validity and flexibility of the homotopy analysis method, as shown later in this book.

When $q = 0$, Equation (2.25) becomes

$$\mathcal{L}\left[\Phi(t;0) - V_0(t)\right] = 0, \qquad t \geq 0, \tag{2.27}$$

subject to the initial condition

$$\Phi(0;0) = 0. \tag{2.28}$$

According to Equations (2.22) and (2.23), the solution of Equations (2.27) and (2.28) is simply

$$\Phi(t;0) = V_0(t). \tag{2.29}$$

When $q = 1$, Equation (2.25) becomes

$$\hbar\, H(t)\, \mathcal{N}[\Phi(t;1)] = 0, \qquad t \geq 0, \tag{2.30}$$

subject to the initial condition

$$\Phi(0;1) = 0. \tag{2.31}$$

Since $\hbar \neq 0$, $H(t) \neq 0$ and by means of the definition (2.24), Equations (2.30) and (2.31) are equivalent to the original equations (2.5) and (2.6), provided

$$\Phi(t;1) = V(t). \tag{2.32}$$

Therefore, according to Equations (2.29) and (2.32), $\Phi(t;q)$ varies from the initial guess $V_0(t)$ to the exact solution $V(t)$ as the embedding parameter q increases from 0 to 1. In topology, this kind of variation is called *deformation*, and Equations (2.25) and (2.26) construct the homotopy $\Phi(t;q)$. For brevity, Equations (2.25) and (2.26) are called *the zero-order deformation equations.*

Having the freedom to choose the auxiliary parameter $\hbar$, the auxiliary function $H(t)$, the initial approximation $V_0(t)$, and the auxiliary linear operator $\mathcal{L}$, we can assume that all of them are properly chosen so that the solution $\Phi(t;q)$ of the zero-order deformation equations (2.25) and (2.26) exists for $0 \leq q \leq 1$, and besides its mth-order derivative with respect to the embedding parameter q, i.e.,

$$V_0^{[m]}(t) = \left.\frac{\partial^m \Phi(t;q)}{\partial q^m}\right|_{q=0} \tag{2.33}$$

exists, where $m = 1, 2, 3, \cdots$. For brevity, $V_0^{[m]}(t)$ is called *the mth-order deformation derivative.* Define

$$V_m(t) = \frac{V_0^{[m]}(t)}{m!} = \frac{1}{m!} \left.\frac{\partial^m \Phi(t;q)}{\partial q^m}\right|_{q=0}. \tag{2.34}$$

By Taylor's theorem, we expand $\Phi(t;q)$ in a power series of the embedding parameter q as follows:

$$\Phi(t;q) = \Phi(t;0) + \sum_{m=1}^{+\infty} \frac{1}{m!} \left. \frac{\partial^m \Phi(t;q)}{\partial q^m} \right|_{q=0} q^m. \tag{2.35}$$

From Equations (2.29) and (2.34), the above power series becomes

$$\Phi(t;q) = V_0(t) + \sum_{m=1}^{+\infty} V_m(t)\, q^m. \tag{2.36}$$

Assume that the auxiliary parameter $\hbar$, the auxiliary function $H(t)$, the initial approximation $V_0(t)$, and the auxiliary linear operator $\mathcal{L}$ are so properly chosen that the series (2.36) converges at $q = 1$. Then, at $q = 1$, the series (2.36) becomes

$$\Phi(t;1) = V_0(t) + \sum_{m=1}^{+\infty} V_m(t). \tag{2.37}$$

Therefore, using Equation (2.32), we have

$$V(t) = V_0(t) + \sum_{m=1}^{+\infty} V_m(t). \tag{2.38}$$

The above expression provides us with a relationship between the initial guess $V_0(t)$ and the exact solution $V(t)$ by means of the terms $V_m(t)$ $(m = 1, 2, 3, \cdots)$, which are unknown up to now.

2.3.2 High-order deformation equation

Define the vector

$$\vec{V}_n = \{V_0(t), V_1(t), V_2(t), \cdots, V_n(t)\}.$$

According to the definition (2.34), the governing equation and corresponding initial condition of $V_m(t)$ can be deduced from the zero-order deformation equations (2.25) and (2.26). Differentiating Equations (2.25) and (2.26) m times with respect to the embedding parameter q and then setting $q = 0$ and finally dividing them by m!, we have the so-called mth-order deformation equation

$$\mathcal{L}\left[V_m(t) - \chi_m\, V_{m-1}(t)\right] = \hbar\, H(t)\, R_m(\vec{V}_{m-1}), \tag{2.39}$$

subject to the initial condition

$$V_m(0) = 0, \tag{2.40}$$

where

$$R_m(\vec{V}_{m-1}) = \frac{1}{(m-1)!} \left. \frac{\partial^{m-1} \mathcal{N}[\Phi(t;q)]}{\partial q^{m-1}} \right|_{q=0} \tag{2.41}$$

and

$$\chi_m = \begin{cases} 0 \text{ when } m \leq 1, \\ 1 \text{ otherwise.} \end{cases} \tag{2.42}$$

From Equations (2.24) and (2.41), we have

$$R_m(\vec{V}_{m-1}) = \dot{V}_{m-1}(t) + \sum_{j=0}^{m-1} V_j(t) V_{m-1-j}(t) - (1-\chi_m). \tag{2.43}$$

Notice that $R_m(\vec{V}_{m-1})$ given by the above expression is only dependent upon

$$V_0(t), V_1(t), V_2(t), \cdots, V_{m-1}(t),$$

which are known when solving the mth-order deformation equations (2.39) and (2.40). Thus, according to the definition (2.23) of the auxiliary operator $\mathcal{L}$, Equation (2.39) is a linear first-order differential equation, subject to the linear initial condition (2.40). Therefore, the solution $V_m(t)$ of high-order deformation equations (2.39) and (2.40) can be easily gained, especially by means of computation software such as Mathematica, Maple, MathLab, and so on. According to (2.38), we in essence transfer the original nonlinear problem, governed by Equations (2.5) and (2.6), into an infinite number of linear sub-problems governed by high-order deformation equations (2.39) and (2.40), and then use the sum of the solutions $V_m(t)$ of its first several sub-problems to approximate the exact solution. Note that such a kind of transformation needs not the existence of any small or large parameters in governing equation and initial/boundary conditions.

The mth-order approximation of $V(t)$ is given by

$$V(t) \approx \sum_{n=0}^{m} V_n(t). \tag{2.44}$$

It should be emphasized that the zero-order deformation equation (2.25) is determined by the auxiliary linear operator $\mathcal{L}$, the initial approximation $V_0(t)$, the auxiliary parameter $\hbar$, and the auxiliary function $H(t)$. Theoretically speaking, the solution $V(t)$ given by the above approach is dependent of the auxiliary linear operator $\mathcal{L}$, the initial approximation $V_0(t)$, the auxiliary parameter $\hbar$, and the auxiliary function $H(t)$. Thus, unlike all previous analytic techniques, the convergence region and rate of solution series given by the above approach might not be uniquely determined. This is indeed true, as shown later in this chapter.

2.3.3 Convergence theorem

THEOREM 2.1

As long as the series (2.38) converges, where $V_m(t)$ is governed by the high-order deformation equations (2.39) and (2.40) under the definitions (2.42) and (2.43), it must be the exact solution of Equations (2.5) and (2.6).

Proof: If the series

$$\sum_{m=0}^{+\infty} V_m(t)$$

converges, we can write

$$S(t) = \sum_{m=0}^{+\infty} V_m(t)$$

and it holds

$$\lim_{m \to +\infty} V_m(t) = 0. \tag{2.45}$$

Using the definition (2.42) of χ_m, we have

$$\begin{aligned} &\sum_{m=1}^{n} [V_m(t) - \chi_m \, V_{m-1}(t)] \\ &= V_1 + (V_2 - V_1) + (V_3 - V_2) + \cdots + (V_n - V_{n-1}) \\ &= V_n(t), \end{aligned}$$

which gives us, according to (2.45),

$$\sum_{m=1}^{+\infty} [V_m(t) - \chi_m \, V_{m-1}(t)] = \lim_{n \to +\infty} V_n(t) = 0.$$

Furthermore, using the above expression and the definition (2.23) of $\mathcal{L}$, we have

$$\sum_{m=1}^{+\infty} \mathcal{L} [V_m(t) - \chi_m \, V_{m-1}(t)] = \mathcal{L} \sum_{m=1}^{+\infty} [V_m(t) - \chi_m \, V_{m-1}(t)] = 0.$$

From the above expression and Equation (2.39), we obtain

$$\sum_{m=1}^{+\infty} \mathcal{L} [V_m(t) - \chi_m \, V_{m-1}(t)] = \hbar \, H(t) \sum_{m=1}^{+\infty} R_m(\vec{V}_{m-1}) = 0$$

which gives, since $\hbar \neq 0$ and $H(t) \neq 0$, that

$$\sum_{m=1}^{+\infty} R_m(\vec{V}_{m-1}) = 0. \tag{2.46}$$

From (2.43), it holds

$$\begin{aligned}\sum_{m=1}^{+\infty} R_m(\vec{V}_{m-1}) &= \sum_{m=1}^{+\infty}\left[\dot{V}_{m-1}(t) + \sum_{j=0}^{m-1} V_j(t)V_{m-1-j}(t) - (1-\chi_m)\right]\\ &= \sum_{m=0}^{+\infty}\dot{V}_m(t) - 1 + \sum_{m=1}^{+\infty}\sum_{j=0}^{m-1} V_j(t)V_{m-1-j}(t)\\ &= \sum_{m=0}^{+\infty}\dot{V}_m(t) - 1 + \sum_{j=0}^{+\infty}\sum_{m=j+1}^{+\infty} V_j(t)V_{m-1-j}(t)\\ &= \sum_{m=0}^{+\infty}\dot{V}_m(t) - 1 + \sum_{j=0}^{+\infty} V_j(t)\sum_{i=0}^{+\infty} V_i(t)\\ &= \dot{S}(t) + S^2(t) - 1. \end{aligned} \tag{2.47}$$

From Equations (2.46) and (2.47), we have

$$\dot{S}(t) + S^2(t) - 1 = 0, \qquad t \geq 0.$$

From Equations (2.22) and (2.40), it holds

$$S(0) = \sum_{m=0}^{+\infty} V_m(0) = V_0(0) + \sum_{m=1}^{+\infty} V_m(0) = V_0(0) = 0.$$

Therefore, according to the above two expressions, $S(t)$ must be the exact solution of Equations (2.5) and (2.6). This ends the proof.

Note that the above theorem is valid for the auxiliary linear operator $\mathcal{L}$ defined by (2.23) in a rather general form, where $\gamma_1(t) \neq 0$ and $\gamma_2(t)$ can be different functions, as illustrated later. This convergence theorem is important. It is because of this theorem that we can focus on ensuring that the approximation series converge. It is clear that the convergence of the series (2.38) depends upon the auxiliary parameter $\hbar$, the auxiliary function $H(t)$, the initial guess $V_0(t)$, and the auxiliary linear operator $\mathcal{L}$. Fortunately, the homotopy analysis method provides us with great freedom to choose all of them. Thus, as long as $\hbar$, $H(t)$, $V_0(t)$, and $\mathcal{L}$ are so properly chosen that the series (2.38) converges in a region $0 \leq t \leq t_0$, it *must* converge to the exact solution in this region. Therefore, the combination of the convergence theorem and the freedom of the choice of the auxiliary parameter $\hbar$, the auxiliary function $H(t)$, the initial guess $V_0(t)$, and the auxiliary linear operator $\mathcal{L}$ establishes the cornerstone of the validity and flexibility of the homotopy analysis method.

2.3.4 Some fundamental rules

As mentioned above, we have great freedom to choose the auxiliary linear operator $\mathcal{L}$, the initial approximation $V_0(t)$, and the auxiliary function $H(t)$

to construct the zero-order deformation equation. Theoretically, the foregoing freedom is so great that we can choose a lot of different auxiliary functions $H(t)$, initial approximations $V_0(t)$, and auxiliary linear operators $\mathcal{L}$. However, from the practical point of view, the freedom seems too great and it is necessary for us to have some fundamental rules to direct us.

Given a nonlinear problem, the essence of analytic approximation is to express its solution by a proper set of base functions. It is well known that a real function $f(x)$ can be approximated by many different base functions and thus can be more efficiently approximated by a relatively better set of base functions. The type of base functions is therefore rather important for the efficiency of approximating a nonlinear problem. The key of efficiently approximating a given nonlinear problem is to choose a relatively better set of base functions. Fortunately, using the freedom in the choice of the auxiliary linear operator $\mathcal{L}$, the initial approximation $V_0(t)$, and the auxiliary function $H(t)$, we can obtain many solution expressions of $V(t)$ presented by different base functions from which we might choose a better one to approximate a given nonlinear problem more efficiently.

In many cases, by mean of analyzing its physical background and/or its initial/boundary conditions and/or its type of nonlinearity, we might know what kinds of base functions are proper to represent the solution, even without solving a given nonlinear problem. For example, let

$$\{e_k(t) \ | \ k = 0, 1, 2, \cdots\} \tag{2.48}$$

denote such a set of base functions for the illustrative problem considered in this chapter. We can represent the solution in a series

$$V(t) = \sum_{n=0}^{+\infty} c_n \, e_k(t), \tag{2.49}$$

where c_n is a coefficient. As long as such a set of base functions is determined, the auxiliary function $H(t)$, the initial approximation $V_0(t)$, and the auxiliary linear operator $\mathcal{L}$ must be chosen in such a way that all solutions of the corresponding high-order deformation equations exist and can be expressed by this set of base functions. This provides us with a fundamental rule to direct the choice of the auxiliary function $H(t)$, the initial approximation $V_0(t)$, and the auxiliary linear operator $\mathcal{L}$, called the *rule of solution expression*. This rule plays an important role in the frame of the homotopy analysis method, as shown in this chapter.

As mentioned above, a real function $f(x)$ might be expressed by many different base functions. Thus, there might exist some different kinds of *rule of solution expressions* and all of them might give accurate approximations for a given nonlinear problem. In this case we might gain the best one by choosing the best set of base functions.

To further restrict the choice of the auxiliary function $H(t)$, it seems necessary to propose the so-called *rule of coefficient ergodicity*, i.e., all coefficients in the solution expression, such as c_n in (2.49), can be modified to ensure the completeness of the set of base functions. In many cases, by means of the *rule of solution expression* and the *rule of coefficient ergodicity*, auxiliary functions can be uniquely determined. It is clear that the high-order deformation equations should be closed and have solutions. This provides us with the so-called *rule of solution existence.*

The so-called *rule of solution expression, rule of coefficient ergodicity*, and *rule of solution existence* play important roles and greatly simplify the application of the homotopy analysis method.

2.3.5 Solution expressions

Different from the foregoing solution expressions given by the perturbation and nonperturbation methods mentioned above, the solution given by the homotopy analysis method can be represented by many different base functions, as shown in this subsection.

2.3.5.1 Solution expressed by polynomial functions

Note that the perturbation solution (2.12) is a power series of t. So, it is straightforward to use the set of base functions

$$\left\{t^{2m+1} \mid m = 0, 1, 2, 3, \cdots\right\} \tag{2.50}$$

to represent $V(t)$, i.e.,

$$V(t) = \sum_{m=0}^{+\infty} a_m\, t^{2m+1}, \tag{2.51}$$

where a_m is a coefficient. This provides us with the first *rule of solution expression* of the illustrative problem.

Under the first *rule of solution expression* and according to the initial condition (2.22), it is straightforward to choose

$$V_0(t) = t \tag{2.52}$$

as the initial approximation of $V(t)$, and to choose an auxiliary linear operator

$$\mathcal{L}[\Phi(t;q)] = \frac{\partial \Phi(t;q)}{\partial t} \tag{2.53}$$

with the property

$$\mathcal{L}\left(C_1\right) = 0, \tag{2.54}$$

where C_1 is an integral constant. Under the first *rule of solution expression* denoted by (2.51) and from Equation (2.39), the auxiliary function $H(t)$ can be chosen in the form

$$H(t) = t^{2\kappa}. \tag{2.55}$$

According to (2.54), the solution of Equation (2.39) becomes

$$V_m(t) = \chi_m V_{m-1}(t) + \hbar \int_0^t \tau^{2\kappa} \; R_m(\vec{V}_{m-1}) \, d\tau + C_1,$$

where the integral constant C_1 is determined by the initial condition (2.40). It is found that when $\kappa \le -1$ the term t^{-1} appears in the solution expression of $V_m(t)$, which disobeys the first *rule of solution expression* denoted by (2.51). In addition, when $\kappa \ge 1$, the base t^3 always disappears in the solution expression of $V_m(t)$ so that the coefficient of the term t^3 is always zero and thus cannot be modified even if the order of approximation tends to infinity. This, however, disobeys the so-called *rule of coefficient ergodicity.* In order to obey both of the first *rule of solution expression* denoted by (2.51) and the *rule of coefficient ergodicity*, we had to set $\kappa = 0$. This uniquely determines the corresponding auxiliary function

$$H(t) = 1. \tag{2.56}$$

We now successively obtain

$$\begin{aligned}
V_1(t) &= \frac{1}{3}\hbar t^3, \\
V_2(t) &= \frac{1}{3}\hbar(1+\hbar)t^3 + \frac{2}{15}\hbar^2 t^5, \\
V_3(t) &= \frac{1}{3}\hbar(1+\hbar)^2 t^3 + \frac{2}{15}\hbar^2(1+\hbar)t^5 + \frac{17}{315}\hbar^3 t^7, \\
&\vdots
\end{aligned}$$

It is found that the corresponding mth-order approximation can be expressed by

$$V(t) \approx \sum_{k=0}^{m} V_k(t) = \sum_{n=0}^{m} \mu_0^{m,n}(\hbar) \; \left[\alpha_{2n+1} \, t^{2n+1}\right], \tag{2.57}$$

where α_{2n+1} is the same coefficient as that which appeared in the perturbation solution (2.12), and the function $\mu_0^{m,n}(\hbar)$ is defined by

$$\mu_0^{m,n}(\hbar) = (-\hbar)^n \sum_{j=0}^{m-n} \binom{n-1+j}{j} (1+\hbar)^j. \tag{2.58}$$

Although the initial approximation $V_0(t)$, the auxiliary linear operator $\mathcal{L}$, and the auxiliary function $H(t)$ have been determined, we still have freedom to choose a proper value of the auxiliary parameter $\hbar$. Equation (2.57) denotes a *family* of solution expressions in auxiliary parameter $\hbar$. It is easy to prove that the function $\mu_0^{m,n}(\hbar)$ mentioned above has the property

$$\mu_0^{m,n}(-1) = 1, \qquad \text{when } n \le m. \tag{2.59}$$

For any a finite positive integer n, it holds

$$\lim_{m\to+\infty} \mu_0^{m,n}(\hbar) = \begin{cases} 1, & \text{when } |1+\hbar| < 1, \\ \infty, & \text{when } |1+\hbar| > 1. \end{cases} \tag{2.60}$$

These two properties will be proved later in this chapter. So, when $\hbar = -1$, we have from (2.59), (2.57), and (2.12) that

$$V(t) = V_{pert}(t). \tag{2.61}$$

Therefore, the perturbation solution (2.12) is only a special case of the solution expression (2.57) when $\hbar = -1$, as are the solution (2.15) given by Lyapunov's artificial small parameter method and the solution (2.17) given by Adomian's decomposition method. Equation (2.57) logically contains the solution expression given by perturbation method, Lyapunov's artificial small parameter method, and Adomian's decomposition method and thus is more general.

Note that the coefficients of the solution expression (2.57) depend upon the auxiliary parameter $\hbar$. According to (2.60), the necessary condition for the series (2.57) to be convergent is $|1+\hbar| < 1$, i.e.,

$$-2 < \hbar < 0.$$

It is interesting that the convergence region of the solution series (2.57) depends upon the value of $\hbar$. The closer the value of $\hbar$ $(-2 < \hbar < 0)$ is to zero, the larger the convergence region of the series (2.57), as shown in Figure 2.1. It is found that the solution series (2.57) converges in the region

$$0 \le t < \rho_0 \sqrt{\frac{2}{|\hbar|} - 1},$$

where $\rho_0 \approx 3/2$ is the convergence radius of the perturbation solution (2.12). So, as $\hbar$ $(-2 < \hbar < 0)$ tends to zero from below, the solution series (2.57) converges to the exact solution $V(t) = \tanh(t)$ in the whole region

$$0 \le t < +\infty.$$

Unlike all previous analytic techniques, we can adjust and control the convergence region of the solution series (2.57) by assigning $\hbar$ a proper value. The auxiliary parameter $\hbar$ therefore provides us with a convenient way to adjust and control convergence regions of solution series.

2.3.5.2 Solution expressed by fractional functions

Although the solution expression (2.57) represented by the base functions (2.50) can be valid in the whole region

$$0 \le t < +\infty$$

as $\hbar$ $(-2 < \hbar < 0)$ tends to 0, the order of approximation must be very high to give an accurate enough result when the absolute value of $\hbar$ $(-2 < \hbar < 0)$ is small. This kind of approximation is barely efficient, although theoretically it is better and more general than the solution (2.12) given by the perturbation method, Lyapunov's artificial small parameter method, and Adomian's decomposition method. It is therefore necessary to choose a better set of base functions to approximate $V(t)$ more efficiently.

As mentioned before, even without solving Equations (2.5) and (2.6), it is easy to know the limit velocity

$$V(+\infty) = 1.$$

The initial approximation (2.52) obviously does not satisfy this property. Generally, a power series converges in a finite region, therefore, the set (2.50) of polynomial functions is not proper to efficiently approximate $V(t)$ in the whole region $0 \le t < +\infty$.

Notice that it holds

$$\lim_{t\to+\infty} \frac{1}{(1+t)^m} = 0, \qquad m \ge 1.$$

Thus, a function expressed by the set of base functions

$$\left\{(1+t)^{-m} \mid m = 0, 1, 2, 3, \cdots\right\} \tag{2.62}$$

has a finite value as $t \to +\infty$. We can assume that the solution $V(t)$ can be expressed by

$$V(t) = \sum_{m=0}^{+\infty} \frac{b_m}{(1+t)^m}, \tag{2.63}$$

where b_m is a coefficient to be determined. This provides us with the second *rule of solution expression* of the illustrative problem.

Under the second *rule of solution expression* and using the initial condition (2.6) and the limit velocity (2.7), it is straightforward to choose

$$V_0(t) = 1 - \frac{1}{1+t} \tag{2.64}$$

as the initial approximation of $V(t)$, and to choose the corresponding auxiliary linear operator

$$\mathcal{L}[\Phi(t;q)] = (1+t)\,\frac{\partial \Phi(t;q)}{\partial t} + \Phi(t;q) \tag{2.65}$$

with the property

$$\mathcal{L}\left(\frac{C_2}{1+t}\right) = 0, \tag{2.66}$$

where C_2 is an integral constant. Under the definition (2.65) of $\mathcal{L}$, the solution of the high-order deformation equation (2.39) becomes

$$V_m(t) = \chi_m V_{m-1}(t) + \frac{\hbar}{1+t}\int_0^t H(\tau)\ R_m(\vec{V}_{m-1})\ d\tau + \frac{C_2}{1+t}, \qquad m \geq 1,$$

where the integral constant C_2 is determined by the initial condition (2.40). Under the second *rule of solution expression* denoted by (2.63) and from Equation (2.39), the auxiliary function $H(t)$ should be in the form

$$H(t) = \frac{1}{(1+t)^{\kappa}}, \tag{2.67}$$

where κ is an integer. It is found that, when $\kappa \leq 0$, the solutions of the high-order deformation equations (2.39) contain the term

$$\frac{\ln(1+t)}{1+t}$$

which incidentally disobeys the second *rule of solution expression* denoted by (2.63). When $\kappa > 1$, the base $(1+t)^{-2}$ disappears in the solution expression of $V_m(t)$ so that the coefficient of the term $(1+t)^{-2}$ is always zero and thus cannot be modified even if the order of approximation tends to infinity. This, however, disobeys the so-called *rule of coefficient ergodicity*. Thus, to obey both the second *rule of solution expression* and *rule of coefficient ergodicity*, we had to choose $\kappa = 1$, which uniquely determines the corresponding auxiliary function

$$H(t) = \frac{1}{1+t}. \tag{2.68}$$

Thereafter, we successively obtain

$$\begin{aligned}
V_1(t) &= -\frac{\hbar}{1+t} + \frac{2\hbar}{(1+t)^2} - \frac{\hbar}{(1+t)^3}, \\
V_2(t) &= -\hbar\left(1 + \frac{7}{12}\hbar\right)\frac{1}{1+t} + \frac{2\hbar(1+\hbar)}{(1+t)^2} \\
&\quad -\hbar\left(1+\frac{7}{2}\hbar\right)\frac{1}{(1+t)^3} + \frac{10\hbar^2}{3(1+t)^4} - \frac{5\hbar^2}{4(1+t)^5}, \\
&\ \ \vdots
\end{aligned}$$

It is found that the corresponding mth-order approximation of $V(t)$ can be expressed by

$$V(t) \approx \sum_{n=0}^{2m+1} \frac{\beta_{m,n}(\hbar)}{(1+t)^n}, \tag{2.69}$$

where $\beta_{m,n}(\hbar)$ is a coefficient dependent upon $\hbar$.

Note that we still have freedom to choose the auxiliary parameter $\hbar$. So, (2.69) is in fact a new family of solution expressions. To investigate the influence of $\hbar$ on the solution series (2.69), we can first consider the convergence of some related series such as $V'(0), V''(0), V'''(0)$, and so on. It is found that $V'(0) = 1$ holds for all results at any order of approximations, thus it cannot provide us with any useful information about the choice of $\hbar$. However, $V''(0)$ and $V'''(0)$ are dependent of $\hbar$. Let $\mathbf{R}_\hbar$ denote a set of all possible values of $\hbar$ by means of which the corresponding series of $V''(0)$ converges. For brevity, we call the set $\mathbf{R}_\hbar$ *the valid region* of $\hbar$ for $V''(0)$. According to Theorem 2.1, for each $\hbar \in \mathbf{R}_\hbar$, the corresponding series of $V''(0)$ converges to the same result. The curve $V''(0)$ versus $\hbar$ contains a horizontal line segment above the valid region $\mathbf{R}_\hbar$. We call such a kind of curve *the $\hbar$-curve*, which clearly indicates the valid region $\mathbf{R}_\hbar$ of a solution series. The so-called $\hbar$-curves of $V''(0)$ and $V'''(0)$ given by the solution expression (2.69) are as shown in Figure 2.2. From Figure 2.2 it is clear that the series of $V''(0)$ and $V'''(0)$ given by the solution series (2.69) are convergent when

$$-3/2 \leq \hbar \leq -1/2.$$

This is indeed true. For example, for five different values of $\hbar$ in the region $-3/2 \leq \hbar \leq -1/2$, the series of $V''(0)$ and $V'''(0)$ given by (2.69) converge to the corresponding exact value 0 and -2, respectively, as shown in Tables 2.1 and 2.2. It is interesting that the convergence rate of the approximation series depends upon the value of $\hbar$, and the series of $V''(0)$ and $V'''(0)$ given by (2.69) converge fastest when $\hbar = -1$, as shown in Tables 2.1 and 2.2. This indicates that we can adjust the convergence rate of the solution series (2.69) by means of the auxiliary parameter $\hbar$. It is also true that, as long as the series of $V''(0)$ and $V'''(0)$ are convergent, the series (2.69) converge in the whole region $0 \leq t < +\infty$. Thus, according to Theorem 2.1, all of these convergent series must be the exact solution of the original nonlinear problem. For example, when $\hbar = -1$, the series (2.69) converges to the exact solution in the whole region $0 \leq t < +\infty$, as shown in Table 2.3. In general, by means of the so-called $\hbar$-curves, it is straightforward to know the corresponding valid region of $\hbar$. Choosing a value of $\hbar$ in the valid region, we can ensure that the corresponding solution series is convergent. In this way, we can control and adjust the convergence region and rate of solution series. Thus, the auxiliary parameter $\hbar$ plays an important role within the frame of the homotopy analysis method.

Unlike the solution (2.12) given by the perturbation method, Lyapunov's artificial small parameter method, and Adomian's decomposition method, the solution series (2.69) converges to the exact solution in the whole region $0 \leq t < +\infty$ when $-3/2 \leq \hbar \leq -1/2$. The solution series (2.69) is therefore almost more efficient than (2.57), although, theoretically speaking, both of them may converge to the exact solution in the whole region $0 \leq t < +\infty$. This is mainly because, for the illustrative problem, the base functions (2.62) are better and thus more efficient than (2.50).

Finally, let us investigate the relationship between the solution expression (2.69) with the perturbation solution (2.12). Substituting $\hbar = -1$ and

$$\frac{1}{1+t} = 1 - t + t^2 - t^3 + \cdots$$

into the 10th-order approximation of $V(t)$ given by (2.69), we have

$$V(t) \sim t - \frac{1}{3}\,t^3 + \frac{2}{5}\,t^5 - \frac{17}{315}\,t^7 + \frac{62}{2835}\,t^9 + \cdots,$$

whose first several terms are exactly the same as the perturbation series (2.12).

2.3.5.3 Solution expressed by exponential functions

It is well known that

$$\lim_{t\to+\infty} \exp(-nt) = 0, \qquad n \geq 1.$$

So, a function expressed by the set of base functions

$$\{\ \exp(-nt) \mid n \geq 0\ \} \tag{2.70}$$

is finite as t tends to infinity. Considering the limit velocity (2.7), the above base functions are better than (2.50). Assume that $V(t)$ can be expressed by

$$V(t) = \sum_{n=0}^{+\infty} c_n\ \exp(-nt), \tag{2.71}$$

where c_n is a coefficient. This provides us with the third *rule of solution expression* of the illustrative problem.

Under the third *rule of solution expression* and from (2.6) and (2.7), it is straightforward to choose

$$V_0(t) = 1 - \exp(-t) \tag{2.72}$$

as the initial approximation of $V(t)$, and to choose the auxiliary linear operator

$$\mathcal{L}[\Phi(t;q)] = \frac{\partial \Phi(t;q)}{\partial t} + \Phi(t;q) \tag{2.73}$$

with the property

$$\mathcal{L}\left[C_3\ \exp(-t)\right] = 0, \tag{2.74}$$

where C_3 is an integral coefficient. In this case, the solution of the mth-order deformation equation (2.39) becomes

$$V_m(t) = \chi_m V_{m-1}(t) + \hbar\ \exp(-t) \int_0^t \exp(\tau)\ H(\tau)\ R_m(\vec{V}_{m-1})\ d\tau \\ + C_3 \exp(-t), \qquad m \geq 1,$$

where the integral constant C_3 is determined by (2.40). According to the third *rule of solution expression* denoted by (2.71) and from Equation (2.39), the auxiliary function $H(t)$ should be in the form

$$H(t) = \exp(-\kappa\, t), \tag{2.75}$$

where κ is an integer. It is found that, when $\kappa \leq 0$, the solutions of the high-order deformation equations (2.39) contain the term

$$t\ \exp(-t),$$

which incidentally disobeys the third *rule of solution expression* denoted by (2.71). When $\kappa \geq 2$, the base $\exp(-2t)$ always disappears in the solution expressions of the high-order deformation equation (2.39) so that the coefficient of the term $\exp(-2\,t)$ cannot be modified even if the order of approximation tends to infinity. This, however, disobeys the so-called *rule of coefficient ergodicity*. Thus, to obey both of the third *rule of solution expression* denoted by (2.71) and the *rule of coefficient ergodicity*, we had to set $\kappa = 1$, which uniquely determines the corresponding auxiliary function

$$H(t) = \exp(-t). \tag{2.76}$$

Therefore, we have

$$\begin{aligned}
V_1(t) &= -\frac{\hbar}{2}\, e^{-t} + \hbar\, e^{-2t} - \frac{\hbar}{2}\, e^{-3t},\\
V_2(t) &= -\frac{\hbar}{2}\left(1+\frac{\hbar}{2}\right) e^{-t} + \hbar\left(1+\frac{\hbar}{2}\right) e^{-2t} - \frac{\hbar}{2}\,(1+\hbar)\, e^{-3t}\\
&\quad + \frac{\hbar^2}{2} e^{-4t} - \frac{\hbar^2}{4} e^{-5t},\\
&\ \vdots
\end{aligned}$$

It is found that the corresponding mth-order approximation of $V(t)$ can be generally expressed by

$$V(t) \approx \sum_{n=0}^{2m+1} \gamma_{m,n}(\hbar)\ \exp(-n\,t), \tag{2.77}$$

where $\gamma_{m,n}(\hbar)$ is a coefficient dependent of $\hbar$.

Equation (2.77) is also a family of solution expressions in the auxiliary parameter $\hbar$. To investigate the influence of $\hbar$ on the convergence of the solution series (2.77), we first plot the so-called $\hbar$-curves of $V''(0)$ and $V'''(0)$, as shown in Figure 2.3. According to these $\hbar$-curves, it is easy to discover the valid region of $\hbar$, which correspond to the line segments nearly parallel to the horizontal axis. The so-called valid regions of $\hbar$ are enlarged as the order of approximations increases, as shown in Figure 2.3. So, it is clear that the series of $V''(0)$ and $V'''(0)$ given by (2.77) converge if $\hbar$ belongs to the corresponding valid regions of $\hbar$. According to Theorem 2.1, they must converge to the exact values of $V''(0)$ and $V'''(0)$, respectively. This is indeed true, as shown in Tables 2.4 and 2.5 when $\hbar = -3/2, -5/4, -1, -3/4$ and $-1/2$. Note that the series seems to converge fastest when $\hbar = -1$. Furthermore, as long as the series of $V''(0)$ and $V'''(0)$ converge, the corresponding solution series (2.77) of $V(t)$ also converges to the exact solution (2.8) in the whole region $0 \le t < +\infty$. For example, the approximation result of $V(t)$ given by (2.77), when $\hbar = -1$, agrees well with the exact result (2.8), as shown in Table 2.6. Generally, it is convenient to investigate the influence of $\hbar$ on the convergence of solution series by means of such kinds of $\hbar$-curves.

Note that the solution expression (2.77) is valid in the whole region $0 \le t < +\infty$. Comparing Tables 2.4 to 2.6 with Tables 2.1 to 2.3, respectively, we find that, by means of the same value of $\hbar$, the solution series (2.77) converges faster than (2.69), and even the 10th-order approximation of (2.77) when $\hbar = -1$ agrees very well with the exact solution. So, the solution expression (2.77) is better and thus more efficient than (2.69) and, as mentioned before, (2.69) is more efficient than (2.57). These therefore illustrate that we may approximate a given nonlinear problem more efficiently by choosing a better set of base functions within the frame of the homotopy analysis method.

It is found that the mth-order approximation (2.77) of $V(t)$ can be explicitly expressed by

$$V(t) \approx 1 + 2\sum_{n=1}^{m} \left[(-1)^n \exp(-nt)\right] \mu_0^{m,n}\left(\frac{\hbar}{2}\right) - \exp(-t)\left[\left(1+\frac{\hbar}{2}\right) + \frac{\hbar}{2}\exp(-2t)\right]^m, \tag{2.78}$$

where the function $\mu_0^{m,n}(x)$ is defined by (2.58). It is interesting that the function $\mu_0^{m,n}(\hbar)$ appears again. Due to the property (2.59), the above expression becomes, when $\hbar = -2$,

$$V(t) \approx 1 + 2\sum_{n=1}^{m} (-1)^n \exp(-nt) + (-1)^{m+1}\exp[-(2m+1)\,t]. \tag{2.79}$$

Note that the exact solution (2.8) can be expanded as a series

$$V(t) \approx 1 + 2\sum_{n=1}^{+\infty} (-1)^n \exp(-nt), \tag{2.80}$$

which converges to the exact solution in the region $0 < t < +\infty$ but diverges at the point $t = 0$ where it gives either 1 or -1. However, with an additional term

$$(-1)^{m+1}\exp[-(2m+1)],$$

the expression (2.79) converges to the exact solution in the whole region $0 \leq t < +\infty$ including the point $t = 0$. In fact, even the third-order approximation of $V(t)$ given by (2.79), i.e.,

$$V(t) \approx 1 - 2\exp(-2\,t) + 2\exp(-4\,t) - 2\exp(-6\,t) + \exp(-7\,t), \tag{2.81}$$

agrees very well with the exact solution, as shown in Figure 2.4.

The idea to avoid the appearance of the term such as

$$\ln(1+t)/(1+t), \qquad t\exp(-t)$$

in approximate expansions is not new. To gain uniformly valid approximations, some perturbation techniques were developed to avoid the appearance of the so-called secular terms such as

$$t\sin t, \quad t\cos t$$

in perturbation solutions. This kind of technique goes back to various scientists in the 19th century such as Lindstedt [52], Bohlin [53], Poincaré [54], Gyldén [55], and so on. The idea was further developed by many scientists such as Lighthill [56, 57], Malkin [58], Kuo [59, 60], and Tsien [61]. However, the terms $\ln(1+t)/(1+t)$ and $t\exp(-t)$ tend to zero as $t \to +\infty$. Therefore, these terms do not belong to the so-called secular term in perturbation techniques. Thus, the *rule of solution expression* can be seen as the generalization of this idea.

To show that the term $t\exp(-t)$ indeed does not belong to the so-called secular terms in perturbation methods, we point out that $V(t)$ can be expressed by the base functions

$$\{t^m \ \exp(-nt) \ \mid \ m \geq 0, n \geq 1\}. \tag{2.82}$$

Using the same initial approximation as (2.72), the same auxiliary linear operator as (2.73) but the auxiliary function

$$H(t) = 1 \tag{2.83}$$

different from (2.76), we can obtain in the similar way the corresponding mth-order approximation of $V(t)$, which can be explicitly expressed by

$$V(t) \approx 1 + 2\sum_{n=1}^{m+1} \sum_{k=0}^{m+1-n} \sigma_0^{m,n,k}(\hbar) \left[(-1)^n \,\frac{(-nt)^k}{k!}\, \exp(-nt)\right], \tag{2.84}$$

where

$$\sigma_0^{m,n,k}(\hbar) = \frac{1}{2}\left[\mu_0^{m,n+k}(\hbar) + \mu_0^{m,n+k-1}(\hbar)\right]. \tag{2.85}$$

It is interesting that the function $\mu_0^{m,n}(\hbar)$ appears once again. By means of the so-called $\hbar$-curves of the corresponding $V''(0)$ and $V'''(0)$, it is found that, when $-2 < \hbar < 0$, the solution series (2.84) converges to the exact solution (2.8) in the whole region $0 \leq t < +\infty$, as shown in Table 2.7.

It should be emphasized that, in the frame of the homotopy analysis method, the solution $V(t)$ may be expressed by four different base functions (2.50), (2.62), (2.70), and (2.82), although the illustrative problem has only a unique solution. Correspondingly, we gain four families of solution expressions (2.57), (2.69), (2.78), and (2.84). Theoretically, all of them can converge to the same exact solution $V(t) = \tanh(t)$ in the whole region $0 \leq t < +\infty$. However, the solution expression (2.57) is least efficient among the four solution expressions and is therefore the worst because it is convergent in a finite region for a given value of $-2 < \hbar < 0$. By means of comparing Tables 2.3, 2.6, and 2.7 with each other, the solution expression (2.78) based on the pure exponential functions is more efficient than the solution expression (2.69) based on fractional functions and the solution expression (2.84) based on combined polynomial and exponential functions, and thus is the best. The solution expression (2.84) is more efficient than the solution expression (2.69). This example clearly illustrates that, in the frame of the homotopy analysis method, the solution of a given nonlinear problem can be expressed by many different base functions and thus can be more efficiently approximated by a better set of base function, even if the solution is unique.

Indeed, this illustrative example is very simple and the exact solution is known. However, it clearly illustrates that, by means of the homotopy analysis method, convergence region and rate of solution series can be adjusted and controlled by means of plotting the so-called $\hbar$-curves and then choosing $\hbar$ in the corresponding valid regions of $\hbar$. Even when it is unnecessary to enlarge convergence regions, we can give a more efficient solution series by assigning $\hbar$ a proper value. This illustrative example also shows the important roles of the *rule of solution expression* and *rule of coefficient ergodicity* in choosing the initial approximation, the auxiliary linear operator, and the auxiliary function.

2.3.6 The role of the auxiliary parameter $\hbar$

As mentioned before, the homotopy analysis method is based on the homotopy, a basic concept of topology. The nonzero auxiliary parameter $\hbar$ is introduced to construct the so-called zero-order deformation equation, which gives a more general homotopy than the traditional one. Thus, unlike all previous analytic techniques, the homotopy analysis method provides us with a family of solution expressions in the auxiliary parameter $\hbar$. As a result, the convergence region and rate of solution series are dependent upon the auxiliary parameter $\hbar$ and thus can be greatly enlarged by means of choosing a proper value for $\hbar$. This provides us with a convenient way to adjust and control convergence region and rate of solution series given by the homotopy analysis method, as illustrated above.

In this subsection we prove in a completely different way that convergence regions of series can be indeed adjusted and controlled by introducing an auxiliary parameter. This proof can provide us with a rational base for the validity of the homotopy analysis method.

First, we emphasize that the definition (2.58) of $\mu_0^{m,n}(\hbar)$ is gained in the homotopy analysis method, which appears in the solution expressions (2.57), (2.78), and (2.84). It is interesting that the same definition can be deduced directly from the famous Newtonian binomial theorem. To show this, consider a series

$$\frac{1}{1+t} = 1 - t + t^2 - t^3 + \cdots = \lim_{m \to +\infty} \sum_{n=0}^{m} (-1)^n t^n, \qquad |t| < 1. \tag{2.86}$$

Define

$$x = 1 + \hbar + \hbar\, t,$$

which gives

$$\frac{1}{1+t} = -\frac{\hbar}{(1-x)}.$$

When $|x| = |1 + \hbar + \hbar\, t| < 1$ and $|1 + \hbar| < 1$, i.e.,

$$-1 < t < \frac{2}{|\hbar|} - 1, \qquad -2 < \hbar < 0,$$

it holds

$$\frac{1}{1+t} = -\frac{\hbar}{1-x} = -\hbar\, \left(1 + x + x^2 + x^3 + \cdots\right) = -\hbar \sum_{n=0}^{+\infty} (1 + \hbar + \hbar\, t)^n.$$

Thus,

$$\frac{1}{1+t} = \lim_{m \to +\infty} \left[-\hbar\, \sum_{n=0}^{m} (1 + \hbar + \hbar\, t)^n \right]$$

is valid in the region

$$-1 < t < \frac{2}{|\hbar|} - 1 \qquad (-2 < \hbar < 0).$$

We have

$$\begin{aligned} & -\hbar \sum_{n=0}^{m} (1 + \hbar + \hbar\, t)^n \\ = & -\hbar \sum_{n=0}^{m} \sum_{k=0}^{n} \binom{n}{k} (1+\hbar)^{n-k}\, (\hbar\, t)^k \\ = & -\hbar \sum_{k=0}^{m} \sum_{n=k}^{m} \binom{n}{k} (1+\hbar)^{n-k}\, \hbar^k\, t^k \end{aligned}$$

$$= \sum_{k=0}^{m} (-1)^k \, t^k (-\hbar)^{k+1} \sum_{i=0}^{m-k} \binom{k+i}{k} (1+\hbar)^i$$

$$= \sum_{k=0}^{m} (-1)^k \, t^k \left[(-\hbar)^{k+1} \sum_{i=0}^{m-k} \binom{k+i}{i} (1+\hbar)^i \right]$$

$$= \sum_{n=0}^{m} (-1)^n \, t^n \, \mu_{-1}^{m,n}(\hbar),$$

where

$$\mu_{-1}^{m,n}(\hbar) = (-\hbar)^{n+1} \sum_{j=0}^{m-n} \binom{n+j}{j} (1+\hbar)^j. \tag{2.87}$$

Comparing the above to the definition (2.58), we gain the relationship

$$\mu_{-1}^{m,n}(\hbar) = \mu_0^{m+1,n+1}(\hbar). \tag{2.88}$$

Thus, it holds

$$\frac{1}{1+t} = \lim_{m \to +\infty} \sum_{n=0}^{m} \mu_0^{m+1,n+1}(\hbar) \; [(-1)^n \, t^n] \tag{2.89}$$

in the region

$$-1 < t < \frac{2}{|\hbar|} - 1 \qquad (-2 < \hbar < 0).$$

Obviously, the convergence region is $-1 < t < 1$ when $\hbar = -1$, $-1 < t < 3$ when $\hbar = -1/2$, and $-1 < t < 99$ when $\hbar = -1/50$, respectively. In particular, the convergence region becomes

$$-1 < t < +\infty$$

as $\hbar$ tends to zero from below. Thus, the convergence region of the series (2.89) can be indeed adjusted and controlled by the auxiliary parameter $\hbar$. What we should emphasize here is that the same definition $\mu_0^{m,n}(\hbar)$ is first obtained in the frame of the homotopy analysis method and then deduced from the famous Newtonian binomial theorem. This fact logically shows the validity and reasonability of the homotopy analysis method.

The foregoing ideas can be employed to give such a generalized theorem as the one below.

THEOREM 2.2

It holds

$$(1+t)^\alpha = \lim_{m \to +\infty} \sum_{n=0}^{m} \mu_\alpha^{m,n}(\hbar) \binom{\alpha}{n} \; t^n \tag{2.90}$$

for a real number α ($\alpha \neq 0, 1, 2, 3, \cdots$) *in the region*

$$-1 < t < \frac{2}{|\hbar|} - 1 \qquad (-2 < \hbar < 0),$$

where

$$\binom{\alpha}{n} = \frac{\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-n+1)}{n!}$$

and

$$\mu_\alpha^{m,n}(\hbar) = (-\hbar)^{n-\alpha} \sum_{j=0}^{m-n} (-1)^j \binom{\alpha-n}{j} (1+\hbar)^j. \tag{2.91}$$

Proof: Write $x = 1 + \hbar + \hbar\, t$. Let $|x| < 1$ and $|1+\hbar| < 1$, i.e.,

$$-1 < t < \frac{2}{|\hbar|} - 1, \qquad -2 < \hbar < 0.$$

By the traditional Newton binomial theorem [62], it holds when $|x| < 1$ and $|1+\hbar| < 1$ that

$$\begin{aligned}
(1+t)^\alpha = (-\hbar)^{-\alpha}(1-x)^\alpha &= (-\hbar)^{-\alpha} \sum_{n=0}^{+\infty} (-1)^n \binom{\alpha}{n} x^n \\
&= (-\hbar)^{-\alpha} \sum_{n=0}^{+\infty} (-1)^n \binom{\alpha}{n} (1+\hbar+\hbar\, t)^n \\
&= \lim_{m\to+\infty} (-\hbar)^{-\alpha} \sum_{n=0}^{m} (-1)^n \binom{\alpha}{n} (1+\hbar+\hbar\, t)^n.
\end{aligned}$$

The sum of the first m terms of above series is given by

$$\begin{aligned}
&(-\hbar)^{-\alpha} \sum_{n=0}^{m} (-1)^n \binom{\alpha}{n} (1+\hbar+\hbar\, t)^n \\
&= (-\hbar)^{-\alpha} \sum_{n=0}^{m} (-1)^n \binom{\alpha}{n} \sum_{j=0}^{n} \binom{n}{j} (1+\hbar)^{n-j}\, \hbar^j\, t^j \\
&= (-\hbar)^{-\alpha} \sum_{j=0}^{m} t^j \sum_{n=j}^{m} (-1)^n \binom{\alpha}{n} \binom{n}{j}\ (1+\hbar)^{n-j}\, \hbar^j \\
&= (-\hbar)^{-\alpha} \sum_{j=0}^{m}\ t^j \sum_{i=0}^{m-j} (-1)^{i+j} \binom{\alpha}{i+j} \binom{i+j}{j}\ (1+\hbar)^i\, \hbar^j \\
&= (-\hbar)^{-\alpha} \sum_{j=0}^{m}\ t^j \sum_{i=0}^{m-j} (-1)^{i+j} \binom{\alpha}{j} \binom{\alpha-j}{i}\ (1+\hbar)^i\, \hbar^j
\end{aligned}$$

$$= \sum_{j=0}^{m} \left[\binom{\alpha}{j} \, t^j \right] \sum_{i=0}^{m-j} (-1)^i \binom{\alpha - j}{i} \, (1+\hbar)^i \, (-\hbar)^{j-\alpha}$$

$$= \sum_{n=0}^{m} \mu_\alpha^{m,n}(\hbar) \left[\binom{\alpha}{n} \, t^n \right],$$

where

$$\mu_\alpha^{m,n}(\hbar) = (-\hbar)^{n-\alpha} \sum_{j=0}^{m-n} (-1)^j \binom{\alpha - n}{j} \, (1+\hbar)^j.$$

This ends the proof.

Although the definition (2.91) is deduced for real numbers $-\infty < \alpha < +\infty$ except integers $\alpha = 0, 1, 2, 3, \cdots$, it is valid for all real numbers. For any integer k, we have using the definition (2.91) that

$$\mu_k^{m,n}(\hbar) = (-\hbar)^{n-k} \sum_{j=0}^{m-n} \binom{n-k-1+j}{j} (1+\hbar)^j, \tag{2.92}$$

which contains the definition (2.58) of $\mu_0^{m,n}(\hbar)$ and the definition (2.87) of $\mu_{-1}^{m,n}(\hbar)$. It can be proved that for any real number $\alpha \in (-\infty, +\infty)$ it holds

$$\mu_\alpha^{m,n}(-1) = 1 \tag{2.93}$$

and

$$\lim_{m \to +\infty} \mu_\alpha^{m,n}(\hbar) = 1, \qquad \text{when } |1+\hbar| < 1 \tag{2.94}$$

for any finite positive integer n. According to the definitions (2.58) and (2.91), it holds for the integer $l \geq 0$ that

$$\mu_{-l}^{m,n}(\hbar) = \mu_0^{m+l,n+l}(\hbar). \tag{2.95}$$

The proof of (2.93) is straightforward. When $|1+\hbar| < 1$ it holds from the definition (2.91) that

$$\begin{aligned}
&\lim_{m \to +\infty} \mu_\alpha^{m,n}(\hbar) \\
&= (-\hbar)^{n-\alpha} \sum_{k=0}^{+\infty} (-1)^k \binom{\alpha - n}{k} (1+\hbar)^k \\
&= (-\hbar)^{n-\alpha} \sum_{k=0}^{+\infty} \binom{\alpha - n}{k} (-1-\hbar)^k \\
&= (-\hbar)^{n-\alpha} \left[1 + (-1-\hbar)\right]^{\alpha - n} \\
&= 1.
\end{aligned}$$

This ends the proof of (2.94).

It should be emphasized that $\mu_0^{m,n}(\hbar)$ defined by (2.58) and $\mu_{-1}^{m,n}(\hbar)$ defined by (2.87) are only special cases of (2.91) when $\alpha = 0$ and $\alpha = -1$, respectively. All of these logically verify the reasonableness and validity of the solutions (2.57), (2.78) and (2.84) given by the homotopy analysis method.

The rationality of solutions (2.57), (2.78), and (2.84) can be explained another way. It is known that the Taylor series of any a given function is unique. According to the property (2.60), when $|1+\hbar| < 1$, it holds for any a given finite positive integer N that

$$\lim_{m\to+\infty} \sum_{n=0}^{N} \left(\alpha_{2n+1} t^{2n+1}\right) \mu_0^{m,n}(\hbar) = \sum_{n=0}^{N} \alpha_{2n+1} t^{2n+1}.$$

Therefore, given any finite positive integer N, the sum of the first N terms of the solution (2.57) is the same as the sum of the first N terms of the perturbation solution (2.12), if the order of approximation tends to infinity. Thus, the series (2.57) obeys the uniqueness of the Taylor series, but now in a more general meaning. Let

$$(\alpha_1, \alpha_3, \alpha_5, \alpha_7, \cdots)$$

denote a point in a space $\mathcal{S}$, where α_k $(k = 1, 3, 5, \cdots)$ is the coefficient in the perturbation solution (2.12). The perturbation solution (2.12) can be regarded as an approach to the point $(\alpha_1, \alpha_3, \alpha_5, \alpha_7, \cdots)$ along such a traditional path Γ_0 defined by:

$$\begin{aligned}
&(\alpha_1, 0, 0, 0, \cdots),\\
&(\alpha_1, \alpha_3, 0, 0, \cdots),\\
&(\alpha_1, \alpha_3, \alpha_5, 0, \cdots),\\
&\qquad\vdots
\end{aligned}$$

However, the solution series (2.57) can be regarded as an approach to the same point

$$(\alpha_1, \alpha_3, \alpha_5, \alpha_7, \cdots)$$

but along such a more general path $\Gamma(\hbar)$ defined by:

$$\begin{aligned}
&(\alpha_1 \mu_0^{0,0}(\hbar), 0, 0, 0, \cdots),\\
&(\alpha_1 \mu_0^{1,0}(\hbar), \alpha_3 \mu_0^{1,1}(\hbar), 0, 0, \cdots),\\
&(\alpha_1 \mu_0^{2,0}(\hbar), \alpha_3 \mu_0^{2,1}(\hbar), \alpha_5 \mu_0^{2,2}(\hbar), 0, \cdots),\\
&\qquad\qquad\vdots
\end{aligned}$$

Notice that the path $\Gamma(\hbar)$ depends on the auxiliary parameter $\hbar$. According to (2.59), the path $\Gamma(-1)$ (when $\hbar = -1$) is exactly the same as the traditional one Γ_0. When $|1+\hbar| < 1$ but $\hbar \neq -1$, the path $\Gamma(\hbar)$ is different from the traditional path Γ_0. Even in this case, according to the property (2.60), all of them approach to the same point $(\alpha_1, \alpha_3, \alpha_5, \alpha_7, \cdots)$. Therefore, the solution series (2.57) can be regarded as a kind of limit process along an infinite number of approaching paths $\Gamma(\hbar)$ to the same point $(\alpha_1, \alpha_3, \alpha_5, \alpha_7, \cdots)$. It is well known that the result of such kind of limit process often depends upon the approaching path. For example, consider the limit

$$\lim_{(x,y)\to(0,0)} \frac{\sqrt{x^2+y^2}}{|x|}.$$

There are a lot of approaching paths to $(0,0)$. For simplicity, let us consider the path $y = \beta x$, where β is a real number. It obviously holds that

$$\lim_{(x,y)\to(0,0)} \frac{\sqrt{x^2+y^2}}{|x|} = \sqrt{1+\beta^2}.$$

The limit is therefore dependent upon the approaching path to the point (0,0). This clearly explains why the convergence region of the solution series (2.57) is dependent upon the auxiliary parameter $\hbar$, because the function $\mu_0^{m,n}(\hbar)$ defines different approaching paths by different values of $\hbar$.

According to above explanation, the function $\mu_\alpha^{m,n}(\hbar)$ can be used to define different approaching paths for a limit process by different values of α and $\hbar$. For the sake of this reason, the function $\mu_\alpha^{m,n}(\hbar)$ is called *the approach function of the first kind*. To generalize the definition (2.85) of $\sigma_0^{m,n}(\hbar)$, we define

$$\sigma_\alpha^{m,n,k}(\hbar) = \frac{1}{2}\left[\mu_\alpha^{m,n+k}(\hbar) + \mu_\alpha^{m,n+k-1}(\hbar)\right] \tag{2.96}$$

as *the approach function of the second kind*, where $|1+\hbar| < 1$ and $-\infty < \alpha < +\infty$. It is easy to prove that, for $\alpha \in (-\infty, +\infty)$ and $0 \le n \le m+1$, it holds

$$\sigma_\alpha^{m,n,k}(-1) = \begin{cases} 1, & \text{when } 0 \le k < m+1-n, \\ 1/2, & \text{when } k = m+1-n, \end{cases} \tag{2.97}$$

and

$$\lim_{m\to+\infty} \sigma_\alpha^{m,n,k}(\hbar) = \begin{cases} 1, & \text{when } |1+\hbar| < 1, \\ \infty, & \text{when } |1+\hbar| > 1, \end{cases} \tag{2.98}$$

where n and k are finite positive integers.

The approach functions $\mu_\alpha^{m,n}(\hbar)$ and $\sigma_\alpha^{m,n,k}(\hbar)$ have rather general meaning and therefore could be employed to greatly enlarge convergence regions of approximation series. For example, from the traditional Taylor series

$$\sum_{n=0}^{+\infty} \frac{f^{(n)}(z_0)}{n!}(z-z_0)^n$$

of a function $f(z)$, we can define *the generalized Taylor series of the first kind*

$$\lim_{m\to+\infty} \sum_{n=0}^{m} \mu_\alpha^{m,n}(\hbar)\left[\frac{f^{(n)}(z_0)}{n!}(z-z_0)^n\right]$$

and *the generalized Taylor series of the second kind*

$$\lim_{m\to+\infty}\sum_{n=0}^{m}\sigma_\alpha^{m,n,0}(\hbar)\left[\frac{f^{(n)}(z_0)}{n!}(z-z_0)^n\right],$$

where $\mu_\alpha^{m,n}(\hbar)$ and $\sigma_\alpha^{m,n,0}(\hbar)$ are defined by (2.91) and (2.96), respectively. The convergence regions of these generalized Taylor series might be greatly enlarged by choosing proper values of $\hbar$ and α. To illustrate this, we can generalize the mth-order approximations (2.57), (2.78), and (2.84) by

$$V(t)\approx\sum_{n=0}^{m}\mu_\alpha^{m,n}(\hbar)\ \left[\alpha_{2n+1}\ t^{2n+1}\right], \tag{2.99}$$

$$V(t)\approx 1+2\sum_{n=1}^{m}\left[(-1)^n\exp(-nt)\right]\mu_\alpha^{m,n}\left(\frac{\hbar}{2}\right)$$
$$-\exp(-t)\left[\left(1+\frac{\hbar}{2}\right)+\frac{\hbar}{2}\ \exp(-2t)\right]^m \tag{2.100}$$

and

$$V(t)\approx 1+2\sum_{n=1}^{m+1}\ \sum_{k=0}^{m+1-n}\sigma_\alpha^{m,n,k}(\hbar)\ \left[(-1)^n\ \frac{(-nt)^k}{k!}\ \exp(-nt)\right], \tag{2.101}$$

respectively, where $|1+\hbar|<1$ and $\alpha\in(-\infty,+\infty)$. For example, when $\alpha=\pi/4$, the convergence region of the approximation (2.99) becomes larger and larger as $\hbar$ tends to zero from below, as shown in Figure 2.5. And when $\hbar=-1$, all of the 20th-order approximations of (2.100) given by $\alpha=\pm1/2,\pm\pi/4$ agree with the exact solution, as shown in Table 2.8. When $\hbar=-1/2$, all of the 20th-order approximations of (2.101) given by $\alpha=\pm1/2,\pm\pi/4$ agree with the exact solution, as shown in Table 2.9. So, the functions $\mu_\alpha^{m,n}(\hbar)$ and $\sigma_\alpha^{m,n,k}(\hbar)$ have indeed rather general meaning.

In this subsection we point out that the definition (2.58) of $\mu_0^{m,n}(\hbar)$, which is first obtained in the homotopy analysis method, can be independently deduced from the Newtonian binomial theorem. Furthermore, we prove that convergence region and rate of a series can be indeed adjusted and controlled by introducing an auxiliary parameter. We also point out that the functions $\mu_\alpha^{m,n}(\hbar)$ and $\sigma_\alpha^{m,n.k}(\hbar)$ define different approaching paths by different values of $\hbar$ and α. All of these provide us with a rational base for the validity of the homotopy analysis method.

2.3.7 Homotopy-Padé method

The homotopy analysis method is based on such an assumption that the series (2.36) of $\Phi(t;q)$ converges at $q=1$ for the illustrative problem. Fortunately, as mentioned above, we have great freedom to choose the initial approximation

$V_0(t)$, the auxiliary linear operator $\mathcal{L}$, the auxiliary function $H(t)$, and the auxiliary parameter $\hbar$ in the frame of the homotopy analysis method. If all of them are properly chosen, the series (2.36) can be convergent at $q = 1$, as shown above. Besides, the convergence region and rate of the solution series given by the homotopy analysis method depend upon the auxiliary parameter $\hbar$. Therefore, the auxiliary parameter $\hbar$ provides us with a convenient way to adjust and control the convergence region and rate of solution series, as shown above.

There exist some techniques to accelerate the convergence of a given series. Among them, the so-called Padé technique is widely applied. For a given series

$$\sum_{n=0}^{+\infty} c_n\, x^n,$$

the corresponding $[m, n]$ Padé approximant is expressed by

$$\frac{\sum_{k=0}^{m} a_{m,k}\, x^k}{\sum_{k=0}^{n} b_{m,k}\, x^k},$$

where $a_{m,k}, b_{m,k}$ are determined by the coefficients c_j $(j = 0, 1, 2, 3, \cdots, m+n)$. In many cases the traditional Padé technique can greatly increase the convergence region and rate of a given series. For example, employing the traditional Padé technique to the perturbation series (2.12), we have the $[1, 1], [2, 2]$ and $[3, 3]$ Padé approximants

$$t, \quad \frac{3t}{3+t^2}, \quad \frac{t(15+t^2)}{15+6t^2},$$

respectively. In general, the $[m, m]$ Padé approximant can be expressed by

$$\frac{\sum_{n=0}^{m} a_0^{m,n}\, t^n}{\sum_{n=0}^{m-1} b_0^{m,n}\, t^n}, \qquad \text{when } m \text{ is an odd number}, \tag{2.102}$$

or

$$\frac{\sum_{n=0}^{m-1} a_0^{m,n}\, t^n}{\sum_{n=0}^{m} b_0^{m,n}\, t^n}, \qquad \text{when } m \text{ is an even number}, \tag{2.103}$$

where $a_0^{m,n}$ and $b_0^{m,n}$ are coefficients. Note that all of these traditional Padé approximants tend to either infinity or zero as $t \to +\infty$. The $[4, 4]$ and $[10, 10]$ Padé approximants of the perturbation solution (2.12) are as shown in Figure 2.6.

The so-called homotopy-Padé technique [50] was proposed by means of combining the above-mentioned traditional Padé technique with the homotopy analysis method. To ensure that the series (2.36) is convergent at $q = 1$, we first employ the traditional $[m, n]$ Padé technique about the embedding parameter q to obtain the $[m, n]$ Padé approximant

$$\frac{\sum_{k=0}^{m} A_{m,k}(t)\; q^k}{\sum_{k=0}^{n} B_{m,k}(t)\; q^k}, \tag{2.104}$$

where the coefficients $A_{m,k}(t)$ and $B_{m,k}(t)$ are determined by the first several approximations

$$V_0(t), V_1(t), V_2(t), \cdots, V_{m+n}(t).$$

Then, setting $q = 1$ in (2.104) and using (2.32), we have the so-called $[m, n]$ homotopy-Padé approximant

$$\frac{\sum_{k=0}^{m} A_{m,k}(t)}{\sum_{k=0}^{n} B_{m,k}(t)}. \tag{2.105}$$

For the illustrative problem, the coefficients $A_{m,n}(t)$ and $B_{m,n}(t)$ are dependent of the base functions used to present the solution $V(t)$. Using the base functions denoted by (2.62), we have the corresponding $[1, 1]$ homotopy-Padé approximant

$$\frac{t(12 + 16t + 7t^2)}{(1 + t)(12 + 4t + 7t^2)}$$

and the $[2, 2]$ homotopy-Padé approximant

$$\frac{t(168000 + 362880\, t + 238000\, t^2 + 14160\, t^3 - 47124\, t^4 - 36308\, t^5 - 13419\, t^6)}{3(1 + t)(56000 + 64960\, t + 33040\, t^2 + 12000\, t^3 - 2508\, t^4 - 9076\, t^5 - 4473\, t^6)},$$

respectively. In general, the $[m, m]$ homotopy-Padé approximation can be expressed by

$$\frac{\sum_{n=1}^{m^2+m+1} a_2^{m,n}\; t^n}{\sum_{n=0}^{m^2+m+1} b_2^{m,n}\; t^n}, \tag{2.106}$$

where $a_2^{m,n}$ and $b_2^{m,n}$ are coefficients. It is very interesting that $a_2^{m,n}$ and $b_2^{m,n}$ are found to be independent of the auxiliary parameter $\hbar$. Comparing (2.106) with (2.102) and (2.103), we find that in accuracy the $[m, m]$ homotopy-Padé approximant is equivalent to the traditional $[m^2 + m + 1, m^2 + m + 1]$ Padé approximant. Unlike the traditional Padé approximants (2.102) and (2.103) which tend to either infinity or zero as $t \to +\infty$, all of the homotopy-Padé approximant (2.106) correctly tend to 1 as $t \to +\infty$. Thus, for a given m, the $[m,m]$ homotopy-Padé approximant (2.106) is much more accurate than the traditional $[m,m]$ Padé approximants (2.102) and (2.103). For example, the [4,4] homotopy-Padé approximant is more accurate and much better than the traditional [4,4] Padé approximant and is even better than the [10,10] traditional Padé approximant, as shown in Figure 2.6. In particular, using the base functions denoted by (2.70), we have the $[1,1]$ homotopy-Padé approximant

$$\frac{1-\exp(-2t)}{1+\exp(-2t)}, \tag{2.107}$$

which is just the exact solution $V(t) = \tanh(t)$. Thus, the so-called homotopy-Padé method is indeed much more efficient than the traditional Padé technique.

Similarly, the so-called homotopy-Padé technique can be applied to accelerate the convergence of the related series. For example, to accelerate the series of $V''(0)$ and $V'''(0)$, we first apply the traditional Padé technique to the series

$$\left.\frac{\partial^2 \Phi(t;q)}{\partial\, t^2}\right|_{t=0} = \sum_{n=0}^{+\infty} V_n''(0)\, q^n$$

and

$$\left.\frac{\partial^3 \Phi(t;q)}{\partial\, t^3}\right|_{t=0} = \sum_{n=0}^{+\infty} V_n'''(0)\, q^n$$

to get their $[m,n]$ Padé approximants about the embedding parameter q, respectively, and then set $q=1$ to obtain the corresponding $[m,n]$ homotopy-Padé approximants. The homotopy-Padé approximations of $V''(0)$ and $V'''(0)$, corresponding to the solution expression (2.69) expressed by the fractional functions, are listed in Table 2.10. The homotopy-Padé approximations of $V''(0)$ and $V'''(0)$, corresponding to the solution expression (2.78) expressed by the exponential functions are listed in Table 2.11. In both cases, the homotopy-Padé technique greatly accelerates the convergence of $V''(0)$ and $V'''(0)$.

For the illustrative problem, it is found that all of the $[m,m]$ homotopy-Padé approximants do not depend upon the auxiliary parameter $\hbar$. Thus, even if we choose a bad value of $\hbar$ such that the corresponding solution series diverges, we can still employ the homotopy-Padé technique to get a convergent result. As shown later in this book for other nonlinear problems, the $[m,m]$ homotopy-Padé approximants are often independent of the auxiliary parameter $\hbar$. However, up to now, we cannot give a mathematical proof about it in general cases.

All of these illustrate that the so-called homotopy-Padé technique can greatly enlarge the convergence region and rate of the solution series given by the homotopy analysis method.

In summary, we introduce in this chapter the basic ideas of the homotopy analysis method by means of a simple example. We show that, unlike all previous analytic techniques, the homotopy analysis method always gives a family of solution expressions in the auxiliary parameter $\hbar$, which may be expressed by different base functions. Using the freedom in choosing the initial guess, the auxiliary linear operator, and the auxiliary function, we can express the solution in many different base functions, and thus approximate a nonlinear problem more efficiently by choosing a better set of base functions. The *rule of solution expression*, the *rule of coefficient ergodicity*, and the *rule of solution existence* are proposed to direct the choice of the initial guess, the auxiliary linear operator, and the auxiliary function. These rules greatly simplify the application of the homotopy analysis method. We demonstrate that the convergence region and rate of the solution series may be adjusted and controlled by means of the auxiliary parameter $\hbar$. By plotting the so-called $\hbar$-curves, it is easy to find out a proper value of $\hbar$ to ensure that the solution series converge. Furthermore, the so-called homotopy-Padé technique is proposed to accelerate the convergence of solution series, which is often much more efficient than the traditional Padé technique.

TABLE 2.1
Approximations of $V''(0)$ given by (2.69) for different values of $\hbar$.

order	$\hbar=-1/2$	$\hbar=-3/4$	$\hbar=-1$	$\hbar=-5/4$	$\hbar=-3/2$
5	-0.062500	-0.001953	0	-0.001953	0.062500
10	-0.001953	-1.9×10^{-6}	0	-1.9×10^{-6}	-0.001953
15	-0.000061	-1.9×10^{-9}	0	1.9×10^{-9}	0.000061
20	-1.9×10^{-6}	-1.9×10^{-12}	0	-1.9×10^{-12}	-1.9×10^{-6}
25	-6.0×10^{-8}	-1.8×10^{-15}	0	1.8×10^{-15}	6.0×10^{-8}
30	-1.9×10^{-9}	-1.7×10^{-18}	0	-1.7×10^{-18}	-1.9×10^{-9}
35	-5.8×10^{-11}	-1.7×10^{-21}	0	1.7×10^{-21}	5.8×10^{-11}
40	-1.8×10^{-12}	-1.7×10^{-24}	0	-1.7×10^{-24}	-1.9×10^{-12}

TABLE 2.2
Approximations of $V'''(0)$ given by (2.69) for different values of $\hbar$.

order	$\hbar=-1/2$	$\hbar=-3/4$	$\hbar=-1$	$\hbar=-5/4$	$\hbar=-3/2$
5	-3.312500	-2.138672	-2	-2.251953	-6.937500
10	-2.089844	-2.000278	-2	-1.999516	-1.699219
15	-2.004333	-2.000000	-2	-2.000001	-2.013977
20	-2.000183	-2.000000	-2	-2.000000	-1.99942
25	-2.000007	-2.000000	-2	-2.000000	-2.000023
30	-2.000000	-2.000000	-2	-2.000000	-1.999999
35	-2.000000	-2.000000	-2	-2.000000	-2.000000
40	-2.000000	-2.000000	-2	-2.000000	-2.000000

TABLE 2.3
Comparison of the exact solution (2.8) with the mth-order approximations of $V(t)$ given by (2.69) when $\hbar=-1$.

t	10th-order approx.	20th-order approx.	40th-order approx.	60th-order approx.	exact result
1/4	0.2449	0.2449	0.2449	0.2449	0.2449
1/2	0.4621	0.4621	0.4621	0.4621	0.4621
3/4	0.6349	0.6351	0.6351	0.6351	0.6351
1	0.7516	0.7616	0.7616	0.7616	0.7616
3/2	0.9082	0.9053	0.9051	0.9051	0.9051
2	0.9720	0.9644	0.9640	0.9640	0.9640
5/2	0.9982	0.9870	0.9866	0.9866	0.9866
3	1.0082	0.9950	0.9950	0.9951	0.9951
4	1.0110	0.9979	0.9992	0.9993	0.9993
5	1.0082	0.9973	0.9997	0.9999	0.9999
10	0.9984	0.9968	1.0003	1.0001	1.0000
100	0.9987	0.9998	1.0001	1.0000	1.0000

TABLE 2.4
Approximations of $V''(0)$ given by (2.77) for different values of $\hbar$.

order	$\hbar=-1/2$	$\hbar=-3/4$	$\hbar=-1$	$\hbar=-5/4$	$\hbar=-3/2$
5	-0.031250	-0.000977	0	-0.000977	0.031250
10	-0.000977	-9.5×10^{-7}	0	-9.5×10^{-7}	-0.000977
15	-0.000031	-9.3×10^{-10}	0	9.3×10^{-10}	0.000031
20	-9.5×10^{-7}	-9.1×10^{-13}	0	-9.1×10^{-13}	-9.5×10^{-7}
25	-3.0×10^{-8}	-8.9×10^{-16}	0	8.8×10^{-16}	3.0×10^{-8}
30	-9.3×10^{-10}	-8.7×10^{-19}	0	-8.7×10^{-19}	-9.3×10^{-10}
35	-2.9×10^{-11}	-8.5×10^{-22}	0	8.5×10^{-22}	2.9×10^{-11}
40	-9.1×10^{-13}	-8.3×10^{-25}	0	-8.3×10^{-25}	-9.1×10^{-13}

TABLE 2.5
Approximations of $V'''(0)$ given by (2.77) for different values of $\hbar$.

order	$\hbar=-1/2$	$\hbar=-3/4$	$\hbar=-1$	$\hbar=-5/4$	$\hbar=-3/2$
5	-2.375000	-2.041016	-2	-2.076172	-3.500000
10	-2.026367	-2.000083	-2	-1.999854	-1.909180
15	-2.001282	-2.000000	-2	-2.000001	-2.004211
20	-2.000054	-2.000000	-2	-2.000000	-1.999825
25	-2.000002	-2.000000	-2	-2.000000	-2.000007
30	-2.000000	-2.000000	-2	-2.000000	-2.000000
35	-2.000000	-2.000000	-2	-2.000000	-2.000000
40	-2.000000	-2.000000	-2	-2.000000	-2.000000

TABLE 2.6
Comparison of the exact solution (2.8) with the approximations of $V(t)$ given by (2.77) when $\hbar=-1$.

t	5th-order approx.	10th-order approx.	15th-order approx.	20th-order approx.	exact result
1/4	0.2449	0.2449	0.2449	0.2449	0.2449
1/2	0.4619	0.4621	0.4621	0.4621	0.4621
3/4	0.6342	0.6351	0.6351	0.6351	0.6351
1	0.7596	0.7616	0.7616	0.7616	0.7616
3/2	0.9020	0.9051	0.9051	0.9051	0.9051
2	0.9612	0.9639	0.9640	0.9640	0.9640
5/2	0.9845	0.9866	0.9866	0.9866	0.9866
3	0.9937	0.9950	0.9951	0.9951	0.9951
4	0.9988	0.9993	0.9993	0.9993	0.9993
5	0.9997	0.9999	0.9999	0.9999	0.9999
10	1.0000	1.0000	1.0000	1.0000	1.0000
100	1.0000	1.0000	1.0000	1.0000	1.0000

TABLE 2.7
Comparison of the exact solution (2.8) with the approximations of $V(t)$ given by (2.84) when $\hbar = -1$.

t	10th-order approx.	20th-order approx.	40th-order approx.	50th-order approx.	exact result
1/4	0.2449	0.2449	0.2449	0.2449	0.2449
1/2	0.4621	0.4621	0.4621	0.4621	0.4621
3/4	0.6351	0.6351	0.6351	0.6351	0.6351
1	0.7616	0.7616	0.7616	0.7616	0.7616
3/2	0.9051	0.9051	0.9051	0.9051	0.9051
2	0.9640	0.9640	0.9640	0.9640	0.9640
5/2	0.9866	0.9866	0.9866	0.9866	0.9866
3	0.9953	0.9950	0.9951	0.9951	0.9951
4	0.9990	0.9993	0.9993	0.9993	0.9993
5	0.9975	0.9999	0.9999	0.9999	0.9999
10	1.0021	0.9982	0.9999	1.0000	1.0000
100	1.0000	1.0000	1.0000	1.0000	1.0000

TABLE 2.8
Comparison of the exact solution (2.8) with the 20th-order approximation of $V(t)$ given by (2.100) when $\hbar = -1$ and $\alpha = \pm 1/2, \pm\pi/4$.

t	when $\alpha = -\pi/4$	when $\alpha = -1/2$	when $\alpha = 1/2$	when $\alpha = \pi/4$	exact result
1/4	0.2449	0.2449	0.2449	0.2449	0.2449
1/2	0.4621	0.4621	0.4621	0.4621	0.4621
3/4	0.6351	0.6351	0.6351	0.6351	0.6351
1	0.7616	0.7616	0.7616	0.7616	0.7616
3/2	0.9051	0.9051	0.9051	0.9051	0.9051
2	0.9640	0.9640	0.9640	0.9640	0.9640
5/2	0.9866	0.9866	0.9866	0.9866	0.9866
3	0.9951	0.9951	0.9951	0.9951	0.9951
4	0.9993	0.9993	0.9993	0.9993	0.9993
5	0.9999	0.9999	0.9999	0.9999	0.9999
10	1.0000	1.0000	0.9999	1.0000	1.0000
100	1.0000	1.0000	1.0000	1.0000	1.0000

TABLE 2.9
Comparison of the exact solution (2.8) with the 20th-order approximation of $V(t)$ given by (2.101) when $\hbar = -1/2$ and $\alpha = \pm 1/2, \pm\pi/4$.

t	when $\alpha = -\pi/4$	when $\alpha = -1/2$	when $\alpha = 1/2$	when $\alpha = \pi/4$	exact result
1/4	0.2449	0.2449	0.2449	0.2449	0.2449
1/2	0.4621	0.4621	0.4621	0.4621	0.4621
3/4	0.6351	0.6351	0.6351	0.6351	0.6351
1	0.7616	0.7616	0.7616	0.7616	0.7616
3/2	0.9051	0.9051	0.9051	0.9051	0.9051
2	0.9640	0.9640	0.9640	0.9640	0.9640
5/2	0.9866	0.9866	0.9866	0.9866	0.9866
3	0.9951	0.9951	0.9951	0.9951	0.9951
4	0.9993	0.9993	0.9993	0.9993	0.9993
5	0.9999	0.9999	0.9999	0.9999	0.9999
10	1.0000	1.0000	0.9999	1.0000	1.0000
100	1.0000	1.0000	1.0000	1.0000	1.0000

TABLE 2.10
The $[m, m]$ homotopy-Padé approximation of $V''(0)$ and $V'''(0)$ corresponding to (2.69).

$[m, m]$	$V''(0)$	$V'''(0)$
$[1, 1]$	0	-3
$[2, 2]$	0	-2
$[3, 3]$	0	-2
$[4, 4]$	0	-2
$[5, 5]$	0	-2
$[10, 10]$	0	-2

TABLE 2.11
The $[m, m]$ homotopy-Padé approximation of $V''(0)$ and $V'''(0)$ corresponding to (2.78).

$[m, m]$	$V''(0)$	$V'''(0)$
$[1, 1]$	0	-5.57143
$[2, 2]$	0	-2
$[3, 3]$	0	-2
$[4, 4]$	0	-2
$[5, 5]$	0	-2
$[10, 10]$	0	-2

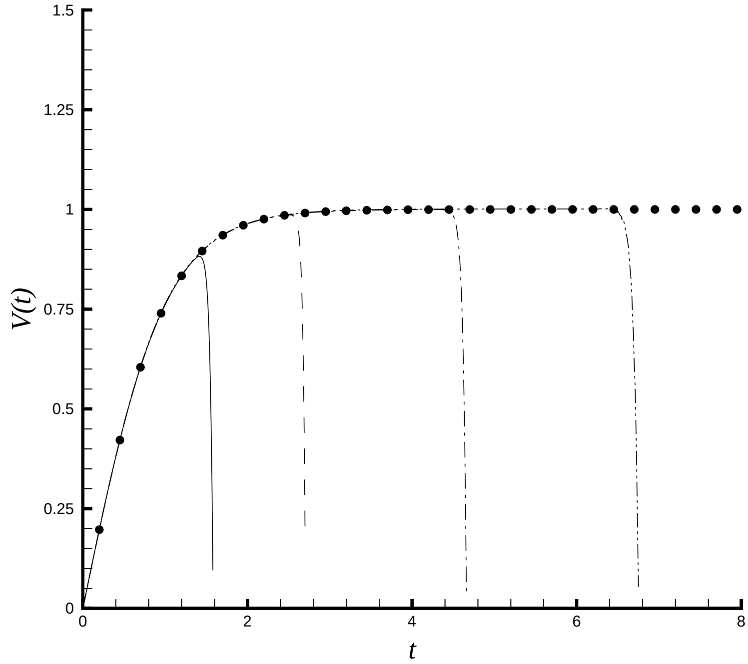

FIGURE 2.1
Comparison of the exact solution (2.8) with the solution expression (2.57). Symbols: exact solution; solid line: perturbation solution (2.12); dashed line: solution (2.57) when $\hbar = -1/2$; dash-dotted line: solution (2.57) when $\hbar = -1/5$; dash-dot-dotted line: solution (2.57) when $\hbar = -1/10$.

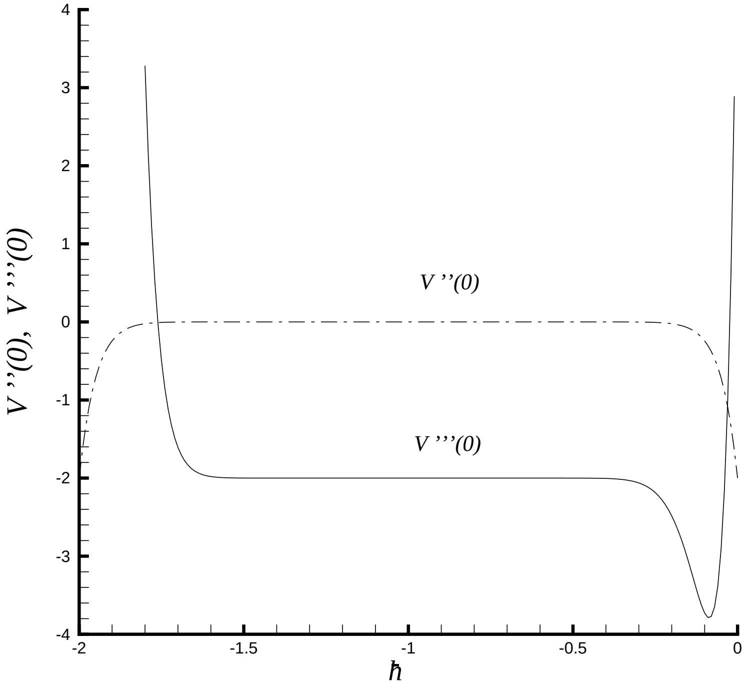

FIGURE 2.2
The $\hbar$-curve of $V''(0)$ and $V'''(0)$ given by (2.69) when $H(t) = 1/(1+t)$. Dash-dotted line: 20th-order approximation of $V''(0)$; solid line: 20th-order approximation of $V'''(0)$.

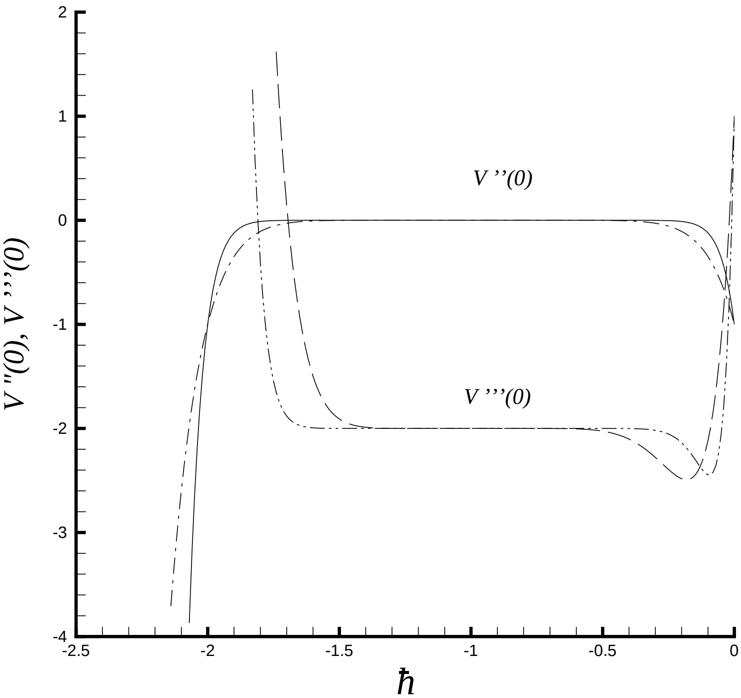

FIGURE 2.3
The $\hbar$-curves of $V''(0)$ and $V'''(0)$ given by (2.77) when $H(t) = \exp(-t)$. Dash-dotted line: 10th-order approximation of $V''(0)$; solid line: 20th-order approximation of $V''(0)$; dashed lined: 10th-order approximation of $V'''(0)$; dash-dot-dotted line: 20th-order approximation of $V'''(0)$.

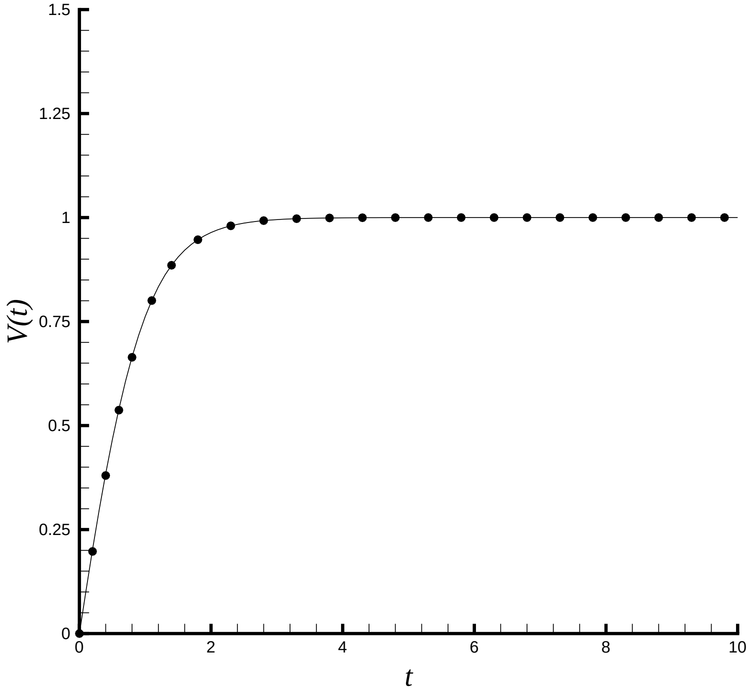

FIGURE 2.4
The comparison of the third-order approximation (2.81) of $V(t)$ with the exact solution (2.8). Solid line: third-order approximation (2.81); symbols: exact solution (2.8).

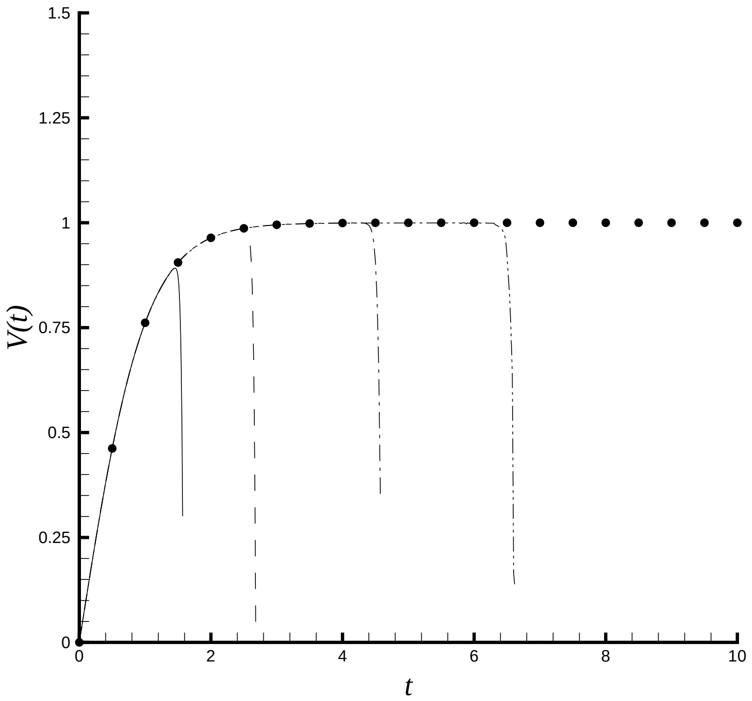

FIGURE 2.5
Comparison of the exact solution (2.8) with the solution expression (2.99) at the 31st-order of approximation when $\alpha = \pi/4$. Symbols: exact solution; solid line: (2.99) when $\hbar = -1$; dashed line: (2.99) when $\hbar = -1/2$; dash-dotted line: (2.99) when $\hbar = -1/5$; dash-dot-dotted line: (2.99) when $\hbar = -1/10$.

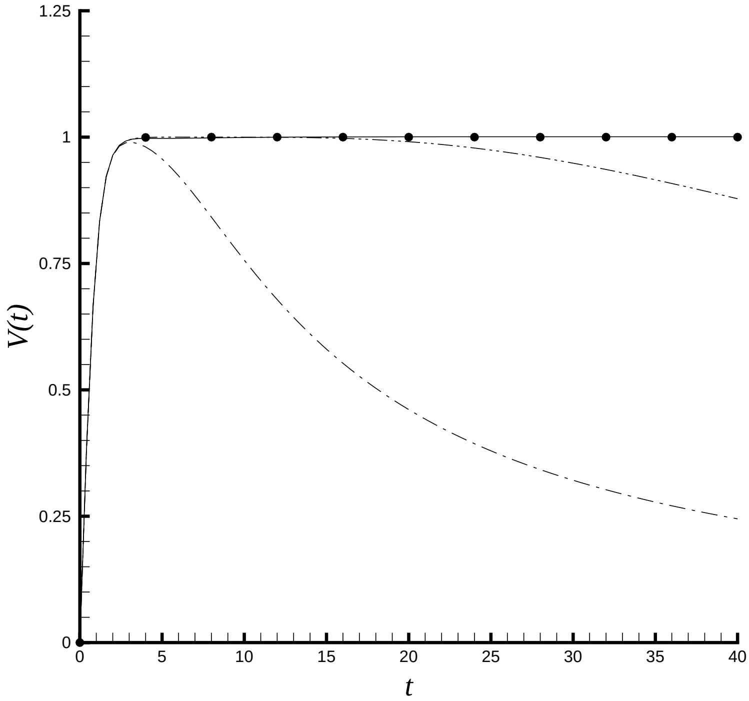

FIGURE 2.6
Comparison of the exact solution (2.8) with the homotopy-Padé approximation (2.106) and traditional Padé approximant (2.103) of $V(t)$. Symbols: exact solution; solid line: [4,4] homotopy-Padé approximant of $V(t)$; dash-dotted line: [4,4] traditional Padé approximant of $V(t)$; dash-dot-dotted line: [10,10] traditional Padé approximant.