PART I

BASIC IDEAS

The way that can be spoken of is not the constant way;
The name that can be named is not the constant name.

Lao Tzu, an ancient Chinese philosopher

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Introduction

Most phenomena in our world are essentially nonlinear and are described by nonlinear equations. Since the appearance of high-performance digit computers, it becomes easier and easier to solve a linear problem. However, generally speaking, it is still difficult to obtain accurate solutions of nonlinear problems. In particular, it is often more difficult to get an analytic approximation than a numerical one of a given nonlinear problem, although we now have high-performance supercomputers and some high-quality symbolic computation software such as Mathematica, Maple, and so on. The numerical techniques generally can be applied to nonlinear problems in complicated computation domain; this is an obvious advantage of numerical methods over analytic ones that often handle nonlinear problems in simple domains. However, numerical methods give discontinuous points of a curve and thus it is often costly and time consuming to get a complete curve of results. Besides, from numerical results, it is hard to have a whole and essential understanding of a nonlinear problem. Numerical difficulties additionally appear if a nonlinear problem contains singularities or has multiple solutions. The numerical and analytic methods of nonlinear problems have their own advantages and limitations, and thus it is unnecessary for us to do one thing and neglect another. Generally, one delights in giving analytic solutions of a nonlinear problem.

There are some analytic techniques for nonlinear problems, such as perturbation techniques [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12] that are well known and widely applied. By means of perturbation techniques, a lot of important properties and interesting phenomena of nonlinear problems have been revealed. One of the astonishing successes of perturbation techniques is the discovery of the ninth planet in the solar system, found in the vast sky at a predicted point. Recently, the singular perturbation techniques are considered to be one of the top 10 progresses of theoretical and applied mechanics in the 20th century [13]. It is therefore out of question that perturbation techniques play important roles in the development of science and engineering. For further details, the reader is referred to the foregoing textbooks of perturbation methods.

Perturbation techniques are essentially based on the existence of small or large parameters or variables called perturbation quantity. Briefly speaking, perturbation techniques use perturbation quantities to transfer a nonlinear problem into an infinite number of linear sub-problems and then approximate it by the sum of solutions of the first several sub-problems. The existence of

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perturbation quantities is obviously a cornerstone of perturbation techniques, however, it is the perturbation quantity that brings perturbation techniques some serious restrictions. Firstly, it is impossible that every nonlinear problem contains such a perturbation quantity. This is an obvious restriction of perturbation techniques. Secondly, analytic approximations of nonlinear problems often break down as nonlinearity becomes strong, and thus perturbation approximations are valid only for nonlinear problems with weak nonlinearity. Consider the drag of a sphere in a uniform stream, a classical nonlinear problem in fluid mechanics governed by the famous Navier-Stokes equation, for example. Since 1851 when Stokes [14] first considered this problem, many scientists have attacked it by means of linear theories [15, 16], straightforward perturbation technique [17], and matching perturbation method [18, 19]. However, all these previous theoretical drag formulae agree with experimental data only for small Reynolds number, as shown in Figure 1.1. Thus, as pointed out by White [20], "the idea of using creeping flow to expand into the high Reynolds number region has not been successful". This might be partly due to the fact that perturbation techniques do not provide us with any ways to adjust convergence region and rate of perturbation approximations.

There are a few nonperturbation techniques. The dependence of perturbation techniques on small/large parameters can be avoided by introducing a so-called artificial small parameter. In 1892 Lyapunov [21] considered the equation

$$\frac{dx}{dt} = A(t)\, x,$$

where $A(t)$ is a time periodic matrix. Lyapunov [21] introduced an artificial parameter ϵ to replace this equation with the equation

$$\frac{dx}{dt} = \epsilon\, A(t)\, x$$

and then calculated power series expansions over ϵ for the solutions. In many cases Lyapunov proved that series converge for $\epsilon = 1$, and therefore we can put in the final expression by setting $\epsilon = 1$. The above approach is called Lyapunov's artificial small parameter method [21]. This idea was further employed by Karmishin et al. [22] to propose the so-called δ-expansion method. Karmishin et al. [22] introduced an artificial parameter δ to replace the equation

$$x^5 + x = 1 \tag{1.1}$$

with the equation

$$x^{1+\delta} + x = 1 \tag{1.2}$$

and then calculated power series expansions over δ and finally gained the approximations by converting the series to [3,3] Padé approximants and setting $\delta = 4$. In essence, the δ-expansion method is equivalent to the Lyapunov's artificial small parameter method. Note that both methods introduce an artificial parameter, although it appears in a different place and is denoted by

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different symbol in a given nonlinear equation. We additionally have great freedom to replace Equation (1.1) by many different equations such as

$$\delta\, x^5 + x = 1. \tag{1.3}$$

As pointed out by Karmishin et al. [22], the approximation given by the above equation is much worse than that given by Equation (1.2). Both the artificial small parameter method and the δ-expansion method obviously need some fundamental rules to determine the place where the artificial parameter ϵ or δ should appear. Like perturbation techniques, both the artificial small parameter method and the δ-expansion method themselves do not provide us with a convenient way to adjust convergence region and rate of approximation series.

Adomian's decomposition method [23, 24, 25] is a powerful analytic technique for strongly nonlinear problems. The basic ideas of Adomian's decomposition method is simply described in §4.1. Adomian's decomposition method is valid for ordinary and partial differential equations, no matter whether they contain small/large parameters, and thus is rather general. Moreover, the Adomian approximation series converge quickly. However, Adomian's decomposition method has some restrictions. Approximates solutions given by Adomian's decomposition method often contain polynomials. In general, convergence regions of power series are small, thus acceleration techniques are often needed to enlarge convergence regions. This is mainly due to the fact that power series is often not an efficient set of base functions to approximate a nonlinear problem, but unfortunately Adomain's decomposition method does not provide us with freedom to use different base functions. Like the artificial small parameter method and the δ-expansion method, Adomian's decomposition method itself also does not provide us with a convenient way to adjust convergence region and rate of approximation solutions.

In summary, neither perturbation techniques nor nonperturbation methods such as the artificial small parameter methods, the δ-expansion method, and Adomian's decomposition method can provide us with a convenient way to *adjust* and *control* convergence region and rate of approximation series. The efficiency to approximate a nonlinear problem has not been taken into enough account, therefore it is necessary to develop some new analytic methods such that they

1. Are valid for strongly nonlinear problems even if a given nonlinear problem does not contain any small/large parameters

2. Provide us with a convenient way to adjust the convergence region and rate of approximation series

3. Provide us with freedom to use different base functions to approximate a nonlinear problem.

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A kind of analytic technique, namely the homotopy analysis method [26, 27, 28, 29, 30], was proposed by means of homotopy [31], a fundamental concept of topology [32]. The idea of the homotopy is very simple and straightforward. For example, consider a differential equation

$$\mathcal{A}[u(t)] = 0, \tag{1.4}$$

where $\mathcal{A}$ is a nonlinear operator, t denotes the time, and $u(t)$ is an unknown variable. Let $u_0(t)$ denote an initial approximation of $u(t)$ and $\mathcal{L}$ denote an auxiliary linear operator with the property

$$\mathcal{L}f = 0 \qquad \text{when } f = 0. \tag{1.5}$$

We then construct the so-called homotopy

$$\mathcal{H}[\phi(t;q);q] = (1-q)\,\mathcal{L}\,[\phi(t;q) - u_0(t)] + q\,\mathcal{A}[\phi(t;q)], \tag{1.6}$$

where $q \in [0,1]$ is an embedding parameter and $\phi(t;q)$ is a function of t and q. When $q = 0$ and $q = 1$, we have

$$\mathcal{H}[\phi(t;q);q]\big|_{q=0} = \mathcal{L}[\phi(t;0) - u_0(t)]$$

and

$$\mathcal{H}[\phi(t;q);q]|_{q=1} = \mathcal{A}[\phi(t;1)],$$

respectively. Using (1.5), it is clear that

$$\phi(t;0) = u_0(t)$$

is the solution of the equation

$$\mathcal{H}[\phi(t;q);q]|_{q=0} = 0.$$

And

$$\phi(t;1) = u(t)$$

is therefore obviously the solution of the equation

$$\mathcal{H}[\phi(t;q);q]|_{q=1} = 0.$$

As the embedding parameter q increases from 0 to 1, the solution $\phi(t;q)$ of the equation

$$\mathcal{H}[\phi(t;q);q] = 0$$

depends upon the embedding parameter q and varies from the initial approximation $u_0(t)$ to the solution $u(t)$ of Equation (1.4). In topology, such a kind of continuous variation is called deformation.

Based on the idea of homotopy, some numerical techniques such as the continuation method [33] and the homotopy continuation method [34] were

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developed. In fact, the artificial small parameter method and the δ-expansion method can be described by the homotopy if we replace the artificial parameter ϵ or δ by the embedding parameter q, as shown in Chapter 4. However, although the above-mentioned traditional way to construct the homotopy (1.6) might be enough from viewpoints of numerical techniques, it is not good enough from viewpoints of analytic ones. This is manly because we have great freedom to choose the so-called auxiliary operator $\mathcal{L}$ and the initial approximations but lack any rules to direct their choice. More importantly, the traditional way to construct a homotopy cannot provide a convenient way to adjust convergence region and rate of approximation series.

In this book the basic ideas of the homotopy analysis method are described in details and some typical nonlinear problems in science and engineering are employed to illustrate its validity and flexibility. To simply show the validity of the homotopy analysis method, we point out that the 10th-order drag formula of a sphere in a uniform stream given by the homotopy analysis method agrees well with experimental data in a considerably larger region than all previous theoretical drag formulae published in the past 150 years, as shown in Figure 1.1. In short, the homotopy analysis method is based on the concept of homotopy. However, instead of using the traditional homotopy (1.6), we introduce a nonzero auxiliary parameter $\hbar$ and a nonzero auxiliary function $H(t)$ to construct such a new kind of homotopy

$$\tilde{\mathcal{H}}(\Phi; q, \hbar, H) = (1-q)\ \mathcal{L}[\Phi(t; q, \hbar, H) - u_0(t)] - q\ \hbar\ H(t)\ \mathcal{A}[\Phi(t; q, \hbar, H)], \quad (1.7)$$

which is more general than (1.6) because (1.6) is only a special case of (1.7) when $\hbar = -1$ and $H(t) = 1$, i.e.,

$$\mathcal{H}(\phi; q) = \tilde{\mathcal{H}}(\Phi; q, -1, 1). \quad (1.8)$$

Similarly, as q increases from 0 to 1, $\Phi(t; q, \hbar, H)$ varies from the initial approximation $u_0(t)$ to the exact solution $u(t)$ of the original nonlinear problem. However, the solution $\Phi(t; q, \hbar, H)$ of the equation

$$\tilde{\mathcal{H}}[\Phi(t; q, \hbar, H)] = 0 \quad (1.9)$$

depends not only on the embedding parameter q but also on the auxiliary parameter $\hbar$ and the auxiliary function $H(t)$. So, at $q = 1$, the solution still depends upon the auxiliary parameter $\hbar$ and the auxiliary function $H(t)$. Thus, different from the traditional homotopy (1.6), the generalized homotopy (1.7) can provide us with a *family* of approximation series whose convergence region depends upon the auxiliary parameter $\hbar$ and the auxiliary function $H(t)$, as illustrated later in this book. More importantly, this provides us with a simple way to *adjust* and *control* the convergence regions and rates of approximation series.

The homotopy analysis method is rather general and valid for nonlinear ordinary and partial differential equations in many different types. It has been

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successfully applied to many nonlinear problems such as nonlinear oscillations [35, 36, 37, 38, 39], boundary layer flows [28, 29, 40, 41, 42, 43], heat transfer [44, 45], viscous flows in porous medium [46], viscous flows of Oldroyd 6-constant fluids [47], magnetohydrodynamic flows of non-Newtonian fluids [48], nonlinear water waves [49, 50], Thomas-Fermi equation [51], Lane-Emden equation [30], and so on. To show its validity and flexibility, we give many new applications of the homotopy analysis method in this book.

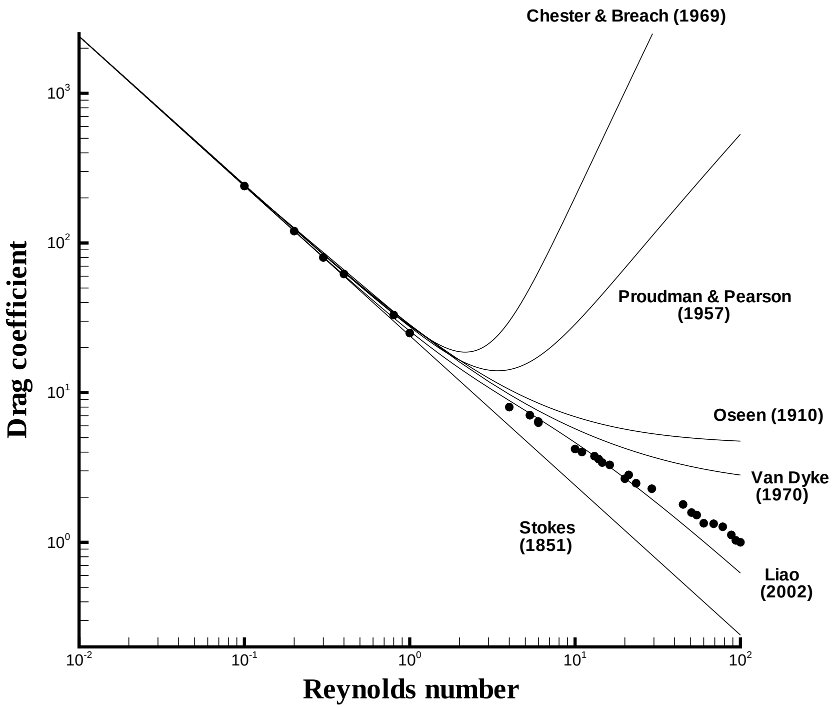

FIGURE 1.1

Comparison of experimental data of drag coefficient of a sphere in a uniform stream with theoretical results. Symbols: experimental data; solid line: theoretical results. (Modified from *International Journal of Non-Linear Mechanics*, 37, Shi-Jun Liao, "An analytic approximation of the drag coefficient for the viscous flow past a sphere", 1-18, Copyright (2002), with permission from Elsevier)

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