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Journal of Functional Analysis 216 (2004) 71–85
JOURNAL OF Functional Analysis
http://www.elsevier.com/locate/jfa

A bounded property for gradients of diffusion semigroups on Euclidean spaces

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Received 30 September 2003; accepted 3 November 2003

Communicated by Paul Malliavin

Abstract

We consider uniformly elliptic diffusion processes $X(t,x)$ on Euclidean spaces $\mathbf{R}^d$, with some conditions in terms of the drift term (see assumptions A2 and A3). By using interpolation theory, we show a bounded property which gives an estimate of $\nabla_x E[f(X(t,x))]$ involving $|x|$ and $||f||_\infty$ but not $||\nabla f||_\infty$, and a power of $\frac{1}{t}$ smaller than 1.

MSC: 60J60; 46B70

Keywords: Diffusion; Euclidean space; Gradient; Interpolation theory

1. Introduction

Consider the stochastic differential equation (SDE) on Euclidean space $\mathbf{R}^d$ given by

$$\begin{cases} dX^i(t,x) = \sum_{j=1}^d \sigma_{ij}(X(t,x))\, dB_t^j + b_i(X(t,x))\, dt, \quad i = 1, \dots, d, \\ X(0,x) = x, \end{cases} \tag{1.1}$$

where $(B_t^1, \dots, B_t^d)$ is a d-dimensional Brownian motion. We assume that the coefficients $\sigma = (\sigma_{ij})_{i,j=1,\dots,d}$ and $b = (b_1, \dots, b_d)$ satisfy the following:

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0022-1236/$ - see front matter
doi:10.1016/j.jfa.2003.11.001

A1. $\sigma \in C_b^\infty(\mathbf{R}^d, \mathbf{R}^{d^2})$, *and* $a := \sigma^t \sigma$ *is uniformly elliptic, i.e., there exist* $c_1, c_2 > 0$ *such that*

$$c_1 \sum_{i=1}^d \xi_i^2 \leqslant \sum_{i,j=1}^d a_{ij}(x)\xi_i \xi_j \leqslant c_2 \sum_{i=1}^d \xi_i^2, \quad \text{for all } x, \xi \in \mathbf{R}^d.$$

A2. $b \in C^\infty(\mathbf{R}^d, \mathbf{R}^d)$ *and there exists a* $c_3 > 0$ *such that*

$$\sum_{i,j=1}^d \xi_i \xi_j \nabla_i b_j(x) \leqslant c_3 \sum_{i=1}^d \xi_i^2, \quad \text{for all } x, \xi \in \mathbf{R}^d.$$

Here $\nabla_i := \frac{\partial}{\partial x_i}$, $i = 1, \ldots, d$.

A3. *There exist constants* $c_4, c_5, c_6 > 0$ *and* $\gamma_2 \geqslant \gamma_1 > 1$ *such that*

$$x \cdot b(x) \leqslant c_4 - c_5 |x|^{\gamma_1 + 1},$$

$$||b(x)||_{\mathbf{R}^d} + ||\nabla b(x)||_{\mathbf{R}^{d^2}} \leqslant c_6(1 + |x|^{\gamma_2}), \quad \text{for all } x \in \mathbf{R}^d.$$

Here $\nabla := (\nabla_1, \ldots, \nabla_d)^t$, *and* ∇b *means the matrix* $(\nabla_i b_j)_{i,j=1}^d$.

Then there exists a unique solution $\{X(t,x)\}_{t \geqslant 0}$ of the SDE (1.1). $\{X(t,x)\}_{t \geqslant 0}$ is a diffusion with generator $L_0 := \frac{1}{2}\sum_{i,j=1}^d a_{ij} \nabla_i \nabla_j + b \cdot \nabla$. Let $\{P_t\}_{t \geqslant 0}$ denote the associated semi-group of operators, i.e., $P_t f(x) = E[f(X(t,x))]$ for any $f \in C(\mathbf{R}^d)$ such that the expectation exists for all $x \in \mathbf{R}^d$.

Our main result of the paper is the following:

Theorem 1.1. *Assume* A1–A3. *Then for any* $\alpha \geqslant 0$ *and* $\beta > 0$, *there exist constants* $d_\beta \in (0,1)$ *and* $C_{\alpha,\beta} > 0$ *such that*

$$|\nabla P_t f(x)| \leqslant (1 + |x|^2)^{\frac{\gamma_2 - \gamma_1 + 1 + \beta + \alpha}{2}} \frac{C_{\alpha,\beta}}{t^{d_\beta}} ||(1 + |\cdot|^2)^{-\frac{\alpha}{2}} f||_\infty$$

for any $t \in (0,1]$, $x \in \mathbf{R}^d$ *and* $f \in C(\mathbf{R}^d)$ *satisfying* $(1 + |\cdot|^2)^{-\frac{\alpha}{2}} f \in C_b(\mathbf{R}^d)$.

The result of Theorem 1.1 is interesting in itself, since it gives a domination of $\nabla P_t f(x)$ in terms of $||x||_{\mathbf{R}^d}$ and $||f||_\infty$ (but with no $||\nabla f||_\infty$), with power of $\frac{1}{t}$ smaller than 1. This is vital if we want to use it to give an estimate of the derivative of the Green operator, as in, for example, Liang [5]. Since this is one of most important motivations of this work, let us explain it in more details. In [5], we discuss about the asymptotic behavior of $E[e^{T\Phi(\frac{1}{T}\int_0^T \delta_{X(t,x)}\, dt)} | X(T,x) = y]$ as $T \to \infty$ for $x, y \in \mathbf{R}^d$, where $\delta_\cdot$ denotes the delta measure, and Φ is some "good" function on the set of all signed measures on $\mathbf{R}^d$. The idea is to show some kind of Ito's formula for Green function Gf first and then use it to prove some L^p-bounded property ($p > 1$) with respect to $T > 0$, in order to show the L^1-convergence as $T \to \infty$. In the proof, we need to estimate the expectation of $e^{\nabla G f}$ (the term $\nabla G f$ comes out when we apply Ito's formula to Gf). For example, let G denote the Green operator corresponding to P_t, then formally

$$\nabla G = \int_0^1 \nabla P_t \, dt + \nabla P_1 G.$$

In order to use this to give an estimate of G, the power of $\frac{1}{t}$ in the estimate of ∇P_t have to be strictly smaller than 1.

For the case when the drift term is 0, we can get some estimate of $\nabla P_t f(x)$ essentially by Kusuoka–Stroock [3] (see Lemma 3.5). In this paper, by using the interpolation theory, we succeeded to get the results for general diffusion processes satisfying conditions A2 and A3.

The related problems have been discussed by other authors, under different settings. Li–Yau [4] gave an estimation of $|\nabla P_t f|$ for the heat equation, with the power of $\frac{1}{t}$ as **1**. In some more general setting, Taniguchi [6] showed the smoothness of the density function, and Kusuoka–Stroock [3, Part II] gave an estimation of the derivatives with no precise expression for the powers of either $||x||_{\mathbf{R}^d}$ or $\frac{1}{t}$ (cf. also the references therein.) But all of these are not usable to give the estimation of the differential of the Green function, which, as mentioned before, is needed in Liang [5]. The question is also related to Elworthy–Li [2].

The main idea of the paper is to use the interpolation theory. In order to explain it more clearly, let us take a look at Lemma 4.5. The estimate of Lemma 4.5 is meaningful if and only if $\theta_1 - \theta_0 < 1$. That is, we can estimate P_t as an operator from B_α^0 to $B_{\alpha+\gamma_2-\beta}^\theta$ (see Section 2 for the definition of the sets B_α^θ) as long as $\theta < 1$. However, the dominator $B(\frac{1}{2} - \frac{1}{2}\theta, 1 - \frac{\beta}{\gamma_1 - 1})$ converges to ∞ as $\theta \to 1$. Our idea is: first give an estimate for a range set $B_{\alpha+\gamma_2-\beta}^\theta$ (with $\beta < \gamma_1 - 1$) that is close to our goal range set $B_{\alpha+\gamma_2-\beta+\varepsilon}^1$, then for the remaining part (i.e., from $B_{\alpha+\gamma_2-\beta}^\theta$ to $B_{\alpha+\gamma_2-\beta+\varepsilon}^1$), we keep the constant finite at a cost of the power of $\frac{1}{t}$, which can be small by choosing θ close to 1 enough. In this way, although we do not need interpolation theory for the statement of our final result, we use it in the way of proving. This seems to the author as a quite new way of the usage of interpolation theory.

The rest of this paper is organized as follows: In Section 2 we define the interpolation spaces B_α^θ and discuss the structure of the family $\{B_\alpha^\theta\}_{\alpha,\theta}$. In Section 3 we discuss the properties of the semi-group when the drift term is equal to 0. The proof of Theorem 1.1 is given in Section 4.

2. The spaces B_α^θ

Let ψ be the function on $\mathbf{R}^d$ given by $\psi(x) = \sqrt{1 + |x|^2}, x \in \mathbf{R}^d$. For any $\alpha \in \mathbf{R}$, we define the sets

$$B_\alpha^0 = \left\{ f \in C(\mathbf{R}^d; \mathbf{C}); ||f||_{B_\alpha^0} := \sup_{x \in \mathbf{R}^d} \psi(x)^{-\alpha} |f(x)| < \infty \right\},$$

$$B_\alpha^1 = \left\{ f \in C^1(\mathbf{R}^d; \mathbf{C}); \|f\|_{B_\alpha^1} := \|f\|_{B_\alpha^0} + \sum_{i=1}^d \|\nabla_i f\|_{B_\alpha^0} < \infty \right\}.$$

Also, for any $\theta \in (0, 1)$, define the interpolation space by using the complex method (cf. [1]):

$$B_\alpha^\theta = C_\theta(B_\alpha^0, B_\alpha^1) = (B_\alpha^0, B_\alpha^1)_{[\theta]}.$$

Note that the space B_α^θ is a Banach space and is an intermediate space with respect to (B_α^0, B_α^1), and the interpolation functor C_θ is an exact interpolation functor of exponent θ.

In this section, we study the structure of the family of spaces $\{B_\alpha^\theta\}_{\alpha,\theta}$.

Lemma 2.1. *For any $\alpha \in \mathbf{R}$, $c \in \mathbf{R}$ and $\theta \in [0, 1]$, the operator $\psi^c : f \mapsto \psi^c f$ is a continuous linear operator from $B_{\alpha-c}^\theta$ to B_α^θ with norm $\|\psi^c\|_{B_{\alpha-c}^\theta \to B_\alpha^\theta} \leqslant 1 + \frac{|c|}{2}$.*

Proof. It is trivial by definition that $\psi^c : B_{\alpha-c}^0 \to B_\alpha^0$ and $\psi^c : B_{\alpha-c}^1 \to B_\alpha^1$ are continuous linear operators with operator norms not greater than $1 + \frac{|c|}{2}$. This accompanied with interpolation theory gives us our assertion. □

Lemma 2.2. $(B_\alpha^0, B_\beta^1)_{[\theta]} = B_{(1-\theta)\alpha+\theta\beta}^\theta$, $(B_\alpha^0, B_\beta^0)_{[\theta]} = B_{(1-\theta)\alpha+\theta\beta}^0$ *and* $(B_\alpha^1, B_\beta^1)_{[\theta]} = B_{(1-\theta)\alpha+\theta\beta}^1$ *for any $\alpha, \beta \in \mathbf{R}$ and $\theta \in (0, 1)$.*

Proof. The proof of the last two equalities are easy. We prove the first one here.

Let $S = \{z \in \mathbf{C}; \Re z \in [0, 1]\}$, where $\Re z$ means the real part of z. Then $S^0 = \{z \in \mathbf{C}; \Re z \in (0, 1)\}$. Let $\mathscr{F} = \{F \in C(S \to C(\mathbf{R}^d, \mathbf{C})); F$ is analytic on S^0, the operators $\mathbf{R} \to B_\alpha^0$, $t \mapsto F(it)$ and $\mathbf{R} \to B_\beta^1$, $t \mapsto F(1 + it)$ are continuous and converge to 0 as $|t| \to \infty\}$. Also, let $\|F\|_{\mathscr{F}} = \max\{\sup_{t\in\mathbf{R}} \|F(it)\|_{B_\alpha^0}, \sup_{t\in\mathbf{R}} \|F(1 + it)\|_{B_\beta^1}\}$ for $F \in \mathscr{F}$.

For any $f \in (B_\alpha^0, B_\beta^1)_{[\theta]}$ and $\varepsilon > 0$, by definition, there exists a $F \in \mathscr{F}$ such that $F(\theta) = f$ and $\|F\|_{\mathscr{F}} \leqslant \|f\|_{(B_\alpha^0, B_\beta^1)_{[\theta]}} + \varepsilon$. In particular, $F(1 + it) \in B_\beta^1$ implies $\psi^{(\alpha-\beta)} F(1 + it) \in B_\alpha^1$. Let $\tilde{F}$ be the function given by

$$\tilde{F}(z) = \psi^{(\alpha-\beta)z} F(z), \quad z \in S.$$

Then $\tilde{F}(\theta) = \psi^{(\alpha-\beta)\theta} f$ and

$$\tilde{F}(it) = \psi^{i(\alpha-\beta)t} F(it) \in B_\alpha^0,$$

$$\tilde{F}(1 + it) = \psi^{(\alpha-\beta)+i(\alpha-\beta)t} F(1 + it) \in B_\alpha^1, \quad \text{for any } t \in \mathbf{R},$$

with $||\tilde{F}(it)||_{B^0_\alpha} = ||F(it)||_{B^0_\alpha}$ and $||\tilde{F}(1+it)||_{B^1_\alpha} = ||\psi^{(\alpha-\beta)}F(1+it)||_{B^1_\alpha} \leqslant (1+\frac{|\alpha-\beta|}{2})||F(1+it)||_{B^1_\beta}$. Therefore, $\psi^{(\alpha-\beta)\theta}f \in B^\theta_\alpha$ with

$$\begin{aligned}
&||\psi^{(\alpha-\beta)\theta}f||_{B^\theta_\alpha}\\
&\quad\leqslant \max\left\{\sup_{t\in\mathbf{R}} ||\tilde{F}(it)||_{B^0_\alpha}, \sup_{t\in\mathbf{R}} ||\tilde{F}(1+it)||_{B^1_\alpha}\right\} \leqslant \left(1+\frac{|\alpha-\beta|}{2}\right)||F||_{\mathscr{F}}\\
&\quad\leqslant \left(1+\frac{|\alpha-\beta|}{2}\right)\left(||f||_{(B^0_\alpha,B^1_\beta)_{[\theta]}} + \varepsilon\right).
\end{aligned}$$

So by Lemma 2.1, $f \in B^\theta_{\alpha-(\alpha-\beta)\theta} = B^\theta_{(1-\theta)\alpha+\theta\beta}$ and

$$\begin{aligned}
||f||_{B^\theta_{(1-\theta)\alpha+\theta\beta}} &\leqslant \left(1+\frac{|\alpha-\beta|}{2}\right)||\psi^{(\alpha-\beta)\theta}f||_{B^\theta_\alpha}\\
&\leqslant \left(1+\frac{|\alpha-\beta|}{2}\right)^2 ||f||_{(B^0_\alpha,B^1_\beta)_{[\theta]}} + \left(1+\frac{|\alpha-\beta|}{2}\right)\varepsilon.
\end{aligned}$$

The arbitrariety of $\varepsilon > 0$ above gives us the inclusion $(B^0_\alpha, B^1_\beta)_{[\theta]} \subset B^\theta_{(1-\theta)\alpha+\theta\beta}$ with domination of norms.

The opposite inclusion can be seen in the same way. $\square$

Lemma 2.3. $(B^{\theta_1}_{\alpha_1}, B^{\theta_2}_{\alpha_2})_{[\theta]} = B^{(1-\theta)\theta_1+\theta\theta_2}_{(1-\theta)\alpha_1+\theta\alpha_2}$ *for any* $\alpha_1, \alpha_2 \in \mathbf{R}$, $\theta \in [0,1]$ *and* $\theta_1, \theta_2 \in (0,1)$ *with* $\theta_1 \neq \theta_2$. *Also*, $(B^\theta_\alpha, B^1_\alpha)_{[\eta]} = B^{(1-\eta)\theta+\eta}_\alpha$ *for any* $\alpha \in \mathbf{R}$ *and* $\theta, \eta \in (0,1)$.

Proof. Since $\theta_1 \neq \theta_2$, we can define

$$\xi_1 = \frac{\theta_2\alpha_1 - \theta_1\alpha_2}{\theta_2 - \theta_1} \quad \text{and} \quad \xi_2 = \frac{(1-\theta_2)\alpha_1 - (1-\theta_1)\alpha_2}{\theta_1 - \theta_2}.$$

Then $(1-(1-\theta)\theta_1 - \theta\theta_2)\xi_1 + ((1-\theta)\theta_1 + \theta\theta_2)\xi_2 = (1-\theta)\alpha_1 + \theta\alpha_2$. Let $A_0 = \overline{B^0_{\xi_1} \cap B^1_{\xi_2}}^{B^0_{\xi_1}}$ and $A_1 = \overline{B^0_{\xi_1} \cap B^1_{\xi_2}}^{B^1_{\xi_2}}$. Then by Bergh–Löfström [1, Theorem 4.2.2], $A_0 \cap A_1$ is dense in the spaces A_0, A_1 and $B^{\theta_1}_{\alpha_1} \cap B^{\theta_2}_{\alpha_2}$. Therefore, by Lemma 2.2, Bergh–Löfström [1, Theorem 4.2.2] and the reiteration theory (cf. [1, Theorem 4.6.1]),

$$\begin{aligned}(B^{\theta_1}_{\alpha_1}, B^{\theta_2}_{\alpha_2})_{[\theta]} &= ((B^0_{\xi_1}, B^1_{\xi_2})_{[\theta_1]}, (B^0_{\xi_1}, B^1_{\xi_2})_{[\theta_2]})_{[\theta]} \\ &= ((A_0, A_1)_{[\theta_1]}, (A_0, A_1)_{[\theta_2]})_{[\theta]} \\ &= (A_0, A_1)_{[(1-\theta)\theta_1+\theta\theta_2]} = (B^0_{\xi_1}, B^1_{\xi_2})_{[(1-\theta)\theta_1+\theta\theta_2]} \\ &= B^{(1-\theta)\theta_1+\theta\theta_2}_{(1-\theta)\alpha_1+\theta\alpha_2}.\end{aligned}$$

By the same method, in order to prove the second assertion, it is sufficient if

$$(B^0_\alpha, B^1_\alpha)_{[1]} = B^1_\alpha, \quad \text{for any } \alpha \in \mathbf{R}. \tag{2.1}$$

Let us show it now. It is trivial from the definition of interpolation spaces that $(B^0_\alpha, B^1_\alpha)_{[1]} \supset B^0_\alpha \cap B^1_\alpha = B^1_\alpha$. Also, by Bergh–Löfström [1, Theorem 4.2.2], $(B^0_\alpha, B^1_\alpha)_{[1]}$ is a subspace of B^1_α. These give us (2.1), and complete the proof. $\square$

Remark 1. Since in general $(B^0_\alpha, B^1_\alpha)_{[0]} \neq B^0_\alpha$, we cannot use this method to calculate $(B^0_\alpha, B^\theta_\alpha)_{[\eta]}$.

3. The semigroup $\{P^0_t\}_{t>0}$

Let $X^0(t,x)$ be the solution of the SDE

$$dX^0(t,x) = \sigma(X^0(t,x))\, dB_t, \quad X^0(0,x) = x,$$

and let $P^0_t f(x) = E[f(X^0(t,x))]$ for any $f \in C(\mathbf{R}^d)$ such that the expectation is finite for any $x \in \mathbf{R}^d$. $\{P^0_t\}_{t>0}$ is the semigroup associated with $\frac{1}{2}\sum_{i,j=1}^d a_{ij}\nabla_i\nabla_j$. Also, let $P^0_t(x, dy)$ denote the distribution of $X^0(t,x)$.

Then we have the following relationship between $\{P_t\}_{t\geqslant 0}$ and $\{P^0_t\}_{t\geqslant 0}$:

Lemma 3.1. *$P_t = P^0_t + \int_0^t P_{t-s} b \cdot \nabla P^0_s\, ds$ for any $t>0$.*

Since $\sigma_{ij} \in C^\infty_b(\mathbf{R}^d)$, the following is well-known (cf., e.g., [3]):

Lemma 3.2. *There exist a function $p^0(t,x,y) \in C^\infty((0,\infty) \times \mathbf{R}^d \times \mathbf{R}^d; \mathbf{R})$ and $C_0, \gamma_0 > 0$ such that*

$$P^0_t(x, dy) = p^0(t,x,y)\, dy,$$

$$|\nabla_x^\beta p^0(t,x,y)| \leqslant C_0 e^{-\frac{\gamma_0|x-y|^2}{2t}} \frac{1}{t^{(d+|\beta|)/2}}$$

for any multi-index β *with* $|\beta| \leqslant 2$, $x, y \in \mathbf{R}^d$ and $t > 0$.

For $i,j = 1, \dots, d$, let $\nabla_i X^0(t,x)$ and $\nabla_i \nabla_j X^0(t,x)$ denote vectors $(\nabla_i X^{0,k}(t,x))_{k=1}^d$ and $(\nabla_i \nabla_j X^{0,k}(t,x))_{k=1}^d$, respectively. Then we have the following:

Lemma 3.3. (1) *There exists a* $c_7 > 0$ *such that*

$$E[||\nabla_i X^0(t,x)||_{\mathbf{R}^d}^4] \leqslant e^{c_7 t}, \quad E[||\nabla X^0(t,x)||_{\mathbf{R}^d \times \mathbf{R}^d}^2] \leqslant d e^{c_7 t/2},$$

for any $i = 1, \dots, d$, $t > 0$ *and* $x \in \mathbf{R}^d$.

(2) *For any* $T > 0$, *there exist* $c_8(T), c_9(T) > 0$ *such that*

$$E[||\nabla_i \nabla_j X^0(t,x)||_{\mathbf{R}^d}^2] \leqslant c_8(T) e^{c_9(T)t}, \quad i,j = 1, \dots, d, t \in [0,T], x \in \mathbf{R}^d.$$

Proof. Denote $\frac{\partial \sigma_{ij}}{\partial x_m}$ by σ_{ij}^m, and $\frac{\partial^2 \sigma_{ij}}{\partial x_m \partial x_n}$ by σ_{ij}^{mn}, for $i,j,m,n = 1, \dots, d$. Let $c_{10} = \max_{i,j,m,n=1,\dots d} ||\sigma_{ij}^{mn}||_\infty$ and $c_{11} = \max_{i,j,m=1,\dots d} ||\sigma_{ij}^m||_\infty$. Then $c_{10}, c_{11} < \infty$ by A1.

We get from the definition of $X^0(t,x)$ that

$$d\nabla_i X^{0,k}(t,x) = \sum_{l,j=1}^d \sigma_{kj}^l (X^0(t,x)) \nabla_i X^{0,l}(t,x)\, dB_t^j. \tag{3.1}$$

So by Ito's formula and condition,

$$\frac{d}{dt} E[||\nabla_i X^0(t,x)||_{\mathbf{R}^d}^4] \leqslant 6 d^3 c_{11}^2 E[||\nabla_i X^0(t,x)||_{\mathbf{R}^d}^4].$$

This combined with $E[||\nabla_i X^0(0,x)||_{\mathbf{R}^d}^4] = 1$ gives us our first assertion.

Let us proof the second assertion. By (3.1)

$$\begin{aligned} d\nabla_i \nabla_j X^{0,k}(t,x) = & \sum_{m,n,l=1}^d \sigma_{kl}^{mn} (X^0(t,x)) \nabla_i X^{0,m}(t,x) \nabla_j X^{0,n}(t,x)\, dB_t^l \\ & + \sum_{m,l=1}^d \sigma_{kl}^m (X^0(t,x)) \nabla_i \nabla_j X^{0,m}(t,x)\, dB_t^l. \end{aligned}$$

For any $t > 0$, let $c_{12}(t) = d^4 c_{10}^2 e^{c_7 t} + d^3 c_{10} c_{11} e^{\frac{1}{2} c_7 t}$ and $c_{13}(t) = d^3 c_{11}^2 + d^3 c_{10} c_{11} e^{\frac{1}{2} c_7 t}$. Then by Ito's formula and the first assertion, we get by Hölder's inequality and a simple calculation that

$$\frac{d}{dt} E[||\nabla_i \nabla_j X^0(t,x)||_{\mathbf{R}^d}^2] \leqslant c_{12}(t) + c_{13}(t) E[||\nabla_i \nabla_j X^0(t,x)||_{\mathbf{R}^d}^2].$$

Note that $E[||\nabla_i \nabla_j X^0(0,x)||^2_{\mathbf{R}^d}] = 0$, and $c_{12}(t)$ and $c_{13}(t)$ are monotone nondecreasing with respect to $t > 0$. Therefore, for any $T > 0$, we get that

$$E[||\nabla_i \nabla_j X^0(t,x)||^2_{\mathbf{R}^d}] \leqslant c_{12}(T)t + c_{13}(T) \int_0^T E[||\nabla_i \nabla_j X^0(s,x)||^2_{\mathbf{R}^d}]\, ds, \quad t \leqslant T.$$

So by Gronwall's Lemma

$$E[||\nabla_i \nabla_j X^0(t,x)||^2_{\mathbf{R}^d}] \leqslant \frac{c_{12}(T)}{c_{13}(T)} e^{c_{13}(T)t}, \quad \text{for any } t \leqslant T. \qquad \square$$

Remark 2. For any $\alpha \in \mathbf{R}$, there exists a $C_\alpha > 0$ such that C_α is monotone nondecreasing with respect to $|\alpha|$ and

$$\sup_{x \in \mathbf{R}^d} \psi(x)^{-\alpha} \int_{\mathbf{R}^d} \psi(y)^\alpha \exp\left(-\frac{\gamma_0 |x-y|^2}{2s} \right) s^{-\frac{d}{2}}\, dy \leqslant C_\alpha, \quad \text{for all } s \in (0,1].$$

The following is a easy corollary of Lemma 3.2 and Remark 2:

Lemma 3.4. *For any $\alpha \in \mathbf{R}$, there exists a $C_\alpha > 0$ (which may be different from before. Same for the following.) such that*

$$||P_s^0||_{B_\alpha^0 \to B_\alpha^0} \leqslant C_\alpha, \quad \textit{for any } s \in (0,1].$$

Lemma 3.5. *For any $\alpha \in \mathbf{R}$, there exists a $C_\alpha > 0$ such that*

$$||\nabla_i P_s^0||_{B_\alpha^{\theta_0} \to B_\alpha^{\theta_1}} \leqslant C_\alpha s^{-(\frac{1}{2} + \frac{\theta_1 - \theta_0}{2})},$$
$$\textit{for any } s \in (0,1], 0 \leqslant \theta_0 \leqslant \theta_1 \leqslant 1, \quad i = 1, \dots, d. \tag{3.2}$$

Proof. The case for $\theta_0 = \theta_1 = 0$ and the case for $\theta_0 = 0, \theta_1 = 1$ are trivial by Lemma 3.2 and Remark 2. Suppose we have shown the case for $\theta_0 = \theta_1 = 1$. Then we get our assertion by interpolation theory since

$$\left((B_\alpha^0, B_\alpha^0)_{[\frac{\theta_1 - \theta_0}{1 - \theta_0}]}, B_\alpha^1 \right)_{[\theta_0]} = (B_\alpha^0, B_\alpha^1)_{[\theta_0]} = B_\alpha^{\theta_0},$$

$$\left((B_\alpha^0, B_\alpha^1)_{[\frac{\theta_1 - \theta_0}{1 - \theta_0}]}, B_\alpha^1 \right)_{[\theta_0]} = \left(B_\alpha^{\frac{\theta_1 - \theta_0}{1 - \theta_0}}, B_\alpha^1 \right)_{[\theta_0]} = B_\alpha^{\theta_1}$$

for any $0<\theta_0<\theta_1<1$ by Lemma 2.3. (The cases for $\theta_0=0$ or $\theta_1=1$ or $\theta_0=\theta_1$ are similar but more simple.)

We show the case for $\theta_0=\theta_1=1$. For any $f\in B^1_\alpha$, we need to estimate $||\nabla_i P^0_s f||_{B^1_\alpha} = ||\nabla_i P^0_s f||_{B^0_\alpha} + \sum_{j=1}^d ||\nabla_j \nabla_i P^0_s||_{B^0_\alpha}$. First, by Hölder's inequality, Lemmas 3.3 and 3.4, there exists a $C_\alpha>0$ such that

$$||\nabla_i P^0_s f||_{B^0_\alpha} \leqslant C_\alpha ||f||_{B^1_\alpha}, \quad \text{for any } s\in(0,1]. \tag{3.3}$$

So it is sufficient if there exists a $C_\alpha>0$ such that

$$||\nabla_j \nabla_i P^0_s f||_{B^0_\alpha} \leqslant C_\alpha s^{-1/2} ||f||_{B^1_\alpha}, \quad \text{for any } s\in(0,1]. \tag{3.4}$$

We first show it for $f\in C^\infty(\mathbf{R}^d)\cap B^1_\alpha$. For any such f,

$$\begin{aligned}\nabla_j \nabla_i P^0_s f(x) &= E[\nabla f(X^0(s,x))\cdot \nabla_j \nabla_i X^0(s,x)]\\ &\quad + E[\nabla_j X^0(s,x)\cdot \nabla\nabla f(X^0(s,x))\nabla_i X^0(s,x)],\end{aligned}$$

and by Lemma 3.3

$$\begin{aligned}&|\psi(x)^{-\alpha} E[\nabla f(X^0(s,x))\cdot \nabla_j \nabla_i X^0(s,x)]|\\ &\quad \leqslant C_\alpha ||f||_{B^1_\alpha}, \quad \text{for any } s\in(0,1], x\in\mathbf{R}^d.\end{aligned}$$

So it is sufficient if

$$|\psi(x)^{-\alpha} E[\nabla_j X^0(s,x)\cdot \nabla\nabla f(X^0(s,x))\nabla_i X^0(s,x)]| \leqslant C_\alpha s^{-1/2} ||f||_{B^1_\alpha} \tag{3.5}$$

for any $x\in\mathbf{R}^d$, $s\in(0,1]$ and $f\in C^\infty(\mathbf{R}^d)\cap B^1_\alpha$. On the other hand, we have

$$\begin{aligned}&|\psi(x)^{-\alpha} E[\nabla_i X^0(s,x)\cdot \nabla\nabla f(X^0(s,x))\nabla_j X^0(s,x)]|\\ &\quad = |\psi(x)^{-\alpha} E[\nabla f(X^0(s,x))\mathscr{R}_1(\det\langle\!\langle X^0(s,x), X^0(s,x)\rangle\!\rangle^{-1}_{\mathscr{L}}\\ &\qquad \nabla_i X^0(s,x)\nabla_j X^0(s,x))]|\\ &\quad \leqslant ||P^0_s||^{1/2}_{B^0_{2\alpha}\to B^0_{2\alpha}} ||f||_{B^1_\alpha} ||\mathscr{R}_1(\det\langle\!\langle X^0(s,x), X^0(s,x)\rangle\!\rangle^{-1}_{\mathscr{L}} \nabla_i X^0(s,x)\\ &\qquad \nabla_j X^0(s,x))||^{(0)}_2\end{aligned}$$

by Kusuoka–Stroock [3, Part I, Theorem 1.20] (with the same notation as there),

$$\begin{aligned}&||\mathscr{R}_1(\det\langle\!\langle X^0(s,x), X^0(s,x)\rangle\!\rangle^{-1}_{\mathscr{L}} \nabla_i X^0(s,x)\nabla_j X^0(s,x))||^{(0)}_2\\ &\quad \leqslant C s^{-1/2} ||\nabla_i X^0(s,x)\nabla_j X^0(s,x)||^{(1)}_{(4d+2)2}\end{aligned}$$

by Kusuoka–Stroock [3, Part I, (3.8)],

$$||\nabla_i X^0(s,x)\nabla_j X^0(s,x)||^{(1)}_{(4d+2)2} \leqslant C||\nabla_i X^0(s,x)||^{(1)}_{(4d+2)4}||\nabla_j X^0(s,x)||^{(1)}_{(4d+2)4}$$

by Kusuoka–Stroock [3, Part I, Lemma 1.14], and

$$\begin{aligned}&||\nabla_i X^0(s,x)||^{(1)}_{(4d+2)4}\\ &= \left|\left|(X^0)^{(1)}(s,x)\right|\right|^{(0)}_{(4d+2)4} + \left|\left|D(X^0)^{(1)}(s,x)\right|\right|^{(0)}_{(4d+2)4}\end{aligned}$$

is bounded for $x \in \mathbf{R}^d$ and $s \in (0,1]$ by Kusuoka–Stroock [3, Part II, (1.12), (1.14)]. These give us (3.5), hence (3.4) holds for any $f \in C^\infty(\mathbf{R}^d) \cap B^1_\alpha$.

We next show that (3.4) holds for any $f \in B^1_\alpha$, too. Choose any $\phi \in C^\infty(\mathbf{R}^d; \mathbf{R}^+)$ with compact support satisfying $\int_{\mathbf{R}^d} \phi(x)\,dx = 1$. Let $\phi_n(x) = n\phi(nx)$, and let $g_n(x) = \int f(y)\phi_n(x-y)\,dy$. Then $g_n(x) \to f(x)$ as $n \to \infty$ for any $x \in \mathbf{R}^d$, and $||g_n||_{B^1_\alpha} \leqslant 2^{|\alpha|+d+1}||f||_{B^1_\alpha}$ for any $n \in \mathbf{N}$. So by Lemma 3.2, the dominated convergence theorem and the fact $f \in B^1_\alpha$, we get

$$\nabla_j \nabla_i P^0_t g_n(x) \to \nabla_j \nabla_i P^0_t f(x), \quad \text{for any } x \in \mathbf{R}^d, t \in (0,1].$$

Therefore, by re-choosing $C_\alpha > 0$ if necessary, (3.4) holds for any $f \in B^1_\alpha$.

This completes the proof. □

The following is a corollary of Lemmas 3.4, 3.5, (3.3) and interpolation theory:

Corollary 3.6. *For any $\alpha \in \mathbf{R}$, there exists a $C_\alpha > 0$ such that C_α is monotone nondecreasing with respect to $|\alpha|$ and*

$$||P^0_s||_{B^{\theta_0}_\alpha \to B^{\theta_1}_\alpha} \leqslant C_\alpha s^{-\frac{\theta_1-\theta_0}{2}} \quad \textit{for any } s \in (0,1], 0 \leqslant \theta_0 \leqslant \theta_1 \leqslant 1.$$

4. Proof of Theorem 1.1

We give the proof of Theorem 1.1 in this section.

As in Section 1, let $X(t,x)$ be the solution of the SDE (1.1), and $\{P_t\}_{t>0}$ the associated semi-group. Let $\nabla_k X(t,x)$ denote the vector $(\nabla_k X^i(t,x))^d_{i=1}$, $k = 1, \ldots, d$.

Note that $||\nabla_k X(0,x)||^2_{\mathbf{R}^d} = 1$. So we have the following by definition and Ito's formula:

Lemma 4.1. *There exists a $c_{14} > 0$ such that*

$$E[||\nabla_k X(t,x)||^2_{\mathbf{R}^d}] \leqslant e^{c_{14}t}, \quad \textit{for any } x \in \mathbf{R}^d, t > 0.$$

We first have the following result:

Lemma 4.2. *For any* $\alpha \geqslant 0$*, there exists a* $C_\alpha > 0$ *such that* C_α *is monotone nondecreasing with respect to* α *and*

$$||P_s||_{B^0_\alpha \to B^0_\alpha} \leqslant C_\alpha, \quad ||P_s||_{B^1_\alpha \to B^1_\alpha} \leqslant C_\alpha e^{c_{14}s}, \quad \textit{for any } s > 0.$$

Proof. For any $\alpha \geqslant 2$, let $K^1_\alpha = 2c_4 + c_2 d + (\alpha - 2)c_2$ and $K^2_\alpha = (\frac{K^1_\alpha}{2c_5})^{\frac{2}{\gamma_1+1}}$. Also, let $v_{\alpha,x}(t) = E^{P_x}[|X_t|^\alpha]^{\frac{2}{\alpha}}$, $t > 0$, $x \in \mathbf{R}^d$. Then by a simple calculation, we get by condition and Ito's formula that

$$\frac{d}{dt} v_{\alpha,x}(t) \leqslant K^1_\alpha - 2c_5 v_{\alpha,x}(t)^{\frac{\gamma_1+1}{2}}. \tag{4.1}$$

Notice that $v_{\alpha,x}(0) = |x|^2$. So (4.1) implies $E^{P_x}[|X_t|^\alpha] \leqslant \max\{|x|^\alpha, (K^2_\alpha)^{\frac{\alpha}{2}}\}, x \in \mathbf{R}^d$. Therefore,

$$\begin{aligned} ||P_s||_{B^0_\alpha \to B^0_\alpha} &= \sup_{x \in \mathbf{R}^d} \psi(x)^{-\alpha} E^{P_x}[\psi(X_t)^\alpha] \\ &\leqslant 2^{\frac{\alpha}{2}+1} + 2^{\frac{\alpha}{2}}(K^2_\alpha)^{\frac{\alpha}{2}}, \quad \text{for any } s > 0. \end{aligned}$$

This gives us our first assertion for any $\alpha \geqslant 2$. Notice that $||P_s||_{B^0_\alpha \to B^0_\alpha} \leqslant (||P_s||_{B^0_2 \to B^0_2})^{\frac{\alpha}{2}}$ for any $\alpha \in [0, 2]$. So we get our first assertion for any $\alpha \geqslant 0$.

Now, our second assertion is easy by Lemma 4.1 since $f \in B^1_\alpha$,

$$\psi(x)^{-\alpha} \nabla_i P_t f(x) \leqslant ||f||_{B^1_\alpha} \psi(x)^{-\alpha} E^{P_x}[\psi(X_t)^{2\alpha}]^{1/2} E[|\nabla_i X(t, x)|^2]^{1/2}. \qquad \square$$

Remark 3. In the proof of Lemma 4.2, we did not use the fact that γ_1 is strictly greater than 1, either the existence of γ_2. The same proof works as long as $\gamma_1 \geqslant 1$. So the results of Lemma 4.2 still holds even if we only assume $x \cdot b(x) \leqslant c_4 - c_5|x|^2, x \in \mathbf{R}^d$, instead of assumption A3.

Lemma 4.3. *For any* $\alpha \geqslant 0$*, there exists a* $C_\alpha > 0$ *such that* C_α *is monotone nondecreasing with respect to* α *and*

$$||P_s||_{B^\theta_\alpha \to B^\theta_{\alpha-\beta}} \leqslant C_\alpha s^{-\frac{\beta}{\gamma_1 - 1}}, \quad \textit{for any } s \in (0, 1], \beta \in [0, \alpha], \theta \in [0, 1].$$

Proof. We first proof the assertion with $\theta = 0$. In this case, the assertion for $\beta = 0$ is nothing but the first estimation in Lemma 4.2. Since $B^0_{\alpha-\beta} = (B^0_0, B^0_\alpha)_{[\frac{\alpha-\beta}{\alpha}]}$ for any $\beta \in (0, \alpha)$ by Lemma 2.3, by interpolation theory, we only need to show it with $\beta = \alpha$, $\alpha \geqslant 0$. We do this now. Notice that $||P_s||_{B^0_\alpha \to B^0_0} = \sup_{x \in \mathbf{R}^d} E^{P_x}[(1 + |X_s|^2)^{\alpha/2}]$.

Use the same notations as in the proof of Lemma 4.2. Let $\tau_{\alpha,x} = \inf\left\{t; K_\alpha^1 - 2c_5 v_{\alpha,x}(t)^{\frac{\gamma_1+1}{2}} \geqslant 0\right\}$. Then we get by (4.1)

$$\int_{v_{\alpha,x}(t)}^{v_{\alpha,x}(0)} \frac{ds}{2c_5 v_{\alpha,x}(t)^{\frac{\gamma_1+1}{2}} - K_\alpha^1} \geqslant t, \quad t \in (0, \tau_{\alpha,x}].$$

On the other hand, since $\gamma_1 > 1$, there exists a $t_0 > 0$ (depending on α) such that for any $t \in (0, t_0]$, there exists a $a(t) > 0$ (depending on α) such that

$$\int_{a(t)}^{\infty} \frac{ds}{2c_5 s^{\frac{\gamma_1+1}{2}} - K_\alpha^1} = t.$$

So

$$v_{\alpha,x}(t) \leqslant a(t), \quad t \in (0, \tau_{\alpha,x} \wedge t_0]. \tag{4.2}$$

Also, it is trivial that

$$v_{\alpha,x}(t) \leqslant K_\alpha^2, \quad \text{if } t \geqslant \tau_{\alpha,x}. \tag{4.3}$$

Note that if $c_5 a(t)^{\frac{\gamma_1+1}{2}} \geqslant K_\alpha^1$, then

$$t = \int_{a(t)}^{\infty} \frac{ds}{2c_5 s^{\frac{\gamma_1+1}{2}} - K_\alpha^1} \leqslant \int_{a(t)}^{\infty} \frac{ds}{c_5 s^{\frac{\gamma_1+1}{2}}} = \frac{1}{c_5} \frac{2}{\gamma_1 - 1} a(t)^{\frac{1-\gamma_1}{2}}.$$

So

$$a(t) \leqslant \max\left\{ 2^{\frac{2}{\gamma_1+1}} K_\alpha^2, \left(\frac{2}{c_5(\gamma_1 - 1)}\right)^{\frac{2}{\gamma_1-1}} t^{-\frac{2}{\gamma_1-1}} \right\}. \tag{4.4}$$

By (4.2)–(4.4), we get that there exists a $t_0 > 0$ (depending on α) such that

$$v_{\alpha,x}(t) \leqslant \max\left\{ 2^{\frac{2}{\gamma_1+1}} K_\alpha^2, \left(\frac{2}{c_5(\gamma_1 - 1)}\right)^{\frac{2}{\gamma_1-1}} t^{-\frac{2}{\gamma_1-1}} \right\}, \quad t \in (0, t_0]. \tag{4.5}$$

This gives us that our assertion for the case $\theta = 0$ and $\beta = \alpha$ with $\alpha \geqslant 2$ holds for any $t \in (0, t_0]$. So by the semigroup property of $\{P_t\}_{t>0}$, by re-choosing C_α if necessary, we can extend it to $t \in (0, 1]$. Since $||P_s||_{B_\alpha^0 \to B_0^0} \leqslant ||P_s||_{B_2^0 \to B_0^0}$ for any $\alpha \in [0, 2]$, we have that our assertion holds for $\theta = 0$ and $\beta = \alpha$ for any $\alpha \geqslant 0$. This completes the proof for the case $\theta = 0$.

We next deal with the case $\theta = 1$. Notice that

$$\begin{aligned}
&\psi(x)^{-(\alpha-\beta)}\nabla_k P_s f(x) \\
&\leqslant \psi(x)^{-(\alpha-\beta)}||f||_{B^1_\alpha} E[|\nabla_k X(s,x)|^2]^{1/2} E[\psi(X_t)^{2\alpha}]^{1/2} \\
&= ||f||_{B^1_\alpha} E[|\nabla_k X(s,x)|^2]^{1/2} ||P_s||^{1/2}_{B^0_{2\alpha}\to B^0_{2(\alpha-\beta)}}
\end{aligned}$$

for any $f \in B^0_\alpha$, $x \in \mathbf{R}^d$ and $s \in (0,1]$. This combined with Lemma 4.1 and the assertion for the case $\theta = 0$ completes the proof for the case $\theta = 1$.

Now, our assertion for $\theta \in (0,1)$ is easy by interpolation theory again. $\square$

Since $\nabla(b \cdot f) = (\nabla b)f + (\nabla f)b$ for $f \in C^1(\mathbf{R}^d)^{\otimes d}$, the following is easy to see from A3 and interpolation theory:

Lemma 4.4. *The map b given by $f \mapsto b \cdot f$ maps $(B^\theta_\alpha)^{\otimes d}$ to $B^\theta_{\alpha+\gamma_2}$ continuously with norm*

$$||b||_{(B^\theta_\alpha)^{\otimes d}\to B^\theta_{\alpha+\gamma_2}} \leqslant 2c_6$$

for any $\alpha \in \mathbf{R}$ and $\theta \in [0,1]$.

Lemma 4.5. *For any $\alpha \geqslant 0$, there exists a $C_\alpha > 0$ such that C_α is monotone nondecreasing with respect to α and*

$$||P_t||_{B^{\theta_0}_\alpha \to B^{\theta_1}_{\alpha+\gamma_2-\beta}} \leqslant C_\alpha B\left(\frac{1}{2} - \frac{1}{2}(\theta_1 - \theta_0), 1 - \frac{\beta}{\gamma_1 - 1}\right)\left(\frac{1}{t}\right)^{\frac{\beta}{\gamma_1-1}+\frac{\theta_1-\theta_0}{2}-\frac{1}{2}}$$

for any $\beta \in [\frac{\gamma_1-1}{2}, \gamma_1 - 1)$, $t \in (0,1]$ and $0 \leqslant \theta_0 \leqslant \theta_1 \leqslant 1$ satisfying $\theta_1 - \theta_0 < 1$. Here $B(\cdot,\cdot)$ is the Beta function, and is finite under our conditions.

Proof. Apply Lemma 3.5 to $\nabla_i P^0_s : B^{\theta_0}_\alpha \to B^{\theta_1}_\alpha$, $i = 1, \ldots, d$, and apply Lemma 4.3 to $P_{t-s} : B^{\theta_1}_{\alpha+\gamma_2} \to B^{\theta_1}_{\alpha+\gamma_2-\beta}$. Then we get by Lemma 4.4 that there exists a $C_\alpha > 0$ such that

$$\begin{aligned}
&\left|\left|\int_0^t P_{t-s} b \nabla P^0_s \, ds\right|\right|_{B^{\theta_0}_\alpha \to B^{\theta_1}_{\alpha+\gamma_2-\beta}} \\
&\leqslant C_\alpha \int_0^t \frac{ds}{s^{\frac{1}{2}+\frac{1}{2}(\theta_1-\theta_0)}(t-s)^{\frac{\beta}{\gamma_1-1}}} \\
&= C_\alpha B\left(\frac{1}{2} - \frac{1}{2}(\theta_1 - \theta_0), 1 - \frac{\beta}{\gamma_1 - 1}\right)\left(\frac{1}{t}\right)^{\frac{\beta}{\gamma_1-1}+\frac{\theta_1-\theta_0}{2}-\frac{1}{2}}
\end{aligned}$$

for any $t \in (0,1]$. Also, since $\alpha - \beta + \gamma_2 \geqslant \alpha$, we get from Corollary 3.6

$$||P_t^0||_{B_\alpha^{\theta_0} \to B_{\alpha+\gamma_2-\beta}^{\theta_1}} \leqslant ||P_t^0||_{B_\alpha^{\theta_0} \to B_\alpha^{\theta_1}} \leqslant \frac{C_\alpha}{t^{\frac{\theta_1-\theta_0}{2}}}.$$

Since $B(\frac{1}{2} - \frac{1}{2}(\theta_1 - \theta_0), 1 - \frac{\beta}{\gamma_1 - 1}) \geqslant 1$ and $\frac{\beta}{\gamma_1-1} - \frac{1}{2} \geqslant 0$, these combined with Lemma 3.1 completes the proof by re-choosing $C_\alpha > 0$ if necessary. □

Now, we are ready to proof the following

Lemma 4.6. *Use the same notations. Then*

$$\begin{aligned} &||P_{2t}||_{B_\alpha^0 \to B^1_{\alpha+\gamma_2-\gamma_1+1+\varepsilon\gamma_2+\varepsilon^2(\gamma_1-1)}} \\ &\quad \leqslant C_\alpha C_{\alpha+\gamma_2} B\left(\frac{\varepsilon}{4}, \varepsilon\right) B\left(\frac{1}{4}, \varepsilon\right)^\varepsilon \frac{1}{t^{1-\frac{\varepsilon}{2}-\varepsilon^2}} \end{aligned}$$

for any $\alpha \geqslant 0$, $t \in (0,1]$ *and* $\varepsilon \in (0, \frac{1}{2})$.

Proof. First, we have by Lemmas 4.2 and 4.5

$$\begin{aligned} &||P_t||_{B_\alpha^\theta \to B^1_{\alpha+\gamma_2-\beta}} \leqslant C_\alpha B\left(\frac{\theta}{2}, 1 - \frac{\beta}{\gamma_1 - 1}\right) \frac{1}{t^{\frac{\beta}{\gamma_1-1} - \frac{\theta}{2}}}, \\ &||P_t||_{B_\alpha^1 \to B_\alpha^1} \leqslant C_\alpha \end{aligned}$$

for $\alpha \geqslant 0$, $\beta \in [\frac{\gamma_1-1}{2}, \gamma_1 - 1)$, $\theta \in (0,1)$ and $s \in (0,1]$. Therefore, by interpolation theory

$$||P_t||_{B_\alpha^{\lambda\theta+1-\lambda} \to B^1_{\alpha+\lambda(\gamma_2-\beta)}} \leqslant C_\alpha B\left(\frac{\theta}{2}, 1 - \frac{\beta}{\gamma_1 - 1}\right)^\lambda \frac{1}{t^{\lambda(\frac{\beta}{\gamma_1-1} - \frac{\theta}{2})}} \tag{4.6}$$

for any $\lambda \in (0,1)$.

Now, let $\tilde{\alpha} = \alpha + \gamma_2 - \beta \in [0, \alpha + \gamma_2)$ and $\tilde{\theta} = \lambda\theta + 1 - \lambda \in (0,1)$. Also, choose any $\tilde{\beta} \in [\frac{\gamma_1-1}{2}, \gamma_1 - 1)$. Then $P_{2t} : B_\alpha^0 \to B^1_{\alpha+\gamma_2-\beta+\lambda(\gamma_2-\tilde{\beta})}$ is a composite of the operators $P_t : B_\alpha^0 \to B^{\tilde{\theta}}_{\alpha+\gamma_2-\beta}$ and $P_t : B_{\tilde{\alpha}}^{\lambda\theta+1-\lambda} \to B^1_{\tilde{\alpha}+\lambda(\gamma_2-\tilde{\beta})}$, and

$$||P_t||_{B_\alpha^0 \to B^{\tilde{\theta}}_{\alpha+\gamma_2-\beta}} \leqslant C_\alpha B\left(\frac{1}{2} - \frac{\tilde{\theta}}{2}, 1 - \frac{\beta}{\gamma_1 - 1}\right) \frac{1}{t^{\frac{\beta}{\gamma_1-1} + \frac{\tilde{\theta}}{2} - \frac{1}{2}}},$$

$$||P_t||_{B_{\tilde{\alpha}}^{\lambda\theta+1-\lambda} \to B^1_{\tilde{\alpha}+\lambda(\gamma_2-\tilde{\beta})}} \leqslant C_{\alpha+\gamma_2} B\left(\frac{\theta}{2}, 1 - \frac{\tilde{\beta}}{\gamma_1 - 1}\right)^\lambda \frac{1}{t^{\lambda(\frac{\tilde{\beta}}{\gamma_1-1} - \frac{\theta}{2})}}$$

by Lemma 4.5 and (4.6). Since $\frac{\tilde{\theta}}{2}-\frac{1}{2}-\frac{\lambda\theta}{2}=-\frac{\lambda}{2}$ and $\frac{1}{2}-\frac{\tilde{\theta}}{2}=\frac{1-\theta}{2}\lambda$, these imply

$$\begin{aligned}&||P_{2t}||_{B^0_\alpha\to B^1_{\alpha+\gamma_2-\beta+\lambda(\gamma_2-\tilde{\beta})}}\\ &\leqslant C_\alpha C_{\alpha+\gamma_2}B\left(\frac{1-\theta}{2}\lambda,1-\frac{\beta}{\gamma_1-1}\right)\\ &\quad\times B\left(\frac{\theta}{2},1-\frac{\tilde{\beta}}{\gamma_1-1}\right)^\lambda\frac{1}{t^{\frac{\beta}{\gamma_1-1}+\lambda\frac{\tilde{\beta}}{\gamma_1-1}-\frac{\lambda}{2}}}.\end{aligned}$$

Let $\theta=\frac{1}{2}$, $\lambda=\varepsilon$ and $\beta=\tilde{\beta}=(1-\varepsilon)(\gamma_1-1)$, and we get our assertion. □

Corollary 4.7. *For any* $\alpha\geqslant 0$ *and* $\beta>0$, *there exist constants* $d_\beta\in(0,1)$ *and* $C_{\alpha,\beta}>0$ such that *for any* $t\in(0,1], f\in B^0_\alpha, x\in\mathbf{R}^d$,

$$|\nabla P_tf(x)|\leqslant(1+|x|^2)^{\frac{\gamma_2-\gamma_1+1+\alpha+\beta}{2}}\frac{C_{\alpha,\beta}}{t^{d_\beta}}||f||_{B^0_\alpha}. \tag{4.7}$$

Proof. Let $\beta_0=\frac{\gamma_1-1}{4}+\frac{\gamma_2}{2}>0$. Then for any $\beta\in(0,\beta_0)$, the equation $\varepsilon\gamma_2+\varepsilon^2(\gamma_1-1)=\beta$ with respect to ε has a solution in $\varepsilon\in(0,\frac{1}{2})$. So by Lemma 4.6, (4.7) is true for any $\beta\in(0,\beta_0)$. Since $||P_t||_{B^0_0\to B^1_{\gamma_2-\gamma_1+1+\beta}}$ is monotone nonincreasing with respect to $\beta>0$, this completes the proof. □

Acknowledgments

The author owe her deepest gratitude to Professor Shigeo Kusuoka and Professor Hirofumi Osada for their helpful suggestions and advice.

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