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Contractibility of compact contractions in Hilbert space

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Dedicated to Professor T. Ando on the occasion of his 70th birthday with admiration and affection

Abstract

For a *finite* set Σ of compact contractions in a complex Hilbert space $(H \cdot \|\cdot\|)$, it is shown that $r(A) < 1$ for all A in the multiplicative semigroup generated by Σ if and only if there exists a positive integer N such that $\|A\| < 1$ for all A in the multiplicative semigroup generated by Σ with *length* greater than N. Here $r(A)$ denotes the spectral radius of A. As an application, an answer is given to an infinite-dimensional case of the finiteness conjecture for the generalized spectral radius attributed to J.C. Lagarias and Y. Wang [Linear Algebra Appl. 214 (1995) 17].

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1. Introduction

More than 40 years ago, Rota and Strang [11] introduced the notion of joint spectral radius of a set of elements of a normed algebra. This notion has proved to be of

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fundamental importance in the dynamical behavior of infinite products of matrices [7], in determining the smoothness (the order of Hölder continuity) of wavelets and solutions of two-scale dilation equations [3–6], and in the stability theory of linear semidynamical systems [1]. In 1992, Daubechies and Lagarias [7] introduced the notion of generalized spectral radius of a finite set of $n \times n$ real matrices, and conjectured that the joint spectral radius and the generalized spectral radius are identical. This outstanding conjecture was resolved by Berger and Wang [2] using analytic-algebraic method, especially using the Levitzki theorem from Ring Theory (see [9]). In 1995, Lagarias and Wang [10] introduced and studied the *finiteness conjecture* for the generalized spectral radius of a finite set of $n \times n$ matrices concerning the existence of an algorithm for deciding if the joint spectral radius is less than unity. The finiteness conjecture remains an open problem to the best of our knowledge, and seems to be one of the most difficult problems related to the joint spectral radius. In the present note we propose to prove a result concerning the contractibility of compact contractions in Hilbert space. As an application, we give an answer to an infinite-dimensional case of the finiteness conjecture for the generalized spectral radius attributed to Lagarias and Wang.

2. The principal theorem

Throughout this note, H will denote a complex Hilbert space endowed with the inner product $\langle\cdot,\cdot\rangle$ and the associated norm $\|\cdot\|$. An operator means a bounded linear transformation on H. Denote by I the identity operator on H. For an operator A, $\sigma(A)$ stands for the spectrum of A, and the *spectral radius* of A, denoted by $r(A)$, is defined by

$$r(A) \equiv \sup\left\{|\lambda|;\ \lambda \in \sigma(A)\right\}.$$

Recall that an operator A is called a *contraction* if $\|A\| \leqslant 1$ (in other words, $I - A^*A$ is positive), and A is said to be *compact* if it maps the unit ball of H onto a totally bounded set in H. The set of all compact operators on H is a closed self-adjoint two-sided ideal (see, e.g., [8, p. 91]). It is readily seen from the Schwarz inequality that if A is positive and $\langle Ax, x\rangle = 0$ for some $x \in H$, then $Ax = 0$. Therefore, if A is a contraction, then the null space of $I - A^*A$ becomes

$$\operatorname{Ker}(I - A^*A) = \left\{x \in H;\ \|Ax\| = \|x\|\right\}. \tag{1}$$

For a contraction A, we define $n(A) \equiv \dim \operatorname{Ker}(I - A^*A)$. We list some simple facts which will illustrate the properties of $n(\cdot)$.

Lemma 1. *Suppose A, B are two contractions on H. Then*
(a) $n(A) = n(A^*)$.
(b) $n(AB) \leqslant \min\{n(A), n(B)\}$.
(c) *If A is compact, then $n(A)$ is finite.*
(d) *If A is compact and $n(A) = 0$, then $\|A\| < 1$.*

Proof. For every $x \in H$, we have

$$x \in \operatorname{Ker}(I - A^*A) \Rightarrow A^*Ax = x$$
$$\Rightarrow AA^*Ax = Ax \Rightarrow Ax \in \operatorname{Ker}(I - AA^*).$$

Therefore $n(A) \leqslant n(A^*)$. A similar argument shows that $n(A^*) \leqslant n(A)$, and the assertion follows. This proves (a).

Since A and B are contractions, we have

$$\operatorname{Ker}(I - (AB)^*(AB)) \subset \operatorname{Ker}(I - B^*B). \tag{2}$$

It follows that $n(AB) \leqslant n(B)$. This and (a) together imply that $n(AB) = n(B^*A^*) \leqslant n(A^*) = n(A)$. This proves (b).

Since the restriction of A to the space $\operatorname{Ker}(I - A^*A)$ is an isometry, the compactness of A implies that $\operatorname{Ker}(I - A^*A)$ cannot be infinite-dimensional, that is, $n(A) < \infty$. This proves (c).

Since A^*A is positive and $\sigma(A^*A)$ is closed in $[0, \infty)$, we obtain $r(A^*A) = \|A^*A\| \in \sigma(A^*A)$ (see, e.g., [13, pp. 350–351]). Thus if $A \neq 0$, then by the compactness of A^*A, $\|A^*A\|$ is an eigenvalue of A^*A. Hence there is a unit vector $\hat{x}$ such that $\|A^*A\| = \|A^*A\hat{x}\|$. It follows from (1) and $n(A) = 0$ that

$$\|A\|^2 = \|A^*A\| = \|A^*A\hat{x}\| < \|\hat{x}\| = 1$$

and so $\|A\| < 1$. This proves (d). $\square$

Let Σ be a set of operators on H. For $m = 1, 2, \ldots,$ let Σ^m denote the set of all products of operators in Σ of *length m*, that is,

$$\Sigma^m \equiv \left\{A_m A_{m-1} \cdots A_1;\ A_i \in \Sigma,\ i = 1, \ldots, m\right\}$$

and let $\varphi(\Sigma)$ be the multiplicative semigroup generated by Σ, i.e.,

$$\varphi(\Sigma) \equiv \bigcup_{m=1}^{\infty} \Sigma^m.$$

We now come to the principal theorem of this note.

Theorem 1. *Let Σ be a finite set of compact contractions on H. Then $r(A) < 1$ for all $A \in \varphi(\Sigma)$ if and only if there exists a positive integer N such that $\|A\| < 1$ for all $A \in \Sigma^N$.*

Proof. "$\Rightarrow$": Let $\Sigma \equiv \{A_1, \ldots, A_s\}$. Let us first introduce the notation $\nu(\Sigma)$ as follows. By (2), we have

$$\bigcup_{A \in \varphi(\Sigma)} \operatorname{Ker}(I - A^*A) \subset \bigcup_{k=1}^{s} \operatorname{Ker}(I - A_k^*A_k). \tag{3}$$

Therefore, by (3) and Lemma 1(c), we have

$$\max\left\{n(A);\ A \in \varphi(\Sigma)\right\} \leqslant \max_{1\leqslant k\leqslant s} n(A_k) < \infty. \tag{4}$$

For $k \geqslant 0$, let

$$L_k \equiv \big\{(i_1, i_2, \ldots, i_m); n(A_{i_m}A_{i_{m-1}}\cdots A_{i_1}) = k, \\ \{i_1, i_2, \ldots, i_m\} \subset \{1, 2, \ldots, s\},\ \text{and } m \geqslant 1\big\}.$$

If $A \in \varphi(\Sigma)$ and $n(A) = 0$, by Lemma 1(b) we get $n(A^j) = 0$ for all $j \geqslant 2$. Thus L_0 is infinite. Now let us define $\nu(\Sigma)$ to be the largest non-negative integer k for which L_k is infinite. This number is well defined just as in the above discussion.

Claim: $\nu(\Sigma) = 0$.

Suppose, on the contrary, that $\nu(\Sigma) \geqslant 1$. We put

$$\Delta \equiv \{1, 2, \ldots, s\},\ \ell \equiv \max\left\{n(A);\ A \in \varphi(\Sigma)\right\} \quad (\text{by (4)},\ \ell < \infty)$$

and

$$q \equiv \max\left\{m \geqslant 1;\ (i_1, i_2, \ldots, i_m) \in L_k \text{ for some } k \text{ between } \nu(\Sigma) + 1 \text{ and } \ell\right\}.$$

If $(i_1, i_2, \ldots, i_r) \in L_{\nu(\Sigma)}$ for some $r > q + 1$, then, by Lemma 1(b), we have for every $1 \leqslant j < k \leqslant r$ with $k - j \geqslant q$

$$\nu(\Sigma) = n(A_{i_r}A_{i_{r-1}}\cdots A_{i_1}) \leqslant n(A_{i_k}A_{i_{k-1}}\cdots A_{i_1}) \\ \leqslant n(A_{i_k}A_{i_{k-1}}\cdots A_{i_j}) \leqslant \nu(\Sigma).$$

Accordingly, if $r > q + 1$, then

$$n(A_{i_k}A_{i_{k-1}}\cdots A_{i_j}) = \nu(\Sigma) \quad \text{for all } 1 \leqslant j < k \leqslant r \text{ with } k - j \geqslant q, \tag{5}$$

that is, $(i_j, i_{j+1}, \ldots, i_k) \in L_{\nu(\Sigma)}$ for all $1 \leqslant j < k \leqslant r$ with $k - j \geqslant q$. Since $L_{\nu(\Sigma)}$ is infinite, the set

$$\left\{(i_1, i_2, \ldots, i_m) \in L_{\nu(\Sigma)};\ \text{for some } i_1, \ldots, i_m \in \Delta,\ \text{and } m \geqslant 1\right\}$$

is infinite. Then there are $i_1, i_2, \ldots, i_m \in \Delta$ such that

$$(i_1, i_2, \ldots, i_m) \in L_{\nu(\Sigma)},$$

where $m \equiv (p + 1)(q + 1)$, $p \equiv$ the cardinality of Σ^{q+1}. Therefore,

$$n(A_{i_j}A_{i_{j-1}}\cdots A_{i_1}) = \nu(\Sigma) \quad \text{for } j = q + 1, q + 2, \ldots, m, \tag{6}$$

by (5). We put

$$B_0 \equiv I,\ B_j \equiv A_{i_{j(q+1)}}A_{i_{j(q+1)-1}}\cdots A_{i_{(j-1)(q+1)+1}} \quad \text{for } j = 1, 2, \ldots, p + 1.$$

Then each $B_j\,(j \geqslant 0)$ is a contraction. Now let us write

$$K_j \equiv \operatorname{Ker}(I - (B_jB_{j-1}\cdots B_1)^*(B_jB_{j-1}\cdots B_1)) \quad \text{for } j = 1, 2, \ldots, p + 1.$$

Then, by (1) and (6), $\dim K_j = \nu(\Sigma)$ $(1 \leqslant j \leqslant p + 1)$ and the spaces $\{K_j\}$ are non-increasing. Therefore all the K_j $(1 \leqslant j \leqslant p + 1)$ are the same. We denote K_j $(1 \leqslant j \leqslant p + 1)$ by K. It then follows from (1) that

$$x \in K \iff \|B_j B_{j-1} \cdots B_1 x\| = \|x\|$$
$$\text{for some } j = 1, 2, \ldots, p+1 \text{ (and hence for all } j). \quad (7)$$

Therefore each B_j is an isometry on $B_{j-1}B_{j-2}\cdots B_0K$ $(1 \leqslant j \leqslant p+1)$. By (1), we have $B_{j-1}B_{j-2}\cdots B_0K \subset \mathrm{Ker}(I - B_j^*B_j)$ for $1 \leqslant j \leqslant p+1$. Thus,

$$\nu(\Sigma) = \dim K = \dim B_{j-1}B_{j-2}\cdots B_0K \leqslant \dim \mathrm{Ker}(I - B_j^*B_j) = \nu(\Sigma),$$

where the last equality is obtained by (5). Therefore,

$$B_{j-1}B_{j-2}\cdots B_0K = \mathrm{Ker}(I - B_j^*B_j) \quad \text{for } 1 \leqslant j \leqslant p+1. \quad (8)$$

Since p is the cardinality of Σ^{q+1} and $B_j \in \Sigma^{q+1}$ for $j = 1, 2, \ldots, p+1$, there are $1 \leqslant j < h \leqslant p+1$ such that $B_j = B_h$. Thus, by (8), we have

$$\begin{aligned} B_{j-1}B_{j-2}\cdots B_0K &= \mathrm{Ker}(I - B_j^*B_j) = \mathrm{Ker}(I - B_h^*B_h) \\ &= (B_{h-1}B_{h-2}\cdots B_j)(B_{j-1}B_{j-2}\cdots B_0K). \end{aligned}$$

This shows that $B_{j-1}B_{j-2}\cdots B_0K$ is an invariant subspace for $B_{h-1}B_{h-2}\cdots B_j$. Since $\dim B_{j-1}B_{j-2}\cdots B_0K = \nu(\Sigma) \neq 0$ and $B_{h-1}B_{h-2}\cdots B_j$ is an isometry (isomorphism) from $B_{j-1}B_{j-2}\cdots B_0K$ onto itself, we conclude that $r(B_{h-1}B_{h-2}\cdots B_j) = 1$, contrary to the assumption. This proves the claim. Then, by the definition of $\nu(\Sigma)$ and the choice of q, we have $n(A) = 0$ for all $A \in \Sigma^{q+1}$. It follows from Lemma 1(d) that $\|A\| < 1$ for all $A \in \Sigma^{q+1}$. Thus $N \equiv q+1$ is the required integer.

"$\Leftarrow$": Since $\|A\| < 1$ for all $A \in \Sigma^N$, and each $A \in \varphi(\Sigma)$ is a contraction, we conclude that $\|A\| < 1$ for all $A \in \Sigma^n$ and $n \geqslant N$. Thus if $A \in \varphi(\Sigma)$, then $\|A^m\| < 1$ for some $m \geqslant N$, so that $(r(A))^m = r(A^m) \leqslant \|A^m\| < 1$, by the Spectral Mapping Theorem. Therefore $r(A) < 1$ for all $A \in \varphi(\Sigma)$.

This completes the proof of the theorem. □

Remark. If Σ consists of a single compact contraction A, then from the proof of Theorem 1, we conclude the following fact:

$$\text{If } \|A^{2^{n(A)}}\| = 1, \quad \text{then } r(A) = 1.$$

To see this, we set $K_j \equiv \mathrm{Ker}(I - (A^j)^*A^j)$ for $j = 1, 2, \ldots$. Then $\dim K_1 = n(A)$ and $\{K_j\}$ is a nonincreasing sequence. Suppose there is some positive integer i such that $K_i = \cdots = K_{2i} \neq \{0\}$. Then, by (7), we have

$$A^iK_i = A^iK_{2i} \subset K_{2i} = K_i.$$

Since A^{2i} is an isometry on $K_{2i} = K_i$, $\dim A^iK_i = \dim K_i$. Therefore, $K_i = A^iK_i$ and A^i is an isometry on K_i. This implies that $r(A^i) = 1$, and so $r(A) = 1$. Now, since $\{K_j\}$ is nonincreasing and $\dim K_1 = n(A)$, we have $K_i = \cdots = K_{2i-1} \neq K_{2i}$ to the extreme case. Hence the maximum j such that $K_i \neq K_{2i}$ for some $1 \leqslant i \leqslant j/2$ and $K_j \neq \{0\}$ is less than or equal to $\sum_{i=0}^{n(A)-1} 2^i = 2^{n(A)} - 1$. Therefore, if

$\|A^{2^{n(A)}}\| = 1$ (i.e., $K_{2^{n(A)}} \neq \{0\}$), then there must have some $1 \leqslant i \leqslant 2^{n(A)-1}$ such that $K_i = \cdots = K_{2i}$.

3. Generalized spectral radius

Let Σ be a set of operators on H. Recall that the *joint spectral radius* of Σ, denoted by $\hat{r}(\Sigma)$, is defined to be

$$\hat{r}(\Sigma) \equiv \limsup_{m\to\infty} [\sup_{A\in\Sigma^m} \|A\|]^{1/m}.$$

It is clear that $\hat{r}(\Sigma)$ is independent of the choice of the equivalent norms. The *generalized spectral radius* of Σ, denoted by $r(\Sigma)$, is defined to be

$$r(\Sigma) \equiv \limsup_{m\to\infty} [\sup_{A\in\Sigma^m} r(A)]^{1/m}.$$

The notion of generalized spectral radius was introduced by Daubechies and Lagarias [7] in the case of matrices. Let us write

$$r_m(\Sigma) \equiv \sup_{A\in\Sigma^m} r(A)$$

for $m \geqslant 1$ and remark that by the spectral mapping theorem $r(\Sigma)$ can be written in the following form:

$$r(\Sigma) = \sup_{m\geqslant 1} r_m(\Sigma)^{1/m}. \tag{9}$$

It is clear that $r(\Sigma) \leqslant \hat{r}(\Sigma) \leqslant \max_{A\in\Sigma} \|A\|$.

As an application of Theorem 1, we have:

Theorem 2. *Let Σ_0 be a finite set of contractions on H, A_s a compact contraction on H, and let $\Sigma_1 \equiv \Sigma_0 \cup \{A_s\}$. Suppose*

(a) *there is a positive integer n such that* $\max_{A\in\Sigma_0^n} \|A\| < 1$,

(b) $r(A) < 1$ *for all* $A \in \varphi(\Sigma_1)$.

Then there exists a positive integer N such that $\|A\| < 1$ *for all* $A \in \Sigma_1^N$.

Proof. Since $r(A_s) < 1$, there is a positive integer m such that $\|A_s^m\| < 1$. Let

$$\Sigma \equiv \left\{ A_s B;\ B \in \bigcup_{i=0}^{n-1} \Sigma_0^i \right\},$$

where $\Sigma_0^0 \equiv \{I\}$. Then Σ consists of finitely many compact contractions, and $r(A) < 1$ for all $A \in \varphi(\Sigma)$, since $\varphi(\Sigma) \subset \varphi(\Sigma_1)$. Thus, Theorem 1 ensures the existence of a positive integer m_1 such that $\|A\| < 1$ for all $A \in \Sigma^{m_1}$, and so

$$\|A\| < 1 \quad \text{for all } A \in \bigcup_{i=0}^{\infty} \Sigma^{m_1+i}. \tag{10}$$

Clearly, each $A \in \Sigma$ has length $\leqslant n$. Let $A \in \Sigma_1^{2m_1(m+n)}$. Then A can be written by

$$A \equiv B_0 A_s^{s_1} B_1 A_s^{s_2} \cdots B_{r-1} A_s^{s_r} B_r, \tag{11}$$

where $B_0, B_r = I$ or $B_0, B_1, \ldots, B_r \in \varphi(\Sigma_0)$, and s_j $(j = 1, 2, \ldots, r)$ are positive integers. To prove the assertion, we split the argument into three cases.

Case I. $s_j \geqslant m$ for some $j = 1, 2, \ldots, r$. Then by (11), we have

$$\|A\| \leqslant \|A_s^{s_j}\| \leqslant \|A_s^m\| < 1.$$

Case II. $B_k \in \bigcup_{i=0}^{\infty} \Sigma_0^{n+i}$ for some $k = 0, 1, \ldots, r$. Then by (11) and condition (a), we have

$$\|A\| \leqslant \|B_k\| < 1.$$

Case III. $B_k \in \bigcup_{i=0}^{n-1} \Sigma_0^i$ and $1 \leqslant s_j \leqslant m - 1$ for all $k = 0, 1, \ldots, r$ and $j = 1, 2, \ldots, r$. Then $A_s B_1, \ldots, A_s B_r \in \Sigma$ and

$$\begin{aligned} 2m_1(m+n) - (n-1) &\leqslant \text{the length of } A_s^{s_1} B_1 A_s^{s_2} \cdots B_{r-1} A_s^{s_r} B_r \\ &\leqslant s_1 + (n-1) + s_2 + (n-1) + \cdots + s_r + (n-1) \\ &\leqslant r(m-1) + r(n-1) \\ &\leqslant r(m+n), \end{aligned}$$

so that $r > m_1$, whence

$$A_s^{s_1} B_1 A_s^{s_2} \cdots B_{r-1} A_s^{s_r} B_r \in \bigcup_{i=1}^{\infty} \Sigma^{m_1+i}.$$

Therefore by (10) and (11), we obtain

$$\|A\| \leqslant \|A_s^{s_1} B_1 A_s^{s_2} \cdots B_{r-1} A_s^{s_r} B_r\| < 1.$$

Setting $N \equiv 2m_1(m+n)$, we see that if $A \in \Sigma^N$, then $\|A\| < 1$. The theorem follows. □

When Σ_0 is a singleton, we obtain the following result by repeated application of Theorem 2 on compact contractions.

Theorem 3. *Let Σ be a finite set of compact contractions on H with one possible exception operator not compact. If $r(A) < 1$ for all $A \in \varphi(\Sigma)$, then there exists a positive integer N such that $\|A\| < 1$ for all $A \in \Sigma^N$.*

Theorem 4. *Let Σ be a finite set of compact contractions on H with one possible exception operator not compact. Then the following conditions are mutually equivalent*:

(a) $r(A) < 1$ *for all* $A \in \varphi(\Sigma)$.

(b) *There exists a norm $|||\cdot|||$ on H equivalent to $\|\cdot\|$ such that $|||A||| < 1$ for all $A \in \Sigma$.*

(c) $\hat{r}(\Sigma) < 1$.
(d) $r(\Sigma) < 1$.

Proof. (a) $\Rightarrow$ (b). By Theorem 3, there exists a positive integer N such that $\|A\| < 1$ for all $A \in \Sigma^N$. Choose $0 < \alpha < 1$ so that

$$\max_{A\in\Sigma^N} \|A\| < \alpha^N < 1. \tag{12}$$

Define a new norm $|||\cdot|||$ on H by setting

$$|||x||| = \|x\| + \frac{1}{\alpha}\max_{B\in\Sigma^1}\|Bx\| + \cdots + \frac{1}{\alpha^{N-1}}\max_{B\in\Sigma^{N-1}}\|Bx\| \quad \text{for } x \in H.$$

Then by (12) we have for $A \in \Sigma$ and $x \in H$,

$$\begin{aligned}|||Ax||| &= \|Ax\| + \frac{1}{\alpha}\max_{B\in\Sigma^1}\|BAx\| + \cdots + \frac{1}{\alpha^{N-1}}\max_{B\in\Sigma^{N-1}}\|BAx\| \\ &\leqslant \alpha\left(\frac{1}{\alpha}\max_{B\in\Sigma^1}\|Bx\| + \frac{1}{\alpha^2}\max_{B\in\Sigma^2}\|Bx\| + \cdots \right.\\ &\qquad \left. + \frac{1}{\alpha^{N-1}}\max_{B\in\Sigma^{N-1}}\|Bx\| + \|x\|\right) \\ &= \alpha|||x|||,\end{aligned}$$

so that $|||A||| < 1$ for all $A \in \Sigma$.

(b) $\Rightarrow$ (c). Trivial.

(c) $\Rightarrow$ (d) is immediate, since $r(\Sigma) \leqslant \hat{r}(\Sigma)$.

(d) $\Rightarrow$ (a) is immediate, since by (9) $r(A) \leqslant r(\Sigma)^m \leqslant r(\Sigma) < 1$ for all $A \in \Sigma^m$ and $m = 1, 2, \ldots$.

This completes the proof. □

Remark. For a set Σ of operators on H, Σ is said to be *asymptotically stable* (in the sense of Lyapunov) if there is $0 < \alpha < 1$ such that there are bounded neighborhoods $U, V \subset H$ of the zero vector for which $AV \subset \alpha^m U$ for all $A \in \Sigma^m$ and $m = 1, 2, \ldots$. It can be proved that Σ is asymptotically stable if and only if there exist $0 < \alpha < 1$ and a norm $|||\cdot|||$ on H equivalent to $\|\cdot\|$ such that $|||A||| \leqslant \alpha < 1$ for all $A \in \Sigma$. Thus Theorem 4 shows that if Σ consists of finitely many compact contractions on H with one possible exception operator not compact, then Σ is asymptotically stable if and only if $r(A) < 1$ for all $A \in \varphi(\Sigma)$. Let us remark also that the equivalence (a) $\Leftrightarrow$ (b) gives an answer to an infinite-dimensional case of the problem posed in [12, p. 68].

Theorem 4 implies the following:

Theorem 5. *Let Σ be a finite set of compact contractions on H with one possible exception operator not compact. If $\hat{r}(\Sigma) = 1$, then there exists a positive integer k such that*

$$\hat{r}(\Sigma) = r(\Sigma) = r_k(\Sigma)^{1/k}.$$

Proof. We have $r_k(\Sigma)^{1/k} \leqslant r(\Sigma) \leqslant \hat{r}(\Sigma) = 1$ for all $k \geqslant 1$. If $r_k(\Sigma)^{1/k} < 1$ for all $k \geqslant 1$, then $r(A) < 1$ for all $A \in \varphi(\Sigma)$, and so $\hat{r}(\Sigma) < 1$ by Theorem 4, which contradicts $r(\Sigma) = 1$. Therefore there exists a positive integer k such that $r_k(\Sigma)^{1/k} = 1$, and the proof is complete. □

Theorem 4 gives an answer to an infinite-dimensional case of the Lagarias–Wang conjecture [10].

4. Concluding remark

The following example shows that the "finiteness" assumption is essential in Theorems 1–5 even in the commutative case.

Example. Let $\ell_2^n(\mathbb{C})$ be the n-dimensional complex Hilbert space equipped with the 2-norms $\|\cdot\|_2$ for $n \geqslant 2$. Let

$$H \equiv \left\{ (x_n);\ x_n \in \ell_2^n(\mathbb{C}),\ n \geqslant 2,\ \sum_{n=2}^{\infty} \|x_n\|_2^2 < \infty \right\}$$

be the Hilbert space equipped with the norm $\|\cdot\|$ defined by

$$\|(x_n)\| \equiv \left(\sum_{n=2}^{\infty} \|x_n\|_2^2 \right)^{1/2} \quad \text{for } (x_n) \in H.$$

Let J_n be the nilpotent *Jordan block* of size n for $n \geqslant 2$. Define operators $A_i : H \to H$ $(i = 2, 3, \ldots)$ by

$$A_i(x_n) \equiv (y_n), \quad \text{where } y_n \equiv J_i x_n \text{ if } n = i, \text{ otherwise } 0.$$

Put $\Sigma \equiv \{A_2, A_3, \ldots\}$. Then Σ is a commuting family of finite rank operators such that $\|A_i^k\| = 1$ for all $1 \leqslant k \leqslant i,\ i = 2, 3, \ldots,$ and $r(A) = 0$ for all $A \in \varphi(\Sigma)$. Thus $r(A) = 0$ for all $A \in \varphi(\Sigma)$ but $\hat{r}(\Sigma) = 1$. Also, we have

$$\hat{r}(\Sigma) = 1 > r(\Sigma) = r_k(\Sigma)^{1/k} = 0 \quad \text{for all } k \geqslant 1.$$

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