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Two New Subclasses of Bi-Univalent Functions

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Abstract. In this paper, we introduce two new subclasses of the function class Σ of bi-univalent functions defined in the open unit disc. We find estimates on the coefficients $|a_2|$ and $|a_3|$ for functions in these new subclasses.

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1. Introduction and definitions

Let Ω denote the class of functions of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in the open unit disc $U := \{z : |z| < 1\}$. Let $M(\lambda)$ denote the class of λ -convex functions in U defined as follows (see [1]):

$$M(\lambda) = \left\{ f \in \Omega : \mathrm{Re}\left((1-\lambda)\frac{zf'(z)}{f(z)} + \lambda(1 + \frac{zf''(z)}{f'(z)}) \right) > 0, \lambda \geq 0 \right\}.$$

Further, by δ we shall denote the class of all functions in Ω which are univalent in U (for details, see [2,3,4]).It is well known that every function $f \in \delta$ has an inverse f^{-1}, defined by

$$f^{-1}(f(z)) = z \qquad (z \in U)$$

and

$$f(f^{-1}(w)) = w \qquad \left\{|w| < r_0(f); r_0(f) \geq \frac{1}{4}\right\}$$

where

$$f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \cdots$$

A function $f(z) \in \Omega$ is said to be bi-univalent in U if both $f(z)$ and $f^{-1}(z)$ are univalent in U (see [5]).

Let Σ denote the class of bi-univalent functions in U given by (1.1). Brannan and Taha [6] (see also [7]) introduced certain subclasses of the bi-univalent function class Σ similar to the familiar subclasses $\delta^*(\alpha)$ and $K(\alpha)$ of starlike and convex functions of order $\alpha (0 < \alpha \leq 1)$, respectively (see [8]). Thus,following Brannan and Taha [6] (see also [7]), a function $f(z) \in \Omega$ is in the class $\delta_\Sigma^*(\alpha)$ of strongly bi-starlike functions of order $\alpha (0 < \alpha \leq 1)$ if each of the following conditions is satisfied:

$$f \in \Sigma, \left|\arg\left(\frac{zf'(z)}{f(z)}\right)\right| < \frac{\alpha\pi}{2} \qquad (0 < \alpha \leq 1, z \in U)$$

and

$$\left|\arg\left(\frac{wg'(w)}{g(w)}\right)\right| < \frac{\alpha\pi}{2} \qquad (0 < \alpha \leq 1, w \in U)$$

where g is the extension of f^{-1} to U. The classes $\delta_\Sigma^*(\alpha)$ and $K_\Sigma(\alpha)$ of bi-starlike functions of order α and bi-convex functions of order α, corresponding (respectively) to the function classes $\delta^*(\alpha)$ and $K(\alpha)$, were also introduced analogously. For each of the function classes $\delta_\Sigma^*(\alpha)$ and $K_\Sigma^*(\alpha)$, they found non-sharp estimates on the first two Taylor–Maclaurin coefficients $|a_2|$ and $|a_3|$ (for details, see [6,7]).

The object of the present paper is to introduce two new subclasses of the function class Σ and find estimates on the coefficients $|a_2|$ and $|a_3|$ for functions in these new subclasses of the function class Σ employing the techniques used earlier by Srivastava et al. [5](see also [6,7,8]).

In order to derive our main results, we have to recall here the following lemma [12].

Lemma 1.1. If $h \in \mathrm{P}$,then $|c_k| \le 2$ for each k, where P is the family of all functions h analytic in U for which $\operatorname{Re} h(z) > 0$

$$h(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots \quad \text{for} \quad z \in U$$

2. Coefficient bounds for the function class $M_\Sigma(\alpha, \lambda)$

Definition 2.1. A function $f(z)$ given by (1.1) is said to be in the class $M_\Sigma(\alpha, \lambda)$ if the following conditions are satisfied :

$$f \in \Sigma, \left| \arg\left((1-\lambda)\frac{zf'(z)}{f(z)} + \lambda\left(1 + \frac{zf''(z)}{f'(z)}\right)\right)\right| < \frac{\alpha\pi}{2} \quad (0 < \alpha \le 1, \lambda \ge 0, z \in U) \tag{2.1}$$

and

$$\left| \arg\left((1-\lambda)\frac{wg'(w)}{g(w)} + \lambda\left(1 + \frac{wg''(w)}{g'(w)}\right)\right)\right| < \frac{\alpha\pi}{2} \quad (0 < \alpha \le 1, \lambda \ge 0, w \in U) \tag{2.2}$$

where the function g is given by

$$g(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \cdots \tag{2.3}$$

We note that for $\lambda = 0$, the class $M_\Sigma(\alpha, \lambda)$ reduces to the class $\delta_\Sigma^*(\alpha)$ introduced and studied by Brannan and Taha [9](see also [10]).

We begin by finding the estimates on the coefficients $|a_2|$ and $|a_3|$ for functions in the class $M_\Sigma(\alpha,\lambda)$.

Theorem 2.2. Let $f(z)$ given by (1.1) be in the class $M_\Sigma(\alpha,\lambda)$, $0<\alpha\le 1$ and $\lambda\ge 0$. Then

$$|a_2|\le \frac{2\alpha}{\sqrt{(1+\lambda)(\alpha+1+\lambda-\alpha\lambda)}} \tag{2.4}$$

and

$$|a_3|\le \frac{4\alpha^2}{(1+\lambda)^2}+\frac{\alpha}{1+2\lambda} \tag{2.5}$$

Proof. We can write the argument inequalities in (2.1) and (2.2) equivalently as follows:

$$(1-\lambda)\frac{zf'(z)}{f(z)}+\lambda\left(1+\frac{zf''(z)}{f'(z)}\right)=[p(z)]^\alpha \tag{2.6}$$

$$(1-\lambda)\frac{wg'(w)}{g(w)}+\lambda\left(1+\frac{wg''(w)}{g'(w)}\right)=[q(w)]^\alpha \tag{2.7}$$

Respectively,where $p(z)$ and $q(w)$ satisfy the following inequalities

$$\mathrm{Re}(p(z))>0 \quad (z\in U) \quad \text{and} \quad \mathrm{Re}(q(w))>0 \quad (w\in U)$$

Furthermore,the functions $p(z)$ and $q(w)$ have the forms

$$p(z)=1+p_1z+p_2z^2+p_3z^3+\cdots \tag{2.8}$$

and

$$q(w)=1+q_1w+q_2w^2+q_3w^3+\cdots \tag{2.9}$$

Now, equating the coefficients in (2.6) and (2.7), we get

$$(1+\lambda)a_2 = p_1\alpha \tag{2.10}$$

$$(2+4\lambda)a_3 = p_2\alpha+\frac{\alpha(\alpha-1)p_1^2}{2}+\frac{1+3\lambda}{(1+\lambda)^2}p_1^2\alpha^2 \tag{2.11}$$

and

$$-(1+\lambda)a_2 = q_1\alpha \tag{2.12}$$

$$(2+4\lambda)(2a_2^2 - a_3) = q_2\alpha + \frac{\alpha(\alpha-1)q_1^2}{2} + \frac{1+3\lambda}{(1+\lambda)^2}q_1^2\alpha^2 \tag{2.13}$$

From (2.10) and (2.12), we get

$$p_1 = -q_1 \tag{2.14}$$

and

$$2(1+\lambda)^2 a_2^2 = \alpha^2(p_1^2 + q_1^2) \tag{2.15}$$

Now from (2.11),(2.13) and (2.15),we obtain

$$4(1+2\lambda)a_2^2 = \alpha(p_2+q_2) + \frac{\alpha(\alpha-1)(p_1^2+q_1^2)}{2} + \frac{(1+3\lambda)\alpha^2(p_1^2+q_1^2)}{(1+\lambda)^2} \tag{2.16}$$

Therefore, we have

$$a_2^2 = \frac{\alpha^2(p_2+q_2)}{(1+\lambda)(\alpha+\lambda+1-\alpha\lambda)}.$$

Applying Lemma 1.1 for the coefficients p_2 and q_2, we immediately have

$$|a_2| \le \frac{2\alpha}{\sqrt{(1+\lambda)(\alpha+1+\lambda-\alpha\lambda)}}$$

Next, in order to find the bound on $|a_3|$, by subtracting (2.11) from (2.13), we get

$$2(1+2\lambda)(2a_3 - 2a_2^2) = \alpha(p_2-q_2) + \frac{\alpha(\alpha-1)(p_1^2-q_1^2)}{2} + \frac{(1+3\lambda)\alpha^2(p_1^2-q_1^2)}{(1+\lambda)^2}$$

Upon substituting the value of a_2^2 from (2.15) and observing that $p_1^2 = q_1^2$,it follows that

$$a_3 = a_2^2 + \frac{\alpha(p_2-q_2)}{4(1+2\lambda)}$$

$$= \frac{\alpha^2(p_1^2+q_1^2)}{2(1+\lambda)^2} + \frac{\alpha(p_2-q_2)}{4(1+2\lambda)}$$

Applying Lemma 1.1 once again for the coefficients p_1, p_2, q_1 and q_2 ,we get

$$|a_3| \le \frac{4\alpha^2}{(1+\lambda)^2} + \frac{\alpha}{1+2\lambda}.$$

This completes the proof of Theorem 2.2.
Putting $\lambda = 0$ in Theorem 2.2, we have

Corollary 2.3 . Let $f(z)$ given by (1.1) be in the class $\delta_\Sigma^*(\alpha)$. Then

$$|a_2| \le \frac{2\alpha}{\sqrt{\alpha+1}} \tag{2.17}$$

and

$$|a_3| \le 4\alpha^2 + \alpha \tag{2.18}$$

3. Coefficient bounds for the function class $\mathrm{B}_\Sigma(\beta,\lambda)$

Definition 3.1. A function $f(z)$ given by (1.1) is said to be in the class $\mathrm{B}_\Sigma(\beta,\lambda)$ if the following conditions are satisfied:

$$f \in \Sigma, \mathrm{Re}\left((1-\lambda)\frac{zf'(z)}{f(z)} + \lambda\left(1 + \frac{zf''(z)}{f'(z)}\right)\right) > \beta \quad (0 \le \beta < 1, \lambda \ge 0, z \in U) \tag{3.1}$$

and

$$\mathrm{Re}\left((1-\lambda)\frac{wg'(w)}{g(w)} + \lambda\left(1 + \frac{wg''(w)}{g'(w)}\right)\right) > \beta \quad (0 \le \beta < 1, \lambda \ge 0, w \in U) \tag{3.2}$$

where the function $g(w)$ is defined by (2.3).

We note that for $\lambda = 0$, the class $\mathrm{B}_\Sigma(\beta,\lambda)$ reduces to the class $\mathrm{B}_\Sigma(\beta)$.

Theorem 3.2. Let $f(z)$ given by (1.1) be in the class $\mathrm{B}_\Sigma(\beta,\lambda)$, $0 \le \beta < 1$ and $\lambda \ge 0$. Then

$$|a_2| \le \sqrt{\frac{2(1-\beta)}{1+\lambda}} \tag{3.3}$$

and

$$|a_3| \le \frac{4(1-\beta)^2}{(1+\lambda)^2} + \frac{1-\beta}{1+2\lambda} \tag{3.4}$$

Proof. It follows from (3.1) and (3.2) that there exist $p(z)$ and $q(w)$ such that

$$(1-\lambda)\frac{zf^{/}(z)}{f(z)} + \lambda\left(1 + \frac{zf^{//}(z)}{f^{/}(z)}\right) = \beta + (1-\beta)p(z) \tag{3.5}$$

and

$$(1-\lambda)\frac{wg^{/}(w)}{g(w)} + \lambda\left(1 + \frac{wg^{//}(w)}{g^{/}(w)}\right) = \beta + (1-\beta)q(w) \tag{3.6}$$

where $p(z)$ and $q(w)$ have the forms (2.8) and (2.9), respectively. Equating coefficients in (3.5) and (3.6) yields

$$(1+\lambda)a_2 = p_1(1-\beta) \tag{3.7}$$

$$(2+4\lambda)a_3 = p_2(1-\beta) + \frac{1+3\lambda}{(1+\lambda)^2} p_1^2(1-\beta)^2 \tag{3.8}$$

and

$$-(1+\lambda)a_2 = q_1(1-\beta) \tag{3.9}$$

$$(2+4\lambda)(2a_2^2 - a_3) = q_2(1-\beta) + \frac{1+3\lambda}{(1+\lambda)^2} q_1^2(1-\beta)^2 \tag{3.10}$$

From (3.7) and (3.9), we get

$$p_1 = -q_1 \tag{3.11}$$

and

$$2(1+\lambda)^2 a_2^2 = (1-\beta)^2(p_1^2 + q_1^2) \tag{3.12}$$

Now from (3.8),(3.10) and (3.12),we obtain

$$4(1+2\lambda)a_2^2 = (2+4\lambda)a_3 + q_2(1-\beta) + \frac{(1+3\lambda)}{(1+\lambda)^2} q_1^2 (1-\beta)^2$$

$$= (p_2 + q_2)(1-\beta) + 2(1+3\lambda)a_2^2$$

Therefore, we have

$$a_2^2 = \frac{(p_2+q_2)(1-\beta)}{2(1+\lambda)}.$$

Applying Lemma 1.1 for the coefficients p_2 and q_2, we immediately have

$$|a_2| \le \sqrt{\frac{2(1-\beta)}{1+\lambda}}.$$

Next, in order to find the bound on $|a_3|$, by subtracting (3.8) from (3.10), we get

$$(2+4\lambda)(2a_3 - 2a_2^2) = (1-\beta)(p_2 - q_2)$$

It follows that

$$4(1+2\lambda)a_3 = 4(1+2\lambda)a_2^2 + (1-\beta)(p_2 - q_2)$$

$$= \frac{2(1+2\lambda)(1-\beta)^2 (p_1^2 + q_1^2)}{(1+\lambda)^2} + (1-\beta)(p_2 - q_2)$$

Once again for the coefficients p_1, p_2, q_1 and q_2, we readily get

$$|a_3| \le \frac{4(1-\beta)^2}{(1+\lambda)^2} + \frac{1-\beta}{1+2\lambda}.$$

This completes the proof of Theorem 3.2.
Putting λ = 0 in Theorem 3.2, we have the following corollary.

Corollary 3.3. Let $f(z)$ given by (1.1) be in the class $\mathrm{B}_\Sigma(\beta)$, $(0 \le \beta < 1)$. Then

$$|a_2| \le \sqrt{2(1-\beta)} \tag{3.13}$$

and

$$|a_3| \le 4(1-\beta)^2 + 1 - \beta \tag{3.14}$$

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