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Maximal operator for multilinear Calderón–Zygmund singular integral operators on weighted Hardy spaces

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ABSTRACT

In this paper, the maximal operator associated with multilinear Calderón–Zygmund singular integral operators will be studied by using an improved Coltlar's inequality. Moreover, weighted norm inequalities and some estimates on weighted Hardy spaces are obtained for this maximal operator.

1. Introduction

The study of multilinear singular integral operators of Calderón–Zygmund type continues to attract many researchers' interests, such as [5,10,7,11,2,1,13]. Many results obtained parallel to the linear theory of classical Calderón–Zygmund operators but new interesting phenomena have been observed. See also [9] and the references therein for a detailed description of previous work in the subject.

One aspect of the theory that still is being developed is the one related to the study of maximal operators associated to multilinear singular integrals and appropriate versions of multilinear weighted norm inequalities. So we first recall the definition of multilinear Calderón–Zygmund operators as well as the corresponding maximal operators.

Definition 1.1 *(Multilinear Calderón–Zygmund operators).* Let T be a multilinear operator initially defined on the m-fold product of Schwartz spaces and taking values in the space of tempered distributions

$$T : \mathscr{S}(\mathbb{R}^n) \times \cdots \times \mathscr{S}(\mathbb{R}^n) \longrightarrow \mathscr{S}'(\mathbb{R}^n).$$

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Following [5], we say that T is an m-linear Calderón–Zygmund operator if, for some $1 \leqslant q_j < \infty$, it extends to a bounded multilinear operator from $L^{q_1} \times \cdots \times L^{q_m}$ to L^q, where $\frac{1}{q} = \frac{1}{q_1} + \cdots + \frac{1}{q_m}$, and if there exists a function K, defined off the diagonal $x = y_1 = \cdots = y_m$ in $(\mathbb{R}^n)^{m+1}$, satisfying

$$T(f_1, \ldots, f_m)(x) = \int\limits_{(\mathbb{R}^n)^m} K(x, y_1, \ldots, y_m) f_1(y_1) \cdots f_m(y_m)\, dy_1 \cdots dy_m$$

for all $x \notin \bigcap_{j=1}^m \operatorname{supp} f_j$;

$$\left|K(y_0, y_1, \ldots, y_m)\right| \leqslant \frac{A}{(\sum_{k,l=0}^m |y_k - y_l|)^{mn}}; \tag{1.1}$$

and

$$\left|K(y_0, \ldots, y_j, \ldots, y_m) - K\big(y_0, \ldots, y_j', \ldots, y_m\big)\right| \leqslant \frac{A|y_j - y_j'|^{\varepsilon}}{(\sum_{k,l=0}^m |y_k - y_l|)^{mn+\varepsilon}}, \tag{1.2}$$

for some $\varepsilon > 0$ and all $0 \leqslant j \leqslant m$, whenever $|y_j - y_j'| \leqslant \frac{1}{2} \max_{0 \leqslant k \leqslant m} |y_j - y_k|$.

As in the linear theory, a certain amount of extra smoothness is required for these operators to have such boundedness properties. We will assume that $K(y_0, y_1, \ldots, y_m)$ satisfies the following estimates

$$\left|\partial_{y_0}^{\alpha_0} \cdots \partial_{y_m}^{\alpha_m} K(y_0, y_1, \ldots, y_m)\right| \leqslant \frac{A_\alpha}{(\sum_{k,l=0}^m |y_k - y_l|)^{mn+|\alpha|}}, \tag{1.3}$$

for all $|\alpha| \leqslant N$, where $\alpha = (\alpha_0, \ldots, \alpha_m)$ is an ordered set of m-tuples of nonnegative integers, $|\alpha| = |\alpha_0| + \cdots + |\alpha_m|$, where $|\alpha_j|$ is the order of each multiindex α_j, and N is a large integer to be determined later.

In this article we study maximal multilinear singular integral operator defined by

$$T_*(\vec{f})(x) = \sup_{\delta > 0} \left|T_\delta(f_1, \ldots, f_m)(x)\right|,$$

where T_δ are the smooth truncations of T given by

$$T_\delta(f_1, \ldots, f_m)(x) = \int\limits_{(\mathbb{R}^n)^m} K_\delta(x, y_1, \ldots, y_m) f_1(y_1) \cdots f_m(y_m)\, d\vec{y}.$$

Here, $d\vec{y} = dy_1 \cdots dy_m$, $K_\delta(x, y_1, \ldots, y_m) = \phi(\sqrt{|x - y_1|^2 + \cdots + |x - y_m|^2}/2\delta) K(x, y_1, \ldots, y_m)$ and $\phi(x)$ is a smooth function on $\mathbb{R}^n$, which vanishes if $|x| \leqslant 1/4$ and is equal to 1 if $|x| > 1/2$.

In [8], the authors studied the following operator $\tilde{T}_*$ given by

$$\tilde{T}_*(\vec{f})(x) = \tilde{T}_*(f_1, \ldots, f_m)(x) = \sup_{\delta > 0} \int\limits_{|x-y_1|^2 + \cdots + |x-y_m|^2 > \delta^2} K(x, y_1, \ldots, y_m) f_1(y_1) \cdots f_m(y_m)\, d\vec{y}.$$

Throughout this paper we will let W be the norm of T in the mapping $T: L^1 \times \cdots \times L^1 \to L^{1/m,\infty}$. We list some results for $\tilde{T}_*$ and T_* as follows:

Theorem A. *(See [8].) Let T be an m-linear Calderón–Zygmund operator. Then, for all $\eta > 0$, there exists a constant $C_\eta = C_\eta(n, m) < \infty$ such that for all $\vec{f}$ in any product of $L^{q_i}(\mathbb{R}^n)$ spaces, with $1 \leqslant q_i < \infty$, the following inequality holds for all x in $\mathbb{R}^n$*

$$\tilde{T}_*(\vec{f})(x) \leqslant C_\eta \left(\big(M\big(\left|T(\vec{f})\right|^\eta\big)(x)\big)^{(1/\eta)} + (A + W) \prod_{i=1}^m M f_i(x) \right), \tag{1.4}$$

where M denotes the Hardy–Littlewood maximal function with respect to balls on $\mathbb{R}^n$.

Corollary B. *(See [8].) Let T be an m-linear Calderón–Zygmund operator. Then, for all exponent $1 < q_i \leqslant \infty$, $q < \infty$, and q satisfying $\frac{1}{q} = \frac{1}{q_1} + \cdots + \frac{1}{q_m}$, we have*

$$\left\|\tilde{T}_*(\vec{f})(x)\right\|_{L^q} \leqslant C(A + W) \prod_{i=1}^m \|f_i\|_{L^{q_i}}.$$

Theorem C. *(See [8].) Let $1 \leqslant q_i < \infty$, and q be such that $\frac{1}{q} = \frac{1}{q_1} + \cdots + \frac{1}{q_m}$, and $\omega \in A_{q_1} \cap \cdots \cap A_{q_m}$. Let T be an m-linear Calderón–Zygmund operator. Then there exists a constant $C_{q,n} < \infty$ for all $\vec{f} = (f_1, \ldots, f_m)$ satisfying*

$$\left\| \tilde{T}_*(\vec{f}) \right\|_{L^q_\omega} \leqslant C_{n,q}(A + W) \prod_{i=1}^m \|f_i\|_{L^{q_i}_\omega}.$$

Theorem D. *(See [6].) Let $1 < q_1, \ldots, q_m, q < \infty$ be fixed indices satisfying*

$$\frac{1}{q_1} + \cdots + \frac{1}{q_m} = \frac{1}{q}$$

and let $0 < p_1, \ldots, p_m, p \leqslant 1$ be real numbers satisfying

$$\frac{1}{p_1} + \cdots + \frac{1}{p_m} = \frac{1}{p}.$$

Suppose that K satisfies (1.3) with $N = [n(1/p - 1)]$. Let T be related to K and assume that T admits an extension that maps $L^{q_1} \times \cdots \times L^{q_m}$ into L^q with norm B. Then T_* extends to a bounded operator from $H^{p_1} \times \cdots \times H^{p_m}$ into L^p, and satisfies the norm estimate $\|T_*\|_{H^{p_1} \times \cdots \times H^{p_m} \to L^p} \leqslant C(A + B)$ for some constant $C = C(n, p_i, q_i)$.

Recently, the theory of weighted multilinear Calderón–Zygmund singular integral operators was established in [11] by Lerner, Ombrosi, Pérez, Torres, Trujillo-González and the multiple weights $A_{\vec{p}}$ were constructed. This together with the results for multiple weights $A_{(\vec{p},q)}$ adapted to multilinear fractional integral operators [1,13] answered an open problem in [9]. That is, the existence of multiple weights theory for multilinear Calderón–Zygmund operators and multilinear fractional integral operators. Meanwhile, a new more refined multiple maximal function $\mathcal{M}$

$$\mathcal{M}(\vec{f})(x) = \sup_{Q \ni x} \prod_{i=1}^m \frac{1}{|Q|} \int_Q \left| f_i(y_i) \right| dy_i$$

was used in [11] to characterize the class of $A_{\vec{p}}$ and to obtain some weighted estimates for the multilinear Calderón–Zygmund singular integral operators. So let us recall the definition of $A_{\vec{p}}$ weights.

For m-exponents $p_1, \ldots, p_m$, we will often write p for the number given by $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$, and $\vec{p}$ for the vector $\vec{p} = (p_1, \ldots, p_m)$.

Definition 1.2 *(Multiple $A_{\vec{p}}$ weights).* (See [11].) Let $1 \leqslant p_1, \ldots, p_m < \infty$. Given $\vec{\omega} = (\omega_1, \ldots, \omega_m)$, set

$$\nu_{\vec{\omega}} = \prod_{j=1}^m \omega_j^{p/p_j}.$$

We say that $\vec{\omega}$ satisfies the $A_{\vec{p}}$ condition if

$$\sup_Q \left(\frac{1}{|Q|} \int_Q \prod_{i=1}^m \omega_i^{\frac{p}{p_i}} \right)^{\frac{1}{p}} \prod_{i=1}^m \left(\frac{1}{|Q|} \int_Q \omega_i^{1-p_i'} \right)^{\frac{1}{p_i'}} < \infty. \tag{1.5}$$

When $p_j = 1$, $(\frac{1}{|Q|} \int_Q \omega_i^{1-p_i'})^{\frac{1}{p_i'}}$ is understood as $(\inf_Q \omega_j)^{-1}$.

In particular, when $m = 1$, we note that $A_{\vec{p}}$ will be degenerated to the classical A_p weight. Moreover, if $m = 1$ and $p_i = 1$, then this class of weights coincides with the classical A_1 weights. It is well known that if $\omega \in A_p$ for $1 < p < \infty$, then $\omega \in A_r$ for all $r > p$ and $\omega \in A_q$ for some $1 < q < p$. We thus use $q_\omega := \inf\{q > 1\colon\ \omega \in A_q\}$ to denote the critical index of ω. We will refer to (1.5) as the multilinear $A_{\vec{p}}$ condition.

We list some results in [11] as follows:

Theorem E. *(See [11].) Let $\vec{\omega} = (\omega_1, \ldots, \omega_m)$ and $1 \leqslant p_i < \infty$. Then $\vec{\omega} \in A_{\vec{p}}$ if and only if*

$$\begin{cases} \omega_i^{1-p_i'} \in A_{mp_i'} & i = 1, \ldots, m, \\ \nu_{\vec{\omega}} \in A_{mp} \end{cases}$$

where the condition $\omega_i^{1-p_i'} \in A_{mp_i'}$ in the case $p_i = 1$ is understood as $\omega^{1/m} \in A_1$.

Theorem F. *(See [11].) Let T be an m-linear Calderón–Zygmund operator, satisfying* (1.1), (1.2), $\frac{1}{p}=\frac{1}{p_1}+\cdots+\frac{1}{p_m}$, $1<p_i<\infty$ *and $\vec{\omega}$ satisfy the $A_{\vec{p}}$ condition. Then*

$$\left\|\mathcal{M}(\vec{f})(x)\right\|_{L^p_{v_{\vec{\omega}}}}\leqslant C\prod_{i=1}^m\|f_i\|_{L^{p_i}_{\omega_i}}. \tag{1.6}$$

Theorem G. *(See [11].) Let T be an m-linear Calderón–Zygmund operator, satisfying* (1.1), (1.2), $\frac{1}{p}=\frac{1}{p_1}+\cdots+\frac{1}{p_m}$, *$\vec{\omega}$ satisfy the $A_{\vec{p}}$ condition, and $1<p_i<\infty$. Then*

$$\left\|T(\vec{f})(x)\right\|_{L^p_{v_{\vec{\omega}}}}\leqslant C\prod_{i=1}^m\|f_i\|_{L^{p_i}_{\omega_i}}. \tag{1.7}$$

On the other hand, in 1989, Strömberg and Torchinsky in [14] defined weighted Hardy spaces and obtained some boundedness for Calderón–Zygmund operators.

We will use Garcia-Cuerva's atomic decomposition theory [3] for weighted Hardy spaces. We characterize weighted Hardy spaces in terms of atoms in the following way.

Definition 1.3 *(Weighted Hardy spaces).* Assume that $\omega\in A_q$ with critical index q_ω. Let $[\cdot]$ be the greatest integer function. For $s\in\mathbb{Z}$ satisfying $s\geqslant[n(q_\omega/p-1)]$, a real-valued function $a(x)$ is called (p,q,s)-atom centered at x_0 with respect to ω (or $\omega-(p,q,s)$-atom centered at x_0) if

(a) $a\in L^q_\omega(\mathbb{R}^n)$ and is supported in a cube Q centered at x_0,
(b) $\|a\|_{L^q_\omega}\leqslant\omega(Q)^{\frac{1}{q}-\frac{1}{p}}$,
(c) $\int_{\mathbb{R}^n}a(x)x^\alpha\,dx=0$ for every multi-index α with $|\alpha|\leqslant s$.

When $q=\infty$, L^∞_ω will be taken to mean L^∞ and $\|f\|_{L^\infty_\omega}=\|f\|_\infty$.

Theorem H. *(See [3].) Let $w\in A_q$, $0<p\leqslant1\leqslant q\leqslant\infty$, and $p\neq q$. For each $f\in H^p_\omega(\mathbb{R}^n)$, there exist a sequence a_i of $\omega-(p,q,[n(q_W/p)-1])$-atoms and a sequence λ_i of real numbers with $\sum|\lambda_i|^p\leqslant C\|f\|^p_{H^p_\omega}$ such that $f=\sum\lambda_ia_i$ both in the sense of distributions and in the H^p_ω norm.*

At present, combining the above, we have obtained the boundedness of multilinear Calderón–Zygmund operators on weighted Hardy spaces in [12], so it is natural to ask the following interesting question. Are there any weighted results for maximal operators for multilinear singular integral operators on weighted Hardy spaces? Our result is as follows:

Theorem 1.1. *Let $1<q_1,\dots,q_m,q<\infty$, $0<p_1,\dots,p_m,p\leqslant1$, satisfying*

$$\frac{1}{q_1}+\cdots+\frac{1}{q_m}=\frac{1}{q}$$

and

$$\frac{1}{p_1}+\cdots+\frac{1}{p_m}=\frac{1}{p}.$$

Let T be an m-linear Calderón–Zygmund operators such that K satisfies (1.1), (1.2), (1.3) *with* $N=\max_{1\leqslant i\leqslant m}\{[n((q_i)_\omega/p_i-1)],[(q_i/p_i-1)mn]\}$. *We have the following results*:

(i) *If $\omega\in A_{q_1}\cap\cdots\cap A_{q_m}$, then*

$$\left\|T_*(\vec{f})(x)\right\|_{L^p_\omega}\leqslant C\prod_{i=1}^m\|f_i\|_{H^{p_i}_\omega}. \tag{1.8}$$

(ii) *If for each i, $\omega_i\in A_1$, then*

$$\left\|T_*(\vec{f})(x)\right\|_{L^p_{v_{\vec{\omega}}}}\leqslant C\prod_{i=1}^m\|f_i\|_{H^{p_i}_{\omega_i}}. \tag{1.9}$$

2. Proof of Theorem 1.1

We will use the following facts as in [12]:

(a) Let $q \geqslant 1$ and $\omega \in A_q$. Suppose a is an ω-(p, ∞, s)-atom. Then a is an ω-(p, q, s)-atom.
(b) Let $\omega \in A_p$, $p \geqslant 1$, then for any cube Q and $\lambda > 1$, $\omega(\lambda Q) \leqslant C\lambda^{np}\omega(Q)$, where C does not dependent on Q nor on λ (see [4] for details).

We now turn our attention to Theorem 1.1.

We prove this theorem using the atomic decomposition of H^p_ω spaces. Since finite sums of atoms are dense in H^p_ω, we will work with such sums and obtain estimates independent of the number of terms in each sum.

Since T_* is bounded from $L^{2m} \times \cdots \times L^{2m}$ into L^2 by Lemma 2.2 below when $w_i = 1$, we can assume each f_i, $1 \leqslant i \leqslant m$, as a finite sum of $H^{p_i}_\omega$-atoms, $f_i = \sum_k \lambda_{i,k} a_{i,k}$, where $a_{i,k}$ are (p_i, ∞, s)-atom, this means they are supported in cubes $Q_{i,k}$ and $|a_{i,k}| \leqslant \omega(Q)^{-1/p_i}$. However, by the above fact (a), we know that $a_{i,k}$ satisfies

$$\|a_{i,k}\|_{L^{q_i}_\omega} \leqslant \omega(Q)^{\frac{1}{q_i} - \frac{1}{p_i}}, \tag{2.1}$$

$$\int_{Q_{i,k}} a_{i,k}(x) x^\alpha \, dx = 0, \quad |\alpha| \leqslant s,\ s \geqslant \big[n\big((q_i)_\omega/p_i - 1\big)\big]. \tag{2.2}$$

Denote by $c_{i,k}$ and $|Q_{i,k}|$ the center and the side length of $Q_{i,k}$, and let $\tilde{Q}_{i,k} = 8\sqrt{n} Q_{i,k}$, employing multilinearity we write

$$T_*(\vec{f})(x) = T_*(f_1, \ldots, f_m)(x) \leqslant \sum_{k_1} \cdots \sum_{k_m} \lambda_{1,k_1} \cdots \lambda_{m,k_m} T_*(a_{1,k_1}, \ldots, a_{m,k_m})(x). \tag{2.3}$$

For $x \in \mathbb{R}^n$, we split the right side of (2.3) into two terms $I_1(x) + I_2(x)$, where

$$I_1(x) = \sum_{k_1} \cdots \sum_{k_m} |\lambda_{1,k_1}| \cdots |\lambda_{m,k_m}| \big|T_*(a_{1,k_1}, \ldots, a_{m,k_m})(x)\big| \chi_{\tilde{Q}_{1,k_1} \cap \cdots \cap \tilde{Q}_{m,k_m}}$$

and

$$I_2(x) = \sum_{k_1} \cdots \sum_{k_m} |\lambda_{1,k_1}| \cdots |\lambda_{m,k_m}| \big|T_*(a_{1,k_1}, \ldots, a_{m,k_m})(x)\big| \chi_{\tilde{Q}^c_{1,k_1} \cup \cdots \cup \tilde{Q}^c_{m,k_m}}.$$

Now, let us begin to discuss $I_1(x)$. For fixed $k_1, \ldots, k_m$, assume that $\tilde{Q}_{1,k_1} \cap \cdots \cap \tilde{Q}_{m,k_m} \neq \emptyset$, otherwise, there is nothing need to be proved.

Suppose that $Q_{i^*,k_{i^*}}$, $i^* \in 1, 2, \ldots, m$, has the smallest size among all these cubes. We take a cube $G_{k_1,\ldots,k_m}$ such that

$$\tilde{Q}_{1,k_1} \cap \cdots \cap \tilde{Q}_{m,k_m} \subset G_{k_1,\ldots,k_m} \subset \tilde{G}_{k_1,\ldots,k_m} \subset \tilde{\tilde{Q}}_{1,k_1} \cap \cdots \cap \tilde{\tilde{Q}}_{m,k_m}$$

and

$$\omega(G_{k_1,\ldots,k_m}) \geqslant C\omega(Q_{i^*,k_{i^*}}).$$

By using the Hölder's inequality, Lemma 2.2 when $w_i = w$, and (2.1), we have

$$\begin{aligned}
&\frac{1}{\omega(G_{k_1,\ldots,k_m})} \int_{G_{k_1,\ldots,k_m}} \big|T_*(a_{1,k_1}, \ldots, a_{m,k_m})(x)\big| \omega(x) \, dx \\
&\leqslant \frac{1}{\omega(G_{k_1,\ldots,k_m})} \omega(G_{k_1,\ldots,k_m})^{1/q'} \big\|T_*(a_{1,k_1}, \ldots, a_{m,k_m})(x)\big\|_{L^q_\omega} \\
&\leqslant C\omega(G_{k_1,\ldots,k_m})^{-1/q} \prod_{i=1}^m \|a_{i,k_i}\|_{L^{q_i}_\omega} \\
&\leqslant C\omega(G_{k_1,\ldots,k_m})^{-\frac{1}{q}} \prod_{i=1}^m \omega(Q_{i,k_i})^{\frac{1}{q_i} - \frac{1}{p_i}} \\
&\leqslant C \prod_{i=1}^m \omega(G_{k_1,\ldots,k_m})^{-\frac{1}{q_i}} \prod_{i=1}^m \omega(Q_{i^*,k_{i^*}})^{\frac{1}{q_i} - \frac{1}{p_i}} \\
&= C \prod_{i=1}^m \omega(Q_{i^*,k_{i^*}})^{-\frac{1}{p_i}}.
\end{aligned} \tag{$*$}$$

In order to get the estimate for $I_1(x)$, we need the following lemma:

Lemma 2.1. *Let $0 < p \leqslant 1$. Then there is a constant $C = C(p)$ such that for all finite collections of cubes $\{Q_k\}_{k=1}^m$ in $\mathbb{R}^n$ and all nonnegative functions $g_k \in L_\omega$ with $\operatorname{supp} g_k \subset Q_k$ we have*

$$\left\| \sum_{k=1}^m g_k \right\|_{L^p(\omega)} \leqslant C \left\| \sum_{k=1}^m \frac{1}{\omega(Q_k)} \int_{Q_k} g_k(x)\omega(x)\,dx \chi_{\tilde{Q}_k} \right\|_{L^p(\omega)}. \tag{2.4}$$

The proof of this lemma can be easily obtained by substituting L^p norm by L^p_ω and $\frac{1}{|Q_k|}\int_{Q_k} g_k(x)\,dx$ by $\frac{1}{\omega(Q_k)}\int_{Q_k} g_k(x)\omega\,dx$ in [6].

Using the above lemma and the fact (b), we have

$$\begin{aligned}
\|I_1\|_{L^p_\omega} &\leqslant C \left\| \sum_{k_1} \cdots \sum_{k_m} |\lambda_{1,k_1}| \cdots |\lambda_{m,k_m}| \prod_{i=1}^m \omega(Q_{i^*,k_{i^*}})^{-\frac{1}{p_i}} \chi_{\tilde{G}_{k_1,\dots,k_m}} \right\|_{L^p_\omega} \\
&\leqslant C \left\| \prod_{i=1}^m \left(\sum_{k_i} |\lambda_{i,k_i}| \omega(Q_{i^*,k_{i^*}})^{-\frac{1}{p_i}} \chi_{\tilde{G}_{k_1,\dots,k_m}} \right) \right\|_{L^p_\omega} \\
&\leqslant C \left\| \prod_{i=1}^m \left(\sum_{k_i} |\lambda_{i,k_i}| \omega(Q_{i^*,k_{i^*}})^{-\frac{1}{p_i}} \omega(x)^{\frac{1}{p_i}} \chi_{\tilde{G}_{k_1,\dots,k_m}} \right) \right\|_{L^p} \\
&\leqslant C \prod_{i=1}^m \left(\sum_{k_i} |\lambda_{i,k_i}|^{p_i} \omega(Q_{i^*,k_{i^*}})^{-1} \omega(\tilde{\tilde{Q}}_{i^*,k_{i^*}}) \right)^{\frac{1}{p_i}} \\
&= C \prod_{i=1}^m \left(\sum_{k_i} |\lambda_{i,k_i}|^{p_i} \right)^{\frac{1}{p_i}}.
\end{aligned} \tag{2.5}$$

Secondly, we consider the estimate of $I_2(x)$.

Let A be a nonempty subset of $\{1,\dots,m\}$, and we denote the cardinality of A by $|A|$, then $1 \leqslant |A| \leqslant m$. Let $A^c = \{1,\dots,m\} \setminus A$. If $A = \{1,\dots,m\}$, we define

$$\left(\bigcap_{i\in A} \tilde{Q}^c_{i,k_i} \right) \cap \left(\bigcap_{i\in A^c} \tilde{Q}_{i,k_i} \right) = \bigcap_{i\in A} \tilde{Q}^c_{i,k_i},$$

then we have

$$\tilde{Q}^c_{1,k_1} \cup \cdots \cup \tilde{Q}^c_{m,k_m} = \bigcup_{A\subset\{1,\dots,m\}} \left(\left(\bigcap_{i\in A} \tilde{Q}^c_{i,k_i} \right) \cap \left(\bigcap_{i\in A^c} \tilde{Q}_{i,k_i} \right) \right).$$

We set $E_A = (\bigcap_{i\in A} \tilde{Q}^c_{i,k_i}) \cap (\bigcap_{i\in A^c} \tilde{Q}_{i,k_i})$. For fixed A, we assume that the side length of the cube $Q_{i^*,k_{i^*}}$, $i^* \in A$, is the smallest among the side lengths of the cube Q_{i,k_i}, $i \in A$. Let $P^N_{c_{i*,k_{i*}}}(x, y_1, \dots, y_m)$ be the Nth order Taylor polynomial of $K(x, y_1, \dots, y_m)$ about the variable y_{i^*} at the $c_{i*,k_{i*}}$.

Since $a_{i*,k_{i*}}$ has zero vanishing moments up to N, and observe that the kernels K_δ satisfy (1.3) uniformly in $\delta > 0$. By (1.3) we get

$$\begin{aligned}
&\left| T_*(a_{1,k_1}, \dots, a_{m,k_m})(x) \right| \\
&\quad \leqslant \sup_{\delta>0} \left| \int_{(\mathbb{R}^n)^{m-1}} \prod_{i=1, i\neq i^*}^m a_{i,k_i}(y_i) \int_{\mathbb{R}^n} a_{i*,k_{i*}}(y_{i^*}) \big(K_\delta(x, y_1, \dots, y_m) - P^N_{c_{i*,k_{i*}}}(x, y_1, \dots, y_m) \big)\, d\vec{y} \right| \\
&\quad \leqslant C \int_{(\mathbb{R}^n)^{m-1}} \prod_{i=1, i\neq i^*}^m \left| a_{i,k_i}(y_i) \right| \int_{\mathbb{R}^n} \left| a_{i*,k_{i*}}(y_{i^*}) \right| |y_{i^*} - c_{i*,k_{i*}}|^{N+1} \left(|x-\xi| + \sum_{j=1, j\neq i^*}^m |x - y_j| \right)^{-mn-N-1} d\vec{y},
\end{aligned}$$

where ξ is between y_{i^*} and $c_{i*,k_{i*}}$.

As argument in [12], when $N = \max_{1\leqslant i\leqslant m}[n((q_i)_\omega/p_i - 1)], [(q_i/p_i - 1)mn]$, we have

$$\|I_2\|_{L^p_\omega} \leqslant C\prod_{i=1}^m\Big(\sum_{k_i}|\lambda_{i,k_i}|^{p_i}\Big)^{\frac{1}{p_i}}. \tag{2.6}$$

In conclusion, summing the estimates (2.5) for I_1 and (2.6) for I_2, we can take limit and obtain

$$\big\|T_*(\vec{f})(x)\big\|_{L^p_\omega} \leqslant C\prod_{i=1}^m \|f_i\|_{H^{p_i}_{\omega_i}}.$$

We complete the proof of Theorem 1.1 for case (i).

Now we turn to prove case (ii). Procedure is similar as in proof of case (i), we only show the differences.

To prove $I_1(x)$, we must introduce another lemma about weighted norm inequality with multiple weights for maximal operators.

Lemma 2.2. *Let T be an m-linear Calderón–Zygmund operator, satisfying* (1.1), (1.2), $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$, *and $\vec{\omega}$ satisfy the $A_{\vec{p}}$ condition, and $1 < p_i < \infty$. Then*

$$\big\|T_*(\vec{f})(x)\big\|_{L^p_{\nu_{\vec{\omega}}}} \leqslant C\prod_{i=1}^m \|f_i\|_{L^{p_i}_{\omega_i}}. \tag{2.7}$$

We will postpone its proof until the last section.

We take a cube $G_{k_1,\ldots,k_m}$ such that for each i, $\omega_i(G_{k_1,\ldots,k_m}) \geqslant C\omega_i(Q_{i^*,k_{i^*}})$. Thus by Lemma 2.2, we have

$$\begin{aligned}
&\frac{1}{\nu_{\vec{\omega}}(G_{k_1,\ldots,k_m})}\int_{G_{k_1,\ldots,k_m}} \big|T_*(a_{1,k_1},\ldots,a_{m,k_m})(x)\big|\nu_{\vec{\omega}}(x)\,dx \\
&\quad\leqslant C\nu_{\vec{\omega}}(G_{k_1,\ldots,k_m})^{-1/q}\prod_{i=1}^m \|a_{i,k_i}\|_{L^{q_i}_{\omega_i}} \\
&\quad\leqslant C\prod_{i=1}^m \omega(Q_{i^*,k_{i^*}})^{-\frac{1}{p_i}}.
\end{aligned}$$

By Lemma 2.1, then

$$\begin{aligned}
\|I_1\|_{L^p_{\nu_{\vec{\omega}}}} &\leqslant C\bigg\|\sum_{k_1}\cdots\sum_{k_m}|\lambda_{1,k_1}|\cdots|\lambda_{m,k_m}|\prod_{i=1}^m \omega_i(Q_{i^*,k_{i^*}})^{-\frac{1}{p_i}}\chi_{\tilde{G}_{k_1,\ldots,k_m}}\bigg\|_{L^p_{\nu_{\vec{\omega}}}} \\
&\leqslant C\prod_{i=1}^m\bigg\|\Big(\sum_{k_i}|\lambda_{i,k_i}|\omega_i(Q_{i^*,k_{i^*}})^{-\frac{1}{p_i}}\omega_i(x)^{\frac{1}{p_i}}\chi_{\tilde{G}_{k_1,\ldots,k_m}}\Big)\bigg\|_{L^{p_i}} \\
&\leqslant C\prod_{i=1}^m\Big(\sum_{k_i}|\lambda_{i,k_i}|^{p_i}\Big)^{\frac{1}{p_i}}.
\end{aligned}$$

3. Proof of Lemma 2.2

In order to prove Lemma 2.2, we must give an improved multiple Coltlar's inequality associated to Theorem A. For integrity, we give its proof with some modifications in [8] as follows.

$$T_*(\vec{f})(x) \leqslant C_\eta\bigg(\big(M\big(\big|T(\vec{f})\big|^\eta\big)(x)\big)^{(1/\eta)} + (A+W)\prod_{i=1}^m \mathcal{M}f_i(x)\bigg). \tag{3.1}$$

Now, we prove (3.1) firstly. We will denote by $S_\delta(x)$ the cube $\{\vec{y}\colon \sup_{1\leqslant j\leqslant m}|x-y_j| \leqslant \delta\}$, and denote $U_\delta = \{\vec{y} \in S_{2\delta}\}$.

It is clear that it is enough to prove for η arbitrary small, so we only discuss for $0 < \eta < 1/m$. Fix x in $\mathbb{R}^n$. Then we have

$$\sup_{\delta>0}\bigg|\int_{U_\delta} K_\delta(x,y_1,\ldots,y_m)f_1(y_1)\cdots f_m(y_m)\,d\vec{y}\bigg| \leqslant CA\mathcal{M}(\vec{f})(x).$$

So it suffices to show (3.1) with $T_*(\vec{f})(x)$ replaced by

$$\bar{T}_*(\vec{f})(x) = \sup_{\delta>0}\left|\bar{T}_\delta(f_1,\dots,f_m)(x)\right|, \tag{3.2}$$

where

$$\bar{T}_\delta(f_1,\dots,f_m)(x) = \int_{\vec{y}\notin S_\delta(x)} K_\delta(x,y_1,\dots,y_m)f_1(y_1)\cdots f_m(y_m)\,d\vec{y}. \tag{3.3}$$

Fix $\delta>0$ and let $B(x,\delta)$ be the ball of center x and radius δ. Since T is an m-linear Calderón–Zygmund operator, $T(\vec{f})$ is in $L^p_{\vec{\omega}}$ and hence it is finite almost everywhere. For $z\in B(x,\delta/2\sqrt{m})$ and $\vec{y}\notin S_\delta(x)$, we get

$$\left(|z-y_1|^2+\cdots+|z-y_m|^2\right)^{1/2} \geqslant \left(|x-y_1|^2+\cdots+|x-y_m|^2\right)^{1/2} - \sqrt{m}|x-z| > 2\delta-\delta=\delta.$$

Hence,

$$\bar{T}_\delta(\vec{f})(z) = T(\vec{f})(z) - T(\vec{f}_0)(z), \tag{3.4}$$

where $\vec{f}_0 = (f_1\chi_{B(0,2\delta)},\dots,f_m\chi_{B(0,2\delta)})$.

By (1.2), we obtain

$$\left|\bar{T}_\delta(\vec{f})(x) - \bar{T}_\delta(\vec{f})(z)\right| \leqslant \int_{\vec{y}\notin S_\delta(x)} \frac{A|x-z|^\varepsilon\prod_{j=1}^m|f_j(y_j)|}{(|x-y_1|+\cdots+|x-y_m|)^{mn+\varepsilon}}\,d\vec{y} = I.$$

Now, the right side of the above inequality can be rewritten as a sum of integral for some $\{j_1,\dots,j_m\}\subsetneqq\{1,\dots,m\}$ so that for $\vec{y}$ we have $|x-y_j|\leqslant\delta$ if and only if $j\in\{j_1,\dots,j_l\}$. We denote the set for $\vec{y}$ satisfying $|x-y_j|\leqslant\delta$ by Q. Then $l<m$ and it follows that

$$\begin{aligned}
I &\leqslant \int_{\vec{y}\notin S_\delta(x)} \frac{A|x-z|^\varepsilon\prod_{j=1}^m|f_j(y_j)|}{(|x-y_1|+\cdots+|x-y_m|)^{mn+\varepsilon}}\,d\vec{y}\\
&\leqslant \prod_{j\in\{j_1,\dots,j_m\}}\int_Q |f_j|\,dy_j \int_{(\mathbb{R}^n\setminus Q)^{m-l}} \frac{|x-z|^\varepsilon\prod_{j\notin\{j_1,\dots,j_l\}}|f_j|\,dy_j}{(|x-y_1|+\cdots+|x-y_m|)^{mn+\varepsilon}}\\
&\leqslant \prod_{j\in\{j_1,\dots,j_m\}}\int_Q |f_j|\,dy_j \sum_{k=0}^\infty \frac{|Q|^{\varepsilon/n}}{(2^k|Q|^{1/n})^{nm+\varepsilon}} \int_{(2^{k+1}Q)^{m-l}} \prod_{j\notin\{j_1,\dots,j_l\}} |f_j|\,dy_j\\
&\leqslant CA\sum_{k=0}^\infty \frac{|Q|^{\varepsilon/n}}{(2^k|Q|^{1/n})^{nm+\varepsilon}} \int_{(2^{k+1}Q)^m}\prod_{j=1}|f_j|\,dy_j\\
&\leqslant CA\mathcal{M}(\vec{f})(x).
\end{aligned}$$

In [8], we can still modify the last inequality easily. Other part is the same as in [8].

Now, we begin to prove Lemma 2.2.

If $\vec{\omega}\in A_{\vec{p}}$, then by Theorem E, we have $\nu_{\vec{\omega}}\in A_{mp}$. If we choose $\eta<\frac{1}{m}$, $mp<\frac{p}{\eta}$, then $\nu_{\vec{\omega}}\in A_{\frac{p}{\eta}}$. Employing Theorem F and Theorem G, we get

$$\begin{aligned}
\left\|T_*(\vec{f})\right\|_{L^p_{\nu_{\vec{\omega}}}} &\leqslant C_\eta\left\|\left(M\left(\left|T(\vec{f})\right|^\eta\right)(x)\right)^{(1/\eta)} + (A+W)\mathcal{M}(\vec{f})(x)\right\|_{L^p_{\nu_{\vec{\omega}}}}\\
&\leqslant C_\eta\left\|\left(M\left(\left|T(\vec{f})\right|^\eta\right)(x)\right)\right\|^{(1/\eta)}_{L^p_{\nu_{\vec{\omega}}}} + C_\eta(A+W)\left\|\mathcal{M}(\vec{f})(x)\right\|_{L^p_{\nu_{\vec{\omega}}}}\\
&\leqslant C_\eta\left\|\left(\left|T(\vec{f})\right|^\eta\right)(x)\right\|^{(1/\eta)}_{L^p_{\nu_{\vec{\omega}}}} + C_\eta(A+W)\prod_{i=1}^m\left\|f_i(x)\right\|_{L^{p_i}_{\omega_i}}\\
&\leqslant C_\eta\left\|T(\vec{f})\right\|_{L^p_{\nu_{\vec{\omega}}}} + C_\eta(A+W)\prod_{i=1}^m\left\|f_i(x)\right\|_{L^{p_i}_{\omega_i}}\\
&\leqslant C_\eta(A+W)\prod_{i=1}^m\left\|f_i(x)\right\|_{L^{p_i}_{\omega_i}}.
\end{aligned}$$

Therefore, we complete the proof of Lemma 2.2.

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References

[1] X. Chen, Q. Xue, Weighted estimates for a class of multilinear fractional type operators, J. Math. Anal. Appl. 362 (2) (2010) 355–373.
[2] X.T. Duong, R. Gong, L. Grafakos, J. Li, L. Yan, Maximal operator for multilinear singular integrals with non-smooth kernels, Indiana Univ. Math. J. 58 (2009) 2517–2542.
[3] J. García-Cuerva, Weighted Hp spaces, Dissertationes Math. 162 (1979) 1–63.
[4] J. García-Cuerva, J.L. Rubio de Francia, Weighted Norm Inequalities and Related Topics, North-Holland Math. Stud., vol. 116, North-Holland, Amsterdam, 1985.
[5] L. Grafakos, On multilinear fractional integrals, Studia Math. 102 (1) (1992) 49–56.
[6] L. Grafakos, N. Kalton, Multilinear Calderón–Zygmund operators on Hardy spaces, Collect. Math. 52 (2) (2001) 169–179.
[7] L. Grafakos, R.H. Torres, Multilinear Calderón–Zygmund theory, Adv. Math. 165 (1) (2002) 124–164.
[8] L. Grafakos, R.H. Torres, Maximal operator and weighted norm inequalities for multilinear singular integrals, Indiana Univ. Math. J. 51 (5) (2002) 1261–1276.
[9] L. Grafakos, R.H. Torres, On multilinear singular integrals of Calderón–Zygmund type, in: Proceedings of the 6th International Conference on Harmonic Analysis and Partial Differential Equations, El Escorial, Publ. Mat. Extra Vol. (2002) 57–91.
[10] C.E. Kenig, E.M. Stein, Multilinear estimates and fractional integration, Math. Res. Lett. 6 (1999) 1–15.
[11] A.K. Lerner, S. Ombrosi, C. Pérez, R.H. Torres, R. Trujillo-González, New maximal functions and multiple weights for the multilinear Calderón–Zygmund theory, Adv. Math. 220 (4) (2009) 1222–1264.
[12] W. Li, Q. Xue, K. Yabuta, Multilinear Calderón–Zygmund operators on weighted Hardy spaces, Studia Math. 199 (2010) 1–16.
[13] K. Moen, Weighted inequalities for multilinear fractional integral operators, Collect. Math. 60 (2009) 213–238.
[14] J.O. Strömberg, A. Torchinsky, Weighted Hardy Spaces, Lecture Notes in Math., vol. 1381, 1989.