Applied Mathematics and Computation 217 (2010) 3144–3154

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Products of composition and differentiation operators from Zygmund spaces to Bloch spaces and Bers spaces

Songxiao Li [a], Stevo Stević [b,*]

[a] Department of Mathematics, JiaYing University, 514015, Meizhou, GuangDong, China
[b] Mathematical Institute of the Serbian Academy of Sciences, Knez Mihailova 36/III, 11000 Beograd, Serbia

ARTICLE INFO

Keywords:
Differentiation operator
Composition operator
Bloch space
Zygmund space
Bers space

ABSTRACT

The boundedness and compactness of the products of differentiation and composition operators from Zygmund spaces to Bloch spaces and Bers spaces are discussed in this paper.

1. Introduction

Let $\mathbb{D}$ be the open unit disk in the complex plane $\mathbb{C}$ and let $H(\mathbb{D})$ be the space of analytic functions on $\mathbb{D}$. An analytic function f on $\mathbb{D}$ is said to belong to the Bers space, denoted by H_1^∞, if

$$\|f\|_{H_1^\infty} = \sup_{z\in\mathbb{D}} \left(1-|z|^2\right)|f(z)| < \infty.$$

H_1^∞ is a Banach space with the norm $\|\cdot\|_{H_1^\infty}$. Let $H_{1,0}^\infty$ denote the subspace of H_1^∞ consisting of those $f \in H_1^\infty$ for which

$$\lim_{|z|\to 1} \left(1-|z|^2\right)|f(z)| = 0.$$

This space is called the little Bers space.

An $f \in H(\mathbb{D})$ is said to belong to the Bloch space $\mathcal{B}$ if

$$B(f) = \sup_{z\in\mathbb{D}} (1-|z|^2)|f'(z)| < \infty.$$

The expression $B(f)$ is a seminorm. Under the natural norm given by $\|f\|_{\mathcal{B}} = |f(0)| + B(f)$, $\mathcal{B}$ becomes a Banach space. Let $\mathcal{B}_0$ denote the subspace of $\mathcal{B}$ consisting of those $f \in \mathcal{B}$ such that

$$\lim_{|z|\to 1} \left(1-|z|^2\right)|f'(z)| = 0.$$

This space is called the little Bloch space.

Let $\mathcal{Z}$ denote the space of all $f \in H(\mathbb{D}) \bigcap C(\overline{\mathbb{D}})$ such that

$$b_{\mathcal{Z}}(f) = \sup_{h>0,\theta\in[0,2\pi]} \frac{\left|f(e^{i(\theta+h)}) + f(e^{i(\theta-h)}) - 2f(e^{i\theta})\right|}{h} < \infty. \quad (1)$$

* Corresponding author.
E-mail addresses: jyulsx@163.com, lsx@mail.zjxu.edu.cn (S. Li), sstevic@ptt.rs (S. Stević).

0096-3003/$ - see front matter © 2010 Elsevier Inc. All rights reserved.
doi:10.1016/j.amc.2010.08.047

From a well-known theorem of Zygmund and the Closed Graph Theorem we see that $f \in \mathcal{Z}$ if and only if

$$b_{\mathcal{Z}}(f) \asymp \sup_{z\in\mathbb{D}} \left(1-|z|^2\right)|f''(z)| < \infty. \tag{2}$$

Therefore, $\mathcal{Z}$ is called the Zygmund class. Under the natural norm

$$\|f\|_{\mathcal{Z}} = |f(0)| + |f'(0)| + \sup_{z\in\mathbb{D}} \left(1-|z|^2\right)|f''(z)|, \tag{3}$$

$\mathcal{Z}$ becomes a Banach space. Zygmund class with this norm will be called the Zygmund space. By (2) it is easy to see that

$$|f'(z) - f'(0)| \leqslant C\|f\|_{\mathcal{Z}} \ln\frac{1}{1-|z|^2}, \quad f \in \mathcal{Z}. \tag{4}$$

The little Zygmund space $\mathcal{Z}_0$ was introduced by the authors of this paper in [6] in the following natural way:

$$f \in \mathcal{Z}_0 \quad \Longleftrightarrow \quad \lim_{|z|\to 1} \left(1-|z|^2\right)|f''(z)| = 0. \tag{5}$$

It is easy to see that $\mathcal{Z}_0$ is a closed subspace of $\mathcal{Z}$.

Let φ denote an analytic self-map of $\mathbb{D}$. Associated with φ is the composition operator C_φ defined by $C_\varphi f = f\circ\varphi$ for $f \in H(\mathbb{D})$. It is interesting to provide a function theoretic characterization for φ inducing a bounded or compact composition operator on various spaces (see, e.g. [2,9,10,16,20,23,29,30]). For example, it is well known that C_φ is bounded on the classical Hardy, Bloch and Bergman spaces.

Let D be the differentiation operator. The differentiation operator is typically unbounded on many analytic function spaces. The products of composition operator and differentiation operator $D\,C_\varphi$ and $C_\varphi D$ are defined respectively as follows

$$DC_\varphi(f) = (f\circ\varphi)' = f'(\varphi)\varphi', \quad f \in H(\mathbb{D})$$

and

$$C_\varphi D(f) = f'\circ\varphi, \quad f \in H(\mathbb{D}).$$

Operators $D\,C_\varphi$ and $C_\varphi D$ as well as some other products of linear operators were studied, for example, in [3–8,11,14,15,17–19,21,22,24–28,31–33] (see also the references therein).

In this paper, we study the operators DC_φ and $C_\varphi D$ from Zygmund spaces to Bloch spaces and Bers spaces. Necessary and sufficient conditions for the boundedness and compactness of these operators are given.

Throughout this paper, constants are denoted by C, they are positive and may not be the same at each occurrence.

2. The boundedness and compactness of $DC_\varphi : \mathcal{Z}(\mathcal{Z}_0) \to \mathcal{B}(\mathcal{B}_0)$

In this section, we characterize the boundedness and compactness of the operator $DC_\varphi : \mathcal{Z}(\mathcal{Z}_0) \to \mathcal{B}(\mathcal{B}_0)$.

Theorem 1. *Let φ be an analytic self-map of $\mathbb{D}$. Then the following statements are equivalent.*

(a) $DC_\varphi : \mathcal{Z} \to \mathcal{B}$ *is bounded;*
(b) $DC_\varphi : \mathcal{Z}_0 \to \mathcal{B}$ *is bounded;*
(c)

$$\sup_{z\in\mathbb{D}} \frac{\left(1-|z|^2\right)|\varphi'(z)|^2}{1-|\varphi(z)|^2} < \infty \tag{6}$$

and

$$\sup_{z\in\mathbb{D}} \left(1-|z|^2\right)|\varphi''(z)| \ln\frac{e}{1-|\varphi(z)|^2} < \infty. \tag{7}$$

Proof. $(a) \Rightarrow (b)$ This implication is obvious.

$(b) \Rightarrow (c)$ Assume that $DC_\varphi : \mathcal{Z}_0 \to \mathcal{B}$ is bounded, i.e., there exists a constant C such that

$$\|DC_\varphi f\|_{\mathcal{B}} \leqslant C\|f\|_{\mathcal{Z}}$$

for all $f \in \mathcal{Z}_0$. Taking the functions $f(z) = z \in \mathcal{Z}_0$ and $f(z) = z^2 \in \mathcal{Z}_0$, we get

$$\sup_{z\in\mathbb{D}}(1-|z|^2)|\varphi''(z)| < \infty \quad \text{and} \quad \sup_{z\in\mathbb{D}} \left(1-|z|^2\right)|(\varphi'(z))^2 + \varphi''(z)\varphi(z)| < \infty. \tag{8}$$

Using these facts and the boundedness of function φ, we have that

$$\sup_{z\in\mathbb{D}}\left(1-|z|^2\right)|\varphi'(z)|^2<\infty. \tag{9}$$

Set

$$h(z)=(z-1)\left[\left(1+\ln\frac{1}{1-z}\right)^2+1\right]$$

and

$$h_a(z)=\frac{h(\bar{a}z)}{\bar{a}}\left(\ln\frac{1}{1-|a|^2}\right)^{-1} \tag{10}$$

for $a\in\mathbb{D}\setminus\{0\}$. It is known that $h_a\in\mathcal{Z}_0$ (see [6]). Since

$$h_a'(z)=\left(\ln\frac{1}{1-\bar{a}z}\right)^2\left(\ln\frac{1}{1-|a|^2}\right)^{-1} \tag{11}$$

and

$$h_a''(z)=\frac{2\bar{a}}{1-\bar{a}z}\left(\ln\frac{1}{1-\bar{a}z}\right)\left(\ln\frac{1}{1-|a|^2}\right)^{-1}, \tag{12}$$

for $|\varphi(\lambda)|>1/2$ we have

$$C\|DC_\varphi\|_{\mathcal{Z}_0\to\mathcal{B}}\geqslant\|DC_\varphi h_{\varphi(\lambda)}\|_{\mathcal{B}}\geqslant(1-|\lambda|^2)|\varphi''(\lambda)|\ln\frac{1}{1-|\varphi(\lambda)|^2}-2\frac{(1-|\lambda|^2)|\varphi'(\lambda)|^2|\varphi(\lambda)|}{1-|\varphi(\lambda)|^2}.$$

Hence

$$\left(1-|\lambda|^2\right)|\varphi''(\lambda)|\ln\frac{1}{1-|\varphi(\lambda)|^2}\leqslant C\|DC_\varphi\|_{\mathcal{Z}_0\to\mathcal{B}}+2\frac{\left(1-|\lambda|^2\right)|\varphi'(\lambda)|^2|\varphi(\lambda)|}{1-|\varphi(\lambda)|^2}. \tag{13}$$

For $a\in\mathbb{D}\setminus\{0\}$, set

$$f_a(z)=\frac{h(\bar{a}z)}{\bar{a}}\left(\ln\frac{1}{1-|a|^2}\right)^{-1}-\int_0^z\ln\frac{1}{1-\bar{a}w}\,dw. \tag{14}$$

Then $f_a\in\mathcal{Z}_0$. It is easy to see that

$$f_a'(z)=\left(\ln\frac{1}{1-\bar{a}z}\right)^2\left(\ln\frac{1}{1-|a|^2}\right)^{-1}-\ln\frac{1}{1-\bar{a}z},\quad f_a'(a)=0,$$

and

$$f_a''(z)=\frac{2\bar{a}}{1-\bar{a}z}\left(\ln\frac{1}{1-\bar{a}z}\right)\left(\ln\frac{1}{1-|a|^2}\right)^{-1}-\frac{\bar{a}}{1-\bar{a}z},\quad f_a''(a)=\frac{\bar{a}}{1-|a|^2}.$$

Therefore

$$C\|DC_\varphi\|_{\mathcal{Z}_0\to\mathcal{B}}\geqslant\|DC_\varphi f_{\varphi(\lambda)}\|_{\mathcal{B}}\geqslant\frac{\left(1-|\lambda|^2\right)|\varphi'(\lambda)|^2|\varphi(\lambda)|}{1-|\varphi(\lambda)|^2}. \tag{15}$$

From (13) and (15) we have that

$$\sup_{|\varphi(\lambda)|>1/2}\left(1-|\lambda|^2\right)|\varphi''(\lambda)|\ln\frac{1}{1-|\varphi(\lambda)|^2}<\infty. \tag{16}$$

On the other hand, from the first inequality in (8) we have that

$$\sup_{|\varphi(\lambda)|\leqslant 1/2}\left(1-|\lambda|^2\right)|\varphi''(\lambda)|\ln\frac{1}{1-|\varphi(\lambda)|^2}\leqslant\sup_{\lambda\in\mathbb{D}}\left(1-|\lambda|^2\right)|\varphi''(\lambda)|\ln\frac{4}{3}<\infty. \tag{17}$$

Hence, from (8), (16) and (17), we obtain (7). Further, from (15), we have

$$\sup_{|\varphi(\lambda)|>\frac{1}{2}} \frac{\left(1-|\lambda|^2\right)|\varphi'(\lambda)|^2}{1-|\varphi(\lambda)|^2} \leqslant \sup_{|\varphi(\lambda)|>\frac{1}{2}} 2\frac{\left(1-|\lambda|^2\right)|\varphi'(\lambda)|^2|\varphi(\lambda)|}{1-|\varphi(\lambda)|^2} \leqslant \sup_{|\varphi(\lambda)|>\frac{1}{2}} C\|DC_\varphi\|_{\mathcal{Z}_0\to\mathcal{B}} < \infty. \tag{18}$$

On the other hand, by (9), we have that

$$\sup_{|\varphi(\lambda)|\leqslant\frac{1}{2}} \frac{\left(1-|\lambda|^2\right)|\varphi'(\lambda)|^2}{1-|\varphi(\lambda)|^2} \leqslant \sup_{|\varphi(\lambda)|\leqslant\frac{1}{2}} \frac{4}{3}\left(1-|\lambda|^2\right)|\varphi'(\lambda)|^2 < \infty. \tag{19}$$

Combining (18) and (19), (6) follows.

$(c) \Rightarrow (a)$. Assume that (6) and (7) hold. Then, for every $f \in \mathcal{Z}$, we have

$$\begin{aligned}\left(1-|z|^2\right)|(DC_\varphi f)'(z)| &\leqslant \left(1-|z|^2\right)|(f'(\varphi)\varphi')'(z)| \leqslant \left(1-|z|^2\right)|\varphi'(z)|^2|f''(\varphi(z))| + \left(1-|z|^2\right)|\varphi''(z)||f'(\varphi(z))| \\ &\leqslant \frac{\left(1-|z|^2\right)|\varphi'(z)|^2}{1-|\varphi(z)|^2}\|f\|_{\mathcal{Z}} + C\left(1-|z|^2\right)|\varphi''(z)|\left(\ln\frac{e}{1-|\varphi(z)|^2}\right)\|f\|_{\mathcal{Z}}.\end{aligned}$$

From this, (6) and (7) it follows that the operator $DC_\varphi : \mathcal{Z} \to \mathcal{B}$ is bounded. □

For studying the compactness of the operator $DC_\varphi : \mathcal{Z} \to \mathcal{B}$, we need the following lemma. The proof of the lemma is standard (see, e.g., Proposition 3.11 in [2] or Lemma 3 in [13]). Hence, we omit the details.

Lemma 1. *Let φ be an analytic self-map of $\mathbb{D}$. Let $T = D\,C_\varphi$ or $C_\varphi D$. Then $T : \mathcal{Z}(\text{or } \mathcal{Z}_0) \to \mathcal{B}(\text{or } H_1^\infty)$ is compact if and only if $T : \mathcal{Z}(\text{or } \mathcal{Z}_0) \to \mathcal{B}(\text{or } H_1^\infty)$ is bounded and for any bounded sequence $(f_k)_{k\in\mathbb{N}}$ in $\mathcal{Z}(\text{or } \mathcal{Z}_0)$ which converges to zero uniformly on compact subsets of $\mathbb{D}$, $Tf_k \to 0$ in $\mathcal{B}(\text{or } H_1^\infty)$ as $k \to \infty$.*

Theorem 2. *Let φ be an analytic self-map of $\mathbb{D}$. Then the following statements are equivalent.*

(a) $DC_\varphi : \mathcal{Z} \to \mathcal{B}$ is compact;
(b) $DC_\varphi : \mathcal{Z}_0 \to \mathcal{B}$ is compact;
(c) $DC_\varphi : \mathcal{Z} \to \mathcal{B}$ is bounded,

$$\lim_{|\varphi(z)|\to 1} \frac{\left(1-|z|^2\right)|\varphi'(z)|^2}{1-|\varphi(z)|^2} = 0 \tag{20}$$

and

$$\lim_{|\varphi(z)|\to 1} \left(1-|z|^2\right)|\varphi''(z)|\ln\frac{e}{1-|\varphi(z)|^2} = 0. \tag{21}$$

Proof. $(a) \Rightarrow (b)$ This implication is clear.

$(b) \Rightarrow (c)$ Assume that $DC_\varphi : \mathcal{Z}_0 \to \mathcal{B}$ is compact. Then it is clear that $DC_\varphi : \mathcal{Z}_0 \to \mathcal{B}$ is bounded. By Theorem 1 we know that $DC_\varphi : \mathcal{Z} \to \mathcal{B}$ is bounded. Let $(z_k)_{k\in\mathbb{N}}$ be a sequence in $\mathbb{D}$ such that $|\varphi(z_k)| \to 1$ as $k \to \infty$ and $\varphi(z_k) \neq 0, k \in \mathbb{N}$ (if such a sequence does not exist then (20) and (21) are vacuously satisfied). Set

$$h_k(z) = \frac{h\left(\overline{\varphi(z_k)}z\right)}{\overline{\varphi(z_k)}}\left(\ln\frac{1}{1-|\varphi(z_k)|^2}\right)^{-1}, \quad k \in \mathbb{N}. \tag{22}$$

Then from the proof of Theorem 1 we see that $h_k \in \mathcal{Z}_0$ for each $k \in \mathbb{N}$. Moreover $h_k \to 0$ uniformly on compact subsets of $\mathbb{D}$ as $k \to \infty$ and

$$h_k'(\varphi(z_k)) = \ln\frac{1}{1-|\varphi(z_k)|^2}, \quad h_k''(\varphi(z_k)) = \frac{2\overline{\varphi(z_k)}}{1-|\varphi(z_k)|^2}.$$

Since $DC_\varphi : \mathcal{Z}_0 \to \mathcal{B}$ is compact, by Lemma 1

$$\lim_{k\to\infty} \|DC_\varphi h_k\|_{\mathcal{B}} = 0.$$

On the other hand, similar to the proof of Theorem 1, we have that

$$\|DC_\varphi h_k\|_{\mathcal{B}} \geqslant \left|\frac{2\left(1-|z_k|^2\right)|\varphi'(z_k)|^2|\varphi(z_k)|}{1-|\varphi(z_k)|^2} - \left(1-|z_k|^2\right)|\varphi''(z_k)|\ln\frac{1}{1-|\varphi(z_k)|^2}\right|,$$

which implies that

$$\lim_{k\to\infty}\frac{2\left(1-|z_k|^2\right)|\varphi'(z_k)|^2|\varphi(z_k)|}{1-|\varphi(z_k)|^2}=\lim_{k\to\infty}\left(1-|z_k|^2\right)|\varphi''(z_k)|\ln\frac{1}{1-|\varphi(z_k)|^2},\tag{23}$$

if one of these two limits exists.

Next, set

$$f_k(z)=\frac{h\left(\overline{\varphi(z_k)}z\right)}{\overline{\varphi(z_k)}}\left(\ln\frac{1}{1-|\varphi(z_k)|^2}\right)^{-1}-\int_0^z\ln^3\frac{1}{1-\overline{\varphi(z_k)}w}\,dw\left(\ln\frac{1}{1-|\varphi(z_k)|^2}\right)^{-2}.\tag{24}$$

Then $f_k\in\mathcal{Z}_0$ and f_k converges to 0 uniformly on compact subsets of $\mathbb{D}$ as $k\to\infty$ ([6]). Since

$$f_k'(z)=\left(\ln\frac{1}{1-\overline{\varphi(z_k)}z}\right)^2\left(\ln\frac{1}{1-|\varphi(z_k)|^2}\right)^{-1}-\left(\ln\frac{1}{1-\overline{\varphi(z_k)}z}\right)^3\left(\ln\frac{1}{1-|\varphi(z_k)|^2}\right)^{-2},$$

we have $f_k'(\varphi(z_k))=0$, for every $k\in\mathbb{N}$ and

$$f_k''(\varphi(z_k))=-\frac{\overline{\varphi(z_k)}}{1-|\varphi(z_k)|^2}.$$

Using these facts, since $DC_\varphi:\mathcal{Z}\to\mathcal{B}$ is compact, and by Lemma 1, it follows that

$$0\leqslant\lim_{k\to\infty}\frac{\left(1-|z_k|^2\right)|\varphi'(z_k)|^2|\varphi(z_k)|}{1-|\varphi(z_k)|^2}\leqslant\lim_{k\to\infty}\|DC_\varphi f_k\|_{\mathcal{B}}=0.$$

Therefore

$$\lim_{k\to\infty}\frac{\left(1-|z_k|^2\right)|\varphi'(z_k)|^2}{1-|\varphi(z_k)|^2}=\lim_{k\to\infty}\frac{\left(1-|z_k|^2\right)|\varphi'(z_k)|^2|\varphi(z_k)|}{1-|\varphi(z_k)|^2}=0,$$

which implies (20). From this and (23), we have that

$$\lim_{k\to\infty}\left(1-|z_k|^2\right)|\varphi''(z_k)|\ln\frac{1}{1-|\varphi(z_k)|^2}=0.\tag{25}$$

From (25) easily follows that $\lim_{k\to\infty}(1-|z_k|^2)|\varphi''(z_k)|=0$, which altogether imply (21).

$(c)\Rightarrow(a)$ Suppose that $DC_\varphi:\mathcal{Z}\to\mathcal{B}$ is bounded and that conditions (20) and (21) hold. From Theorem 1 we known that

$$C_1=\sup_{z\in\mathbb{D}}\left(1-|z|^2\right)|\varphi''(z)|<\infty,\quad C_2=\sup_{z\in\mathbb{D}}\left(1-|z|^2\right)|\varphi'(z)|^2<\infty.\tag{26}$$

By the assumption, for every $\varepsilon>0$, there is a $\delta\in(0,1)$, such that

$$\frac{\left(1-|z|^2\right)|\varphi'(z)|^2}{1-|\varphi(z)|^2}<\varepsilon\quad\text{and}\quad\left(1-|z|^2\right)|\varphi''(z)|\ln\frac{e}{1-|\varphi(z)|^2}<\varepsilon,\tag{27}$$

whenever $\delta<|\varphi(z)|<1$.

Assume that $(f_k)_{k\in\mathbb{N}}$ is a sequence in $\mathcal{Z}$ such that $\sup_{k\in\mathbb{N}}\|f_k\|_{\mathcal{Z}}\leqslant L$ and f_k converges to 0 uniformly on compact subsets of $\mathbb{D}$ as $k\to\infty$. Let $K=\{z\in\mathbb{D}:|\varphi(z)|\leqslant\delta\}$. Then by (26) and (27), we have that

$$\begin{aligned}\sup_{z\in\mathbb{D}}\left(1-|z|^2\right)|(DC_\varphi f_k)'(z)|&=\sup_{z\in\mathbb{D}}\left(1-|z|^2\right)\left|(\varphi' f_k'(\varphi))'(z)\right|\\&\leqslant\sup_{z\in\mathbb{D}}\left(1-|z|^2\right)|\varphi'(z)|^2|f_k''(\varphi(z))|+\sup_{z\in\mathbb{D}}\left(1-|z|^2\right)|\varphi''(z)||f_k'(\varphi(z))|\\&\leqslant\sup_{z\in K}(1-|z|^2)|\varphi'(z)|^2\left|f_k''(\varphi(z))\right|+\sup_{z\in K}(1-|z|^2)|\varphi''(z)|\left|f_k'(\varphi(z))\right|\\&\quad+\sup_{z\in\mathbb{D}\setminus K}(1-|z|^2)|\varphi'(z)|^2\left|f_k''(\varphi(z))\right|+\sup_{z\in\mathbb{D}\setminus K}(1-|z|^2)|\varphi''(z)||f_k'(\varphi(z))|\\&\leqslant\sup_{z\in K}\left(1-|z|^2\right)|\varphi'(z)|^2\left|f_k''(\varphi(z))\right|+\sup_{z\in K}(1-|z|^2)|\varphi''(z)||f_k'(\varphi(z))|\\&\quad+\sup_{z\in\mathbb{D}\setminus K}\frac{\left(1-|z|^2\right)|\varphi'(z)|^2}{1-|\varphi(z)|^2}\|f_k\|_{\mathcal{Z}}+C\sup_{z\in\mathbb{D}\setminus K}\left(1-|z|^2\right)|\varphi''(z)|\left(\ln\frac{e}{1-|\varphi(z)|^2}\right)\|f_k\|_{\mathcal{Z}}\\&\leqslant C_2\sup_{|w|\leqslant\delta}\left|f_k''(w)\right|+C_1\sup_{|w|\leqslant\delta}|f_k'(w)|+(C+1)\varepsilon\|f_k\|_{\mathcal{Z}},\end{aligned}$$

i.e. we obtain

$$\|DC_\varphi f_k\|_{\mathcal{B}} \leqslant C_2 \sup_{|w|\leqslant\delta} |f_k''(w)| + C_1 \sup_{|w|\leqslant\delta} |f_k'(w)| + C\varepsilon\|f_k\|_{\mathcal{Z}} + |f_k'(\varphi(0))||\varphi'(0)|. \quad (28)$$

Since f_k converges to 0 uniformly on compact subsets of $\mathbb{D}$ as $k \to \infty$, Cauchy's estimate gives that $f_k' \to 0$ and $f_k'' \to 0$ as $k \to \infty$ on compact subsets of $\mathbb{D}$. Hence, letting $k \to \infty$ in (28), and using the fact that ε is an arbitrary positive number, we obtain

$$\lim_{k\to\infty} \|DC_\varphi f_k\|_{\mathcal{B}} = 0.$$

Combining this with Lemma 1 the result easily follows. □

Theorem 3. *Let φ be an analytic self-map of $\mathbb{D}$. Then, $DC_\varphi : \mathcal{Z}_0 \to \mathcal{B}_0$ is bounded if and only if $DC_\varphi : \mathcal{Z}_0 \to \mathcal{B}$ is bounded and*

$$\lim_{|z|\to 1} \left(1-|z|^2\right)|\varphi''(z)| = 0 \quad \text{and} \quad \lim_{|z|\to 1} \left(1-|z|^2\right)|\varphi'(z)|^2 = 0. \quad (29)$$

Proof. Assume that $DC_\varphi : \mathcal{Z}_0 \to \mathcal{B}_0$ is bounded. Then, it is clear that $DC_\varphi : \mathcal{Z}_0 \to \mathcal{B}$ is bounded. Taking the functions $f(z) = z$ and $f(z) = z^2$, we obtain (29).

Conversely, assume that $DC_\varphi : \mathcal{Z}_0 \to \mathcal{B}$ is bounded and (29) holds. Then for each polynomial p, we have that

$$\left(1-|z|^2\right)|(DC_\varphi p)'(z)| \leqslant \left(1-|z|^2\right)|\varphi'(z)|^2|p''(\varphi(z))| + \left(1-|z|^2\right)|\varphi''(z)p'(\varphi(z))|. \quad (30)$$

In view of the facts that

$$\sup_{w\in\mathbb{D}} |p''(w)| < \infty, \quad \sup_{w\in\mathbb{D}} |p'(w)| < \infty,$$

from (29) and (30), it follows that $DC_\varphi p \in \mathcal{B}_0$. Since the set of all polynomials is dense in $\mathcal{Z}_0$, we have that for every $f \in \mathcal{Z}_0$ there is a sequence of polynomials $(p_n)_{n\in\mathbb{N}}$ such that $\|f - p_n\|_{\mathcal{Z}} \to 0$ as $n \to \infty$. Hence

$$\|DC_\varphi f - DC_\varphi p_n\|_{\mathcal{B}} \leqslant \|DC_\varphi\|_{\mathcal{Z}_0\to\mathcal{B}}\|f - p_n\|_{\mathcal{Z}} \to 0$$

as $n \to \infty$, since the operator $DC_\varphi : \mathcal{Z}_0 \to \mathcal{B}$ is bounded. Hence $DC_\varphi(\mathcal{Z}_0) \subseteq \mathcal{B}_0$, which implies the boundedness of $DC_\varphi : \mathcal{Z}_0 \to \mathcal{B}_0$. □

Next we characterize the compactness of $DC_\varphi : \mathcal{Z} \to \mathcal{B}_0$. For this purpose, we need the following lemma (see [10]).

Lemma 2. *A closed set K in $\mathcal{B}_0$ is compact if and only if it is bounded and satisfies*

$$\lim_{|z|\to 1} \sup_{f\in K} \left(1-|z|^2\right)|f'(z)| = 0.$$

Theorem 4. *Let φ be an analytic self-map of $\mathbb{D}$. Then the following statements are equivalent.*

(a) $DC_\varphi : \mathcal{Z} \to \mathcal{B}_0$ is compact;
(b) $DC_\varphi : \mathcal{Z}_0 \to \mathcal{B}_0$ is compact;
(c)

$$\lim_{|z|\to 1} \frac{\left(1-|z|^2\right)|\varphi'(z)|^2}{1-|\varphi(z)|^2} = 0 \quad (31)$$

and

$$\lim_{|z|\to 1} \left(1-|z|^2\right)|\varphi''(z)| \ln\frac{e}{1-|\varphi(z)|^2} = 0. \quad (32)$$

Proof. $(a) \Rightarrow (b)$ This implication is trivial.

$(b) \Rightarrow (c)$ Assume that $DC_\varphi : \mathcal{Z}_0 \to \mathcal{B}_0$ is compact. Then $DC_\varphi : \mathcal{Z}_0 \to \mathcal{B}_0$ is bounded. From the proof of Theorem 3 we know that

$$\lim_{|z|\to 1} \left(1-|z|^2\right)|\varphi''(z)| = 0 \quad (33)$$

and

$$\lim_{|z|\to 1} \left(1-|z|^2\right)|\varphi'(z)|^2 = 0. \quad (34)$$

Hence, if $\|\varphi\|_\infty < 1$, from (33) and (34), we obtain that

$$\lim_{|z|\to 1} \frac{\left(1-|z|^2\right)|\varphi'(z)|^2}{1-|\varphi(z)|^2} \leqslant \frac{1}{1-\|\varphi\|_\infty^2} \lim_{|z|\to 1}\left(1-|z|^2\right)|\varphi'(z)|^2 = 0$$

and

$$\lim_{|z|\to 1}\left(1-|z|^2\right)|\varphi''(z)|\ln\frac{e}{1-|\varphi(z)|^2} \leqslant \ln\frac{e}{1-\|\varphi\|_\infty^2}\lim_{|z|\to 1}\left(1-|z|^2\right)|\varphi''(z)| = 0,$$

from which the result follows in this case.

Now assume that $\|\varphi\|_\infty = 1$. Let $(\varphi(z_k))_{k\in\mathbb{N}}$ be a sequence such that $|\varphi(z_k)| \to 1$ as $k\to\infty$. Since $DC_\varphi : \mathcal{Z}_0 \to \mathcal{B}$ is compact, by Theorem 2,

$$\lim_{|\varphi(z)|\to 1} \frac{\left(1-|z|^2\right)|\varphi'(z)|^2}{1-|\varphi(z)|^2} = 0 \tag{35}$$

and

$$\lim_{|\varphi(z)|\to 1}\left(1-|z|^2\right)|\varphi''(z)|\ln\frac{e}{1-|\varphi(z)|^2} = 0. \tag{36}$$

It is not difficult to see that (34), together with (35) implies (31). Similar, (33) and (36) imply (32).

$(c) \Rightarrow (a)$ Let $f \in \mathcal{Z}$. We have

$$\left(1-|z|^2\right)|(DC_\varphi f)'(z)| \leqslant C\left(\frac{\left(1-|z|^2\right)|\varphi'(z)|^2}{1-|\varphi(z)|^2} + \left(1-|z|^2\right)|\varphi''(z)|\ln\frac{e}{1-|\varphi(z)|^2}\right)\|f\|_{\mathcal{Z}}.$$

Taking the supremum in this inequality over all $f\in\mathcal{Z}$ such that $\|f\|_{\mathcal{Z}} \leqslant 1$, then letting $|z|\to 1$, and using (31) and (32) we obtain that

$$\lim_{|z|\to 1}\sup_{\|f\|_{\mathcal{Z}}\leqslant 1}\left(1-|z|^2\right)|(DC_\varphi f)'(z)| = 0.$$

Using Lemma 2 we obtain that the operator $DC_\varphi : \mathcal{Z} \to \mathcal{B}_0$ is compact. □

3. The boundedness and compactness of $C_\varphi D : Z(Z_0) \to B(B_0)$

In this section, we characterize the boundedness and compactness of the operator $C_\varphi D : \mathcal{Z}(\mathcal{Z}_0) \to \mathcal{B}(\mathcal{B}_0)$. Since most of the proofs are similar to those in the previous section, we will only point out main differences.

Theorem 5. *Let φ be an analytic self-map of $\mathbb{D}$. Then $C_\varphi D : \mathcal{Z}(\text{or}\,\mathcal{Z}_0) \to \mathcal{B}$ is bounded.*

Proof. Similar to the proof of Theorem 1 (by using the test functions in (14), we obtain that $C_\varphi D : \mathcal{Z}(\text{or}\,\mathcal{Z}_0) \to \mathcal{B}$ is bounded if and only if

$$\sup_{z\in\mathbb{D}} \frac{\left(1-|z|^2\right)|\varphi'(z)|}{1-|\varphi(z)|^2} < \infty.$$

By the Schwarz–Pick Lemma, we see that the above inequality automatically holds. Hence $C_\varphi D : \mathcal{Z} \to \mathcal{B}$ is automatically bounded. □

The proofs of the following three theorems are similar to the proofs of Theorems 2–4, hence they are omitted.

Theorem 6. *Let φ be an analytic self-map of $\mathbb{D}$. Then the following statements are equivalent.*

(a) $C_\varphi D : \mathcal{Z} \to \mathcal{B}$ is compact;
(b) $C_\varphi D : \mathcal{Z}_0 \to \mathcal{B}$ is compact;
(c)

$$\lim_{|\varphi(z)|\to 1} \frac{\left(1-|z|^2\right)|\varphi'(z)|}{1-|\varphi(z)|^2} = 0.$$

Theorem 7. *Let φ be an analytic self-map of $\mathbb{D}$. Then $C_\varphi D : \mathcal{Z}_0 \to \mathcal{B}_0$ is bounded if and only if $\varphi \in \mathcal{B}_0$.*

Theorem 8. *Let φ be an analytic self-map of $\mathbb{D}$. Then the following statements are equivalent.*

(a) $C_\varphi D : \mathcal{Z} \to \mathcal{B}_0$ *is compact;*
(b) $C_\varphi D : \mathcal{Z}_0 \to \mathcal{B}_0$ *is compact;*
(c)

$$\lim_{|z|\to 1} \frac{\left(1-|z|^2\right)|\varphi'(z)|}{1-|\varphi(z)|^2} = 0.$$

Theorem 9. *Let φ be an analytic self-map of $\mathbb{D}$. Then $C_\varphi D : \mathcal{Z} \to \mathcal{B}_0$ is bounded if and only if*

$$\lim_{|z|\to 1} \frac{\left(1-|z|^2\right)|\varphi'(z)|}{1-|\varphi(z)|^2} = 0. \tag{37}$$

Proof. Assume that (37) holds. Then for any $f \in \mathcal{Z}$,

$$(1-|z|^2)|(C_\varphi Df)'(z)| = \left(1-|z|^2\right)|f''(\varphi(z))||\varphi'(z)| \leqslant \frac{\left(1-|z|^2\right)|\varphi'(z)|}{1-|\varphi(z)|^2}\|f\|_{\mathcal{Z}} \to 0$$

as $|z| \to 1$, which implies that $C_\varphi D(\mathcal{Z}) \subseteq \mathcal{B}_0$. This together with Theorem 5 implies the boundedness of $C_\varphi D : \mathcal{Z} \to \mathcal{B}_0$.

Conversely, assume that $C_\varphi D : \mathcal{Z} \to \mathcal{B}_0$ is bounded. From [12], there exist two holomorphic maps $h_1, h_2 \in \mathcal{B}$ such that

$$|h_1'(z)| + |h_2'(z)| \geqslant \frac{1}{1-|z|},$$

for all $z \in \mathbb{D}$.

Let $f_i(z) = \int_0^z h_i(\xi)d\xi,\ i = 1, 2$. Then from the definition of Bloch functions and Zygmund functions, we see that $f_1, f_2 \in \mathcal{Z}$. The last inequality can be written as

$$\left|f_1''(z)\right| + \left|f_2''(z)\right| \geqslant \frac{1}{1-|z|}, \quad z \in \mathbb{D}.$$

Hence

$$\frac{\left(1-|z|^2\right)|\varphi'(z)|}{1-|\varphi(z)|} \leqslant \left(1-|z|^2\right)|(C_\varphi Df_1)'(z)| + (1-|z|^2)|(C_\varphi Df_2)'(z)|.$$

Since $C_\varphi D : \mathcal{Z} \to \mathcal{B}_0$ is bounded, then $C_\varphi Df_1, C_\varphi Df_2 \in \mathcal{B}_0$ and hence the right-hand side of the above inequality tends to zero as $|z| \to 1$. The desired result follows. □

4. The boundedness and compactness of $DC_\varphi : Z(Z_0) \to H_1^\infty(H_{1,0}^\infty)$

In this section, we characterize the boundedness and compactness of the operator $DC_\varphi : \mathcal{Z}(\mathcal{Z}_0) \to H_1^\infty(H_{1,0}^\infty)$.

The proofs of the next four theorems are correspondingly similar to the proofs of Theorems 1–4, hence they will be omitted.

Theorem 10. *Let φ be an analytic self-map of $\mathbb{D}$. Then the following statements are equivalent.*

(a) $DC_\varphi : \mathcal{Z} \to H_1^\infty$ *is bounded;*
(b) $DC_\varphi : \mathcal{Z}_0 \to H_1^\infty$ *is bounded;*
(c)

$$\sup_{z\in\mathbb{D}} \left(1-|z|^2\right)|\varphi'(z)| \ln\frac{e}{1-|\varphi(z)|} < \infty. \tag{38}$$

Theorem 11. *Let φ be an analytic self-map of $\mathbb{D}$. Then the following statements are equivalent.*

(a) $DC_\varphi : \mathcal{Z} \to H_1^\infty$ *is compact;*
(b) $DC_\varphi : \mathcal{Z}_0 \to H_1^\infty$ *is compact;*
(c) $DC_\varphi : \mathcal{Z} \to H_1^\infty$ *is bounded and*

$$\lim_{|\varphi(z)|\to 1}\left(1-|z|^2\right)|\varphi'(z)|\ln\frac{e}{1-|\varphi(z)|}=0.$$

Theorem 12. *Let φ be an analytic self-map of $\mathbb{D}$. Then $DC_\varphi : \mathcal{Z}_0 \to H_{1,0}^\infty$ is bounded if and only if $\varphi \in \mathcal{B}_0$ and condition (38) holds.*

Theorem 13. *Let φ be an analytic self-map of $\mathbb{D}$. Then the following statements are equivalent.*

(a) $DC_\varphi : \mathcal{Z} \to H_{1,0}^\infty$ *is compact;*
(b) $DC_\varphi : \mathcal{Z}_0 \to H_{1,0}^\infty$ *is compact;*
(c)

$$\lim_{|z|\to 1}\left(1-|z|^2\right)|\varphi'(z)|\ln\frac{e}{1-|\varphi(z)|}=0.$$

Theorem 14. *Let φ be an analytic self-map of $\mathbb{D}$. Then $DC_\varphi : \mathcal{Z} \to H_{1,0}^\infty$ is bounded if and only if*

$$\lim_{|z|\to 1}\left(1-|z|^2\right)|\varphi'(z)|\ln\frac{e}{1-|\varphi(z)|}=0. \tag{39}$$

Proof. By Theorem 10, (39) implies that $DC_\varphi : \mathcal{Z} \to H_1^\infty$ is bounded. For each $f \in \mathcal{Z}$, we have that

$$\left(1-|z|^2\right)|(DC_\varphi f)(z)| = \left(1-|z|^2\right)|f'(\varphi(z))||\varphi'(z)| \leqslant C\left(1-|z|^2\right)|\varphi'(z)|\left(\ln\frac{e}{1-|\varphi(z)|}\right)\|f\|_{\mathcal{Z}},$$

from which we see that (39) implies $DC_\varphi(\mathcal{Z}) \subseteq H_{1,0}^\infty$. This together with the boundedness of $DC_\varphi : \mathcal{Z} \to H_1^\infty$ yields the boundedness of $DC_\varphi : \mathcal{Z} \to H_{1,0}^\infty$.

Conversely, suppose that $DC_\varphi : \mathcal{Z} \to H_{1,0}^\infty$ is bounded. Then for $f(z) = z$ we obtain that

$$\lim_{|z|\to 1}\left(1-|z|^2\right)|\varphi'(z)| = 0. \tag{40}$$

If we prove that

$$\lim_{|\varphi(z)|\to 1}\left(1-|z|^2\right)|\varphi'(z)|\ln\frac{e}{1-|\varphi(z)|}=0, \tag{41}$$

then the proof is finished, since (41), together with (40) implies (39).

Now we need to prove that (41) holds. To show this, we argue by contradiction.

If it were not true, then it would exist $\varepsilon_0 > 0$ and a sequence $(z_n)_{n\in\mathbb{N}} \in \mathbb{D}$, such that $\lim_{n\to\infty}|\varphi(z_n)| = 1$ and

$$\left(1-|z_n|^2\right)|\varphi'(z_n)|\ln\frac{e}{1-|\varphi(z_n)|^2} \geqslant \varepsilon_0 > 0$$

for sufficiently large n. We may also assume that

$$\frac{1-|\varphi(z_{n-1})|}{2} > 1-|\varphi(z_n)|, \quad n \in \mathbb{N}.$$

Then, for every non-negative integer t there is at most one z_n such that $1-\frac{1}{2^t} \leqslant |\varphi(z_n)| < 1-\frac{1}{2^{t+1}}$. Hence, there is a $q_0 \in \mathbb{N}$ such that for any box

$$Q = \{re^{i\theta} | 0 < 1-r < l(Q), |\theta-\theta_0| < \ l(Q)\}$$

and $t \in \mathbb{N}$, there are at most q_0 elements in

$$\left\{\varphi(z_n) \in Q | 2^{-(t+1)}l(Q) < 1-|\varphi(z_n)| < 2^{-t}l(Q)\right\}.$$

Therefore, $(\varphi(z_n))_{n\in\mathbb{N}}$ is an interpolating sequence for $\mathcal{B}$ in sense of [1].

Hence, there exist $p \in \mathcal{B}$ such that

$$p(\varphi(z_n)) = \ln\frac{1}{1-|\varphi(z_n)|^2}, \quad n \in \mathbb{N}.$$

Let $f(z) = \int_0^z p(\xi)d\xi$. From the definition of Bloch functions and Zygmund functions, we see that $f \in \mathcal{Z}$. We have

$$\left(1-|z_n|^2\right)|(DC_\varphi f)(z_n)| = \left(1-|z_n|^2\right)|\varphi'(z_n)||f'(\varphi(z_n))| = \left(1-|z_n|^2\right)|\varphi'(z_n)||p(\varphi(z_n))|$$
$$= \left(1-|z_n|^2\right)|\varphi'(z_n)|\left(\ln\frac{e}{1-|\varphi(z_n)|^2}-1\right) \geqslant \varepsilon_0 - \left(1-|z_n|^2\right)|\varphi'(z_n)|,$$

for sufficiently large n. Since $\lim_{n\to\infty}|\varphi(z_n)| = 1$ implies that $\lim_{n\to\infty}|z_n| = 1$, from the above inequality and (40) we obtain that $DC_\varphi f \notin H^\infty_{1,0}$, which is a contradiction. The proof of this theorem is completed. □

5. The boundedness and compactness of $C_\varphi D : Z(Z_0) \to H_1^\infty(H_{1,0}^\infty)$

Theorem 15. *Let φ be an analytic self-map of $\mathbb{D}$. Then the following statements are equivalent.*

(a) $C_\varphi D : \mathcal{Z} \to H_1^\infty$ is bounded;
(b) $C_\varphi D : \mathcal{Z}_0 \to H_1^\infty$ is bounded;
(c) $C_\varphi D : \mathcal{Z}_0 \to H_{1,0}^\infty$ is bounded;
(d)

$$\sup_{z\in\mathbb{D}}\left(1-|z|^2\right)\ln\frac{e}{1-|\varphi(z)|} < \infty.$$

Proof. The following equivalences $(a) \iff (b) \iff (d)$ are proved similarly as in Theorem 1.

The implication $(c) \Rightarrow (b)$ is trivial.

To see $(d) \Rightarrow (c)$, first note that (d) implies that $C_\varphi D : \mathcal{Z}_0 \to H_1^\infty$ is bounded. Hence it is enough to prove that $C_\varphi D(\mathcal{Z}_0) \subseteq H_{1,0}^\infty$.

By Lemma 2.3 in [6] we know that for every $f \in \mathcal{Z}_0$,

$$\lim_{|z|\to 1}\frac{|f'(z)|}{\ln\frac{e}{1-|z|}} = 0$$

and consequently

$$\lim_{|\varphi(z)|\to 1}\frac{|f'(\varphi(z))|}{\ln\frac{e}{1-|\varphi(z)|}} = 0.$$

From this and known inequalities $0 < \frac{1-|z|}{1-|\varphi(z)|} \leqslant \frac{1+|\varphi(0)|}{1-|\varphi(0)|}$, $z \in \mathbb{D}$, it follows that

$$\lim_{|\varphi(z)|\to 1}(1-|z|)|f'(\varphi(z))| = \lim_{|\varphi(z)|\to 1}\frac{1-|z|}{1-|\varphi(z)|}\frac{\ln\frac{e}{1-|\varphi(z)|}}{(1-|\varphi(z)|)^{-1}}\frac{|f'(\varphi(z))|}{\ln\frac{e}{1-|\varphi(z)|}} = 0,$$

that is, for every $\varepsilon > 0$ there is a $\delta \in (0,1)$ such that

$$(1-|z|)|f'(\varphi(z))| < \varepsilon, \tag{42}$$

whenever $|\varphi(z)| > \delta$.

Let $K = \{z \in \mathbb{D} \,|\, |\varphi(z)| \leqslant \delta\}$. Then we have that $\sup_{z\in K}|f'(\varphi(z))| \leqslant \sup_{|w|\leqslant\delta}|f'(w)| < \infty$. Hence

$$\lim_{z\in K, |z|\to 1}(1-|z|)|f'(\varphi(z))| = 0. \tag{43}$$

From (42) and (43), it follows that for each $f \in \mathcal{Z}_0, C_\varphi D(f) \in H_{1,0}^\infty$, as desired. □

The proof of the following theorem is similar to the proof of Theorem 2 and is omitted.

Theorem 16. *Let φ be an analytic self-map of $\mathbb{D}$. Then the following statements are equivalent.*

(a) $C_\varphi D : \mathcal{Z} \to H_1^\infty$ is compact;
(b) $C_\varphi D : \mathcal{Z}_0 \to H_1^\infty$ is compact;
(c) $C_\varphi D : \mathcal{Z} \to H_1^\infty$ is bounded and

$$\lim_{|\varphi(z)|\to 1}\left(1-|z|^2\right)\ln\frac{e}{1-|\varphi(z)|} = 0.$$

Theorem 17. *Let φ be an analytic self-map of $\mathbb{D}$. Then the following statements are equivalent.*

(a) $C_\varphi D : \mathcal{Z} \to H^\infty_{1,0}$ *is compact;*
(b) $C_\varphi D : \mathcal{Z}_0 \to H^\infty_{1,0}$ *is compact;*
(c) $C_\varphi D : \mathcal{Z} \to H^\infty_{1,0}$ *is bounded;*
(d)

$$\lim_{|z|\to 1}(1-|z|^2)\ln\frac{e}{1-|\varphi(z)|}=0.$$

Proof. The proof of $(a) \iff (b) \iff (d)$ is similar to the proof of Theorem 4. The implication $(a) \Rightarrow (c)$ is obvious, while the proof of $(c) \Rightarrow (d)$ is similar to the proof of the necessity part in Theorem 14, hence is omitted. □

References

[1] B. Boe, A. Nikolau, Interpolation by functions in the Bloch space, J. Anal. Math. 94 (2004) 171–194.
[2] C.C. Cowen, B.D. MacCluer, Composition Operators on Spaces of Analytic Functions, CRC Press, Boca Raton, FL, 1995.
[3] R.A. Hibschweiler, N. Portnoy, Composition followed by differentiation between Bergman and Hardy spaces, Rocky Mt. J. Math. 35 (3) (2005) 843–855.
[4] S. Krantz, S. Stević, On the iterated logarithmic Bloch space on the unit ball, Nonlinear Anal. TMA 71 (2009) 1772–1795.
[5] S. Li, S. Stević, Composition followed by differentiation between Bloch type spaces, J. Comput. Anal. Appl. 9 (2) (2007) 195–206.
[6] S. Li, S. Stević, Volterra type operators on Zygmund spaces, J. Inequal. Appl. vol. 2007, Article ID 32124, 2007, 10 pp.
[7] S. Li, S. Stević, Generalized composition operators on Zygmund spaces and Bloch type spaces, J. Math. Anal. Appl. 338 (2008) 1282–1295.
[8] S. Li, S. Stević, Products of composition and integral type operators from H^∞ to the Bloch space, Complex Var. Elliptic Equ. 53 (5) (2008) 463–474.
[9] Z. Lou, Composition operators on Bloch type spaces, Analysis (Munich) 23 (1) (2003) 81–95.
[10] K. Madigan, A. Matheson, Compact composition operators on the Bloch space, Trans. Amer. Math. Soc. 347 (7) (1995) 2679–2687.
[11] S. Ohno, Products of composition and differentiation between Hardy spaces, Bull. Austral. Math. Soc. 73 (2) (2006) 235–243.
[12] W. Ramey, D. Ullrich, Bounded mean oscillation on Bloch pull-back, Math. Anal. 291 (1991) 591–606.
[13] S. Stević, Composition operators between H^∞ and the α-Bloch spaces on the polydisc, Z. Anal. Anwendungen 25 (4) (2006) 457–466.
[14] S. Stević, Generalized composition operators between mixed norm space and some weighted spaces, Numer. Funct. Anal. Optim. 29 (7–8) (2008) 959–978.
[15] S. Stević, Generalized composition operators from logarithmic Bloch spaces to mixed-norm spaces, Util. Math. 77 (2008) 167–172.
[16] S. Stević, Norm of weighted composition operators from Bloch space to H^∞_μ on the unit ball, Ars. Combin. 88 (2008) 125–127.
[17] S. Stević, Norms of some operators from Bergman spaces to weighted and Bloch-type space, Util. Math. 76 (2008) 59–64.
[18] S. Stević, On a new operator from H^∞ to the Bloch-type space on the unit ball, Util. Math. 77 (2008) 257–263.
[19] S. Stević, On a new operator from the logarithmic Bloch space to the Bloch-type space on the unit ball, Appl. Math. Comput. 206 (2008) 313–320.
[20] S. Stević, Essential norms of weighted composition operators from the Bergman space to weighted-type spaces on the unit ball, Ars Combin. 91 (2009) 391–400.
[21] S. Stević, Integral-type operators from the mixed-norm space to the Bloch-type space on the unit ball, Siberian J. Math. 50 (6) (2009) 1098–1105.
[22] S. Stević, Norm and essential norm of composition followed by differentiation from α-Bloch spaces to H^∞_μ, Appl. Math. Comput. 207 (2009) 225–229.
[23] S. Stević, Norm of weighted composition operators from α-Bloch spaces to weighted-type spaces, Appl. Math. Comput. 215 (2009) 818–820.
[24] S. Stević, On an integral operator from the Zygmund space to the Bloch-type space on the unit ball, Glasg. J. Math. 51 (2009) 275–287.
[25] S. Stević, Products of composition and differentiation operators on the weighted Bergman space, Bull. Belg. Math. Soc. Simon Stevin 16 (2009) 623–635.
[26] S. Stević, Products of integral-type operators and composition operators from the mixed norm space to Bloch-type spaces, Siberian J. Math. 50 (4) (2009) 726–736.
[27] S. Stević, Weighted differentiation composition operators from mixed-norm spaces to weighted-type spaces, Appl. Math. Comput. 211 (2009) 222–233.
[28] S. Stević, S.I. Ueki, Integral-type operators acting between weighted-type spaces on the unit ball, Appl. Math. Comput. 215 (2009) 2464–2471.
[29] S.I. Ueki, Composition operators on the Privalov spaces of the unit ball of $\mathbb{C}^n$, J. Korean Math. Soc. 42 (1) (2005) 111–127.
[30] S.I. Ueki, Weighted composition operators on the Bargmann–Fock space, Int. J. Mod. Math. 3 (3) (2008) 231–243.
[31] W. Yang, On an integral-type operator between Bloch-type spaces, Appl. Math. Comput. 215 (3) (2009) 954–960.
[32] X. Zhu, Generalized weighted composition operators from Bloch-type spaces to weighted Bergman spaces, Indian J. Math. 49 (2) (2007) 139–149.
[33] X. Zhu, Multiplication followed by differentiation on Bloch-type spaces, Bull. Allahbad. Math. Soc. 23 (1) (2008) 25–39.