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Products of Volterra type operator and composition operator from H^∞ and Bloch spaces to Zygmund spaces

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ABSTRACT

The boundedness and compactness of the products of Volterra type operators and composition operators from the space of bounded analytic functions and the Bloch space to the Zygmund space are discussed in this paper.

1. Introduction

Let $\mathbb{D}$ be the open unit disk in the complex plane $\mathbb{C}$ and $H(\mathbb{D})$ be the class of all analytic functions on $\mathbb{D}$. Denote by $H^\infty = H^\infty(\mathbb{D})$ the space of bounded analytic functions on $\mathbb{D}$. An analytic function f on $\mathbb{D}$ is said to belong to the Bloch space $\mathcal{B} = \mathcal{B}(\mathbb{D})$, if

$$B(f) = \sup_{z\in\mathbb{D}}\left(1-|z|^2\right)\left|f'(z)\right| < \infty.$$

The expression $B(f)$ defines a seminorm while the natural norm is given by $\|f\|_{\mathcal{B}} = |f(0)| + B(f)$. It makes $\mathcal{B}$ into a Banach space. Let $\mathcal{B}_0$ denote the subspace of $\mathcal{B}$ consisting of those $f \in \mathcal{B}$ for which $\lim_{|z|\to 1}(1-|z|^2)|f'(z)| = 0$. This space is called the little Bloch space.

Let $\mathcal{Z}$ denote the space of all $f \in H(\mathbb{D}) \cap C(\overline{\mathbb{D}})$ such that

$$\|f\|_{\mathcal{Z}} = \sup_{e^{i\theta}\in\partial\mathbb{D},\, h>0} \frac{|f(e^{i(\theta+h)}) + f(e^{i(\theta-h)}) - 2f(e^{i\theta})|}{h} < \infty. \tag{1}$$

By a theorem of Zygmund (see [6, Theorem 5.3]) and the Closed Graph Theorem we have that $f \in \mathcal{Z}$ if and only if

$$\sup_{z\in\mathbb{D}}\left(1-|z|^2\right)\left|f''(z)\right| < \infty. \tag{2}$$

Moreover the following asymptotic relation holds

$$\|f\|_{\mathcal{Z}} \asymp \sup_{z\in\mathbb{D}}\left(1-|z|^2\right)\left|f''(z)\right|.$$

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The class $\mathcal{Z}$ with the norm $|f(0)| + |f'(0)| + \sup_{z\in\mathbb{D}}(1-|z|^2)|f''(z)|$, which again is denoted by $\|\cdot\|_{\mathcal{Z}}$, we call the Zygmund space.

From (2) it is easy to see that

$$\left|f'(z) - f'(0)\right| \leqslant C\|f\|_{\mathcal{Z}} \ln \frac{1}{1-|z|}$$

and $\|f\|_\infty \leqslant C\|f\|_{\mathcal{Z}}$, for every $f \in \mathcal{Z}$, and for some positive constants C independent of f (see [12]).

The little Zygmund space $\mathcal{Z}_0$ was introduced in [12] in the following natural way

$$f \in \mathcal{Z}_0 \quad \Leftrightarrow \quad \lim_{|z|\to 1}\left(1-|z|\right)\left|f''(z)\right| = 0.$$

It is easy to see that $\mathcal{Z}_0$ is a closed subspace of $\mathcal{Z}$.

An analytic self-map $\varphi : \mathbb{D} \to \mathbb{D}$ induces the composition operator C_φ on $H(\mathbb{D})$, defined by

$$C_\varphi(f) = f\left(\varphi(z)\right)$$

for f analytic on $\mathbb{D}$. The composition operator has been studied by many researchers on various spaces (see, for example, [4] and references therein).

Let $g \in H(\mathbb{D})$. For $f \in H(\mathbb{D})$, the Volterra type operator J_g is defined by the following:

$$J_g f(z) = \int_0^z f(\xi)g'(\xi)\,d\xi, \quad z \in \mathbb{D}.$$

Another integral operator I_g is defined by

$$I_g f(z) = \int_0^z f'(\xi)g(\xi)\,d\xi, \quad z \in \mathbb{D}.$$

If $g(z) = z$ then J_g is the integration operator and if $g(z) = \ln \frac{1}{1-z}$ then J_g is the Cesàro operator. Moreover

$$J_g f + I_g f = M_g f - f(0)g(0),$$

where M_g is the multiplication operator

$$(M_g f)(z) = g(z)f(z), \quad f \in H(\mathbb{D}),\ z \in \mathbb{D}.$$

In [15] Pommerenke introduced the operator J_g and showed that $J_g : H^2 \to H^2$ is bounded if and only if $g \in BMOA$. The operators J_g and I_g, as well as the corresponding operators on the unit ball and the polydisk, acting on various function spaces, have been extensively studied recently (see, for example, [1–3,7–12,16–19] and the related references therein).

Here, we consider the products of composition operators and Volterra type operators, which are defined by

$$(C_\varphi J_g f)(z) = \int_0^{\varphi(z)} f(\zeta)g'(\zeta)\,d\zeta, \qquad (C_\varphi I_g f)(z) = \int_0^{\varphi(z)} f'(\zeta)g(\zeta)\,d\zeta \tag{3}$$

and

$$(J_g C_\varphi f)(z) = \int_0^z (f\circ\varphi)(\zeta)g'(\zeta)\,d\zeta, \qquad (I_g C_\varphi f)(z) = \int_0^z (f\circ\varphi)'(\zeta)g(\zeta)\,d\zeta. \tag{4}$$

More precisely, the boundedness and compactness of these operators from H^∞, $\mathcal{B}$ and $\mathcal{B}_0$ to Zygmund spaces are studied in this paper. As some corollaries we obtain the boundedness and compactness for C_φ, J_g and I_g from H^∞ and Bloch spaces to Zygmund spaces. These results are new even for a single operator.

Throughout this paper, constants are denoted by C, they are positive and may differ from one occurrence to other. The notation $A \asymp B$ means that there is a positive constant C such that $C^{-1}B \leqslant A \leqslant CB$.

2. Auxiliary results

In order to prove the main results of this paper, we need some auxiliary results which are incorporated in the following lemmas.

The first lemma can be proved similar to Lemma 1 in [13].

Lemma 1. *A closed set K in $\mathcal{Z}_0$ is compact if and only if it is bounded and satisfies*

$$\lim_{|z|\to 1}\sup_{f\in K}\big(1-|z|^2\big)\big|f''(z)\big|=0.$$

The next lemma can be proved in a standard way (see, for example, Proposition 3.11 in [4]).

Lemma 2. *Suppose that φ is an analytic self-map of the unit disk and $g\in H(\mathbb{D})$. Then the operator $C_\varphi I_g$ (or $C_\varphi J_g$; $I_g C_\varphi$; $J_g C_\varphi$) $: H^\infty$ (or $\mathcal{B}$) $\to \mathcal{Z}$ is compact if and only if for any bounded sequence $(f_k)_{k\in\mathbb{N}}$ in H^∞ (or $\mathcal{B}$) which converges to zero uniformly on compact subsets of $\mathbb{D}$, $C_\varphi I_g f_k \to 0$ (or $C_\varphi J_g f_k$; $I_g C_\varphi f_k$; $J_g C_\varphi f_k \to 0$) in $\mathcal{Z}$ as $k\to\infty$.*

The following lemma is a consequence of the Cauchy integral formula.

Lemma 3. *Assume that $f\in H^\infty(\mathbb{D})$. Then for each $n\in\mathbb{N}$, there is a positive constant C independent of f such that*

$$\sup_{z\in\mathbb{D}}\big(1-|z|\big)^n\big|f^{(n)}(z)\big|\leqslant C\|f\|_\infty. \tag{5}$$

The following lemma is folklore (see, e.g., [20]).

Lemma 4. *Assume that $f\in\mathcal{B}(\mathbb{D})$. Then*

$$\|f\|_{\mathcal{B}}\asymp\big|f(0)\big|+\big|f'(0)\big|+\sup_{z\in\mathbb{D}}\big(1-|z|\big)^2\big|f''(z)\big|. \tag{6}$$

Lemma 5. *Let $g\in H(\mathbb{D})$ and φ be an analytic self-map of $\mathbb{D}$. Suppose that X is a Banach space and $T=C_\varphi I_g$ (or $C_\varphi J_g$; $I_g C_\varphi$; $J_g C_\varphi$). Then $T:\mathcal{B}_0\to X$ is compact if and only if $T:\mathcal{B}_0\to X$ is weakly compact.*

Proof. From [20], we see that the dual space of $\mathcal{B}_0$ is isomorphic to the Bergman space A^1, i.e. $(\mathcal{B}_0)^*\cong A^1$. Assume that $T:\mathcal{B}_0\to X$ is compact. By a well-known theorem, this is equivalent to $T^*:X^*\to A^1$ is compact (see, e.g., [5, Theorem 2, p. 485]). Recall that every weakly convergent sequence in A^1 is norm-convergent (see [5, Theorem 12, p. 295]). Hence, $T^*:X^*\to A^1$ is weakly compact, which is equivalent with $T:\mathcal{B}_0\to X$ is weakly compact. □

3. Boundedness and compactness of $C_\varphi I_g : H^\infty$ (or $\mathcal{B}$) $\to \mathcal{Z}$ (or $\mathcal{Z}_0$)

In this section we study the boundedness and compactness of the operators $C_\varphi I_g : H^\infty$ (or $\mathcal{B}$) $\to \mathcal{Z}$ (or $\mathcal{Z}_0$).

Theorem 1. *Suppose that φ is an analytic self-map of the unit disk and $g\in H(\mathbb{D})$. Then the following statements are equivalent.*

(i) $C_\varphi I_g : H^\infty\to\mathcal{Z}$ *is bounded*;
(ii) $C_\varphi I_g : \mathcal{B}\to\mathcal{Z}$ *is bounded*;
(iii) $C_\varphi I_g : \mathcal{B}_0\to\mathcal{Z}$ *is bounded*;
(iv)

$$\sup_{z\in\mathbb{D}}\frac{(1-|z|^2)|\varphi'(z)|^2|g(\varphi(z))|}{(1-|\varphi(z)|^2)^2}<\infty \tag{7}$$

and

$$\sup_{z\in\mathbb{D}}\frac{(1-|z|^2)|\varphi''(z)g(\varphi(z))+\varphi'^2(z)g'(\varphi(z))|}{1-|\varphi(z)|^2}<\infty. \tag{8}$$

Proof. (iv) ⇒ (i) and (ii). Suppose that (7) and (8) hold. From (3) we have that

$$(C_\varphi I_g f)'(z)=f'\big(\varphi(z)\big)\varphi'(z)g\big(\varphi(z)\big) \tag{9}$$

and

$$(C_\varphi I_g f)''(z)=f''\big(\varphi(z)\big)\varphi'^2(z)g\big(\varphi(z)\big)+f'\big(\varphi(z)\big)\big[\varphi''(z)g\big(\varphi(z)\big)+\varphi'^2(z)g'\big(\varphi(z)\big)\big]. \tag{10}$$

Let $f \in H^\infty$ (or $\mathcal{B}$), then from (10) and by Lemmas 4 and 3 we have

$$\begin{aligned}(1-|z|^2)\big|(C_\varphi I_g f)''(z)\big| &= (1-|z|^2)\big|f''(\varphi(z))\varphi'^2(z)g(\varphi(z)) + f'(\varphi(z))\big[\varphi''(z)g(\varphi(z)) + \varphi'^2(z)g'(\varphi(z))\big]\big| \\ &\leqslant \frac{C\|f\|_{\mathcal{B}}(1-|z|^2)|\varphi'(z)|^2|g(\varphi(z))|}{(1-|\varphi(z)|^2)^2} + \frac{C\|f\|_{\mathcal{B}}(1-|z|^2)|\varphi''(z)g(\varphi(z)) + \varphi'^2(z)g'(\varphi(z))|}{1-|\varphi(z)|^2} \quad (11)\\ &\leqslant \frac{C\|f\|_\infty(1-|z|^2)|\varphi'(z)|^2|g(\varphi(z))|}{(1-|\varphi(z)|^2)^2} + \frac{C\|f\|_\infty(1-|z|^2)|\varphi''(z)g(\varphi(z)) + \varphi'^2(z)g'(\varphi(z))|}{1-|\varphi(z)|^2}. \quad (12)\end{aligned}$$

On the other hand, we have that

$$\begin{aligned}\big|(C_\varphi I_g f)(0)\big| &= \left|\int_0^{\varphi(0)} f'(\zeta)g(\zeta)\,d\zeta\right| \leqslant \max_{|\zeta|\leqslant|\varphi(0)|}\big|f'(\zeta)\big| \max_{|\zeta|\leqslant|\varphi(0)|}\big|g(\zeta)\big| \\ &\leqslant C \max_{|\zeta|\leqslant(1+|\varphi(0)|)/2}\big|f(\zeta)\big| \max_{|\zeta|\leqslant|\varphi(0)|}\big|g(\zeta)\big| \leqslant C\|f\|_\infty \max_{|\zeta|\leqslant|\varphi(0)|}\big|g(\zeta)\big| \quad (13)\end{aligned}$$

and

$$\big|(C_\varphi I_g f)(0)\big| \leqslant \max_{|\zeta|\leqslant|\varphi(0)|}\big|f'(\zeta)\big| \max_{|\zeta|\leqslant|\varphi(0)|}\big|g(\zeta)\big| \leqslant \frac{\|f\|_{\mathcal{B}}}{1-|\varphi(0)|^2} \max_{|\zeta|\leqslant|\varphi(0)|}\big|g(\zeta)\big|. \quad (14)$$

From (9) and Lemma 3, it follows that

$$\begin{aligned}\big|(C_\varphi I_g f)'(0)\big| &= \big|f'(\varphi(0))\varphi'(0)g(\varphi(0))\big| \\ &\leqslant 2\|f\|_{\mathcal{B}}\big(1-|\varphi(0)|\big)^{-1}\big|\varphi'(0)\big|\big|g(\varphi(0))\big| \quad (15)\\ &\leqslant C\|f\|_\infty\big(1-|\varphi(0)|\big)^{-1}\big|\varphi'(0)\big|\big|g(\varphi(0))\big|. \quad (16)\end{aligned}$$

Note that the quantity $\max_{|\zeta|\leqslant|\varphi(0)|}|g(\zeta)|$ is finite since the set $|\zeta| \leqslant |\varphi(0)|$ is compact in view of the fact $|\varphi(0)| < 1$. From (12), (13) and (16) (or (11), (14) and (15)) and by conditions (7) and (8) it follows that the operator $C_\varphi I_g : H^\infty$ (or $\mathcal{B}$) $\to \mathcal{Z}$ is bounded.

(i) ⇒ (iv). Suppose that $C_\varphi I_g : H^\infty \to \mathcal{Z}$ is bounded, i.e. there exists a constant C such that

$$\|C_\varphi I_g f\|_{\mathcal{Z}} \leqslant C\|f\|_\infty$$

for all $f \in H^\infty$. By taking the function given by $f(z) = z$ we obtain

$$M_1 = \sup_{z\in\mathbb{D}}(1-|z|^2)\big|\varphi''(z)g(\varphi(z)) + \varphi'^2(z)g'(\varphi(z))\big| < \infty. \quad (17)$$

On the other hand, by taking the function $f(z) = z^2$, using (17) and the fact that $\|\varphi\|_\infty \leqslant 1$, it follows that

$$M_2 = \sup_{z\in\mathbb{D}}(1-|z|^2)\big|\varphi'(z)\big|^2\big|g(\varphi(z))\big| < \infty. \quad (18)$$

For $\lambda \in \mathbb{D}$, set

$$h_{\varphi(\lambda)}(z) = 2\frac{1-|\varphi(\lambda)|^2}{1-\overline{\varphi(\lambda)}z} - \frac{(1-|\varphi(\lambda)|^2)^2}{(1-\overline{\varphi(\lambda)}z)^2}. \quad (19)$$

It is easy to see that $\sup_{\lambda\in\mathbb{D}}\|h_{\varphi(\lambda)}\|_\infty \leqslant 8$,

$$h'_{\varphi(\lambda)}(z) = \frac{2(1-|\varphi(\lambda)|^2)\overline{\varphi(\lambda)}}{(1-\overline{\varphi(\lambda)}z)^2} - \frac{2(1-|\varphi(\lambda)|^2)^2\overline{\varphi(\lambda)}}{(1-\overline{\varphi(\lambda)}z)^3}, \qquad h'_{\varphi(\lambda)}(\varphi(\lambda)) = 0$$

and

$$h''_{\varphi(\lambda)}(z) = \frac{4(1-|\varphi(\lambda)|^2)(\overline{\varphi(\lambda)})^2}{(1-\overline{\varphi(\lambda)}z)^3} - \frac{6(1-|\varphi(\lambda)|^2)^2(\overline{\varphi(\lambda)})^2}{(1-\overline{\varphi(\lambda)}z)^4}, \qquad h''_{\varphi(\lambda)}(\varphi(\lambda)) = \frac{-2\overline{\varphi(\lambda)}^2}{(1-|\varphi(\lambda)|^2)^2}.$$

Therefore, we have

$$\begin{aligned}8\|C_\varphi I_g\|_{H^\infty\to\mathcal{Z}} &\geqslant \|C_\varphi I_g h_{\varphi(\lambda)}\|_{\mathcal{Z}} \\ &\geqslant (1-|\lambda|^2)\big|h''_{\varphi(\lambda)}(\varphi(\lambda))\varphi'^2(\lambda)g(\varphi(\lambda)) + h'_{\varphi(\lambda)}(\varphi(\lambda))\big[\varphi''(\lambda)g(\varphi(\lambda)) + \varphi'^2(\lambda)g'(\varphi(\lambda))\big]\big| \\ &= \frac{2(1-|\lambda|^2)|\varphi(\lambda)|^2|\varphi'^2(\lambda)g(\varphi(\lambda))|}{(1-|\varphi(\lambda)|^2)^2} \quad (20)\end{aligned}$$

for $\lambda \in \mathbb{D}$. If $|\varphi(\lambda)| > \frac{1}{2}$, then

$$\sup_{|\varphi(\lambda)|>\frac{1}{2}} \frac{(1-|\lambda|^2)|\varphi'^2(\lambda)g(\varphi(\lambda))|}{(1-|\varphi(\lambda)|^2)^2} \leqslant \sup_{|\varphi(\lambda)|>\frac{1}{2}} \frac{4(1-|\lambda|^2)|\varphi(\lambda)|^2|\varphi'^2(\lambda)g(\varphi(\lambda))|}{(1-|\varphi(\lambda)|^2)^2} < \infty. \tag{21}$$

If $|\varphi(\lambda)| \leqslant \frac{1}{2}$, then by (18) we obtain

$$\sup_{|\varphi(\lambda)|\leqslant\frac{1}{2}} \frac{(1-|\lambda|^2)|\varphi'^2(\lambda)g(\varphi(\lambda))|}{(1-|\varphi(\lambda)|^2)^2} \leqslant \sup_{|\varphi(\lambda)|\leqslant\frac{1}{2}} \frac{16}{9}\big(1-|\lambda|^2\big)\big|\varphi'^2(\lambda)g\big(\varphi(\lambda)\big)\big| < \infty. \tag{22}$$

Combining (21) with (22) we obtain (7).

For $\lambda \in \mathbb{D}$, set

$$f_{\varphi(\lambda)}(z) = \frac{1-|\varphi(\lambda)|^2}{1-\overline{\varphi(\lambda)}z}. \tag{23}$$

It is easy to see that $f_{\varphi(\lambda)} \in H^\infty$ and $\|f_{\varphi(\lambda)}\|_\infty \leqslant 2$. Moreover

$$f'_{\varphi(\lambda)}(z) = \frac{(1-|\varphi(\lambda)|^2)\overline{\varphi(\lambda)}}{(1-\overline{\varphi(\lambda)}z)^2}, \qquad f''_{\varphi(\lambda)}(z) = \frac{2(1-|\varphi(\lambda)|^2)\overline{\varphi(\lambda)}^2}{(1-\overline{\varphi(\lambda)}z)^3}.$$

Therefore, we have

$$\begin{aligned} 2\|C_\varphi I_g\|_{H^\infty\to\mathcal{Z}} &\geqslant \|C_\varphi I_g f_{\varphi(\lambda)}\|_{\mathcal{Z}} \\ &\geqslant \big(1-|\lambda|^2\big)\big|f''_{\varphi(\lambda)}\big(\varphi(\lambda)\big)\varphi'^2(\lambda)g\big(\varphi(\lambda)\big) + f'_{\varphi(\lambda)}\big(\varphi(\lambda)\big)\big[\varphi''(\lambda)g\big(\varphi(\lambda)\big) + \varphi'^2(\lambda)g'\big(\varphi(\lambda)\big)\big]\big| \\ &\geqslant \frac{(1-|\lambda|^2)|\varphi(\lambda)||\varphi''(\lambda)g(\varphi(\lambda)) + \varphi'^2(\lambda)g'(\varphi(\lambda))|}{1-|\varphi(\lambda)|^2} - \big(1-|\lambda|^2\big)\frac{2|\varphi^2(\lambda)\varphi'^2(\lambda)g(\varphi(\lambda))|}{(1-|\varphi(\lambda)|^2)^2} \end{aligned}$$

for $\lambda \in \mathbb{D}$. From the above inequality and (20) we get

$$\sup_{\lambda\in\mathbb{D}} \frac{(1-|\lambda|^2)|\varphi(\lambda)||\varphi''(\lambda)g(\varphi(\lambda)) + \varphi'^2(\lambda)g'(\varphi(\lambda))|}{1-|\varphi(\lambda)|^2} < \infty. \tag{24}$$

From (24) similar to (21) and (22) and by using (17) instead of (18) we obtain (8), finishing the proof of the implication.

(ii) ⇒ (i). The result follows from the fact that $H^\infty \subset \mathcal{B}$ (see Lemma 3 with $n = 1$).

(ii) ⇒ (iii). This is obvious.

(iii) ⇒ (iv). Since the test functions in the proof of (i) ⇒ (iv) also belong to the little Bloch space, the implication follows. □

Theorem 2. *Suppose that φ is an analytic self-map of the unit disk and $g \in H(\mathbb{D})$. Then the following statements are equivalent.*

(i) *$C_\varphi I_g : H^\infty \to \mathcal{Z}$ is compact*;
(ii) *$C_\varphi I_g : \mathcal{B} \to \mathcal{Z}$ is compact*;
(iii) *$C_\varphi I_g : \mathcal{B}_0 \to \mathcal{Z}$ is compact*;
(iv) *$C_\varphi I_g : H^\infty \to \mathcal{Z}$ is bounded,*

$$\lim_{|\varphi(z)|\to 1} \frac{(1-|z|^2)|\varphi'(z)|^2|g(\varphi(z))|}{(1-|\varphi(z)|^2)^2} = 0 \tag{25}$$

and

$$\lim_{|\varphi(z)|\to 1} \frac{(1-|z|^2)|\varphi''(z)g(\varphi(z)) + \varphi'^2(z)g'(\varphi(z))|}{1-|\varphi(z)|^2} = 0; \tag{26}$$

(v) *$C_\varphi I_g : \mathcal{B} \to \mathcal{Z}$ is bounded and conditions* (25) *and* (26) *hold.*

Proof. First note that in view of Theorem 1 we have that conditions (iv) and (v) are equivalent.

(iv) ⇒ (i) and (ii). Suppose that conditions (25) and (26) hold. From them it follows that for every $\varepsilon > 0$, there is $\delta \in (0,1)$, such that $\delta < |\varphi(z)| < 1$ implies

$$\frac{(1-|z|^2)|\varphi'(z)|^2|g(\varphi(z))|}{(1-|\varphi(z)|^2)^2} < \varepsilon \tag{27}$$

and

$$\frac{(1-|z|^2)|\varphi''(z)g(\varphi(z))+\varphi'^2(z)g'(\varphi(z))|}{1-|\varphi(z)|^2}<\varepsilon. \tag{28}$$

Assume that $(f_k)_{k\in\mathbb{N}}$ is a sequence in H^∞ (or $\mathcal{B}$) such that $\sup_{k\in\mathbb{N}}\|f_k\|_\infty\leqslant C$ (or $\sup_{k\in\mathbb{N}}\|f_k\|_{\mathcal{B}}\leqslant C$) and f_k converges to 0 uniformly on compact subsets of $\mathbb{D}$ as $k\to\infty$. Let $K=\{z\in\mathbb{D}\colon\ |\varphi(z)|\leqslant\delta\}$. Then, we have that

$$\begin{aligned}\|C_\varphi I_g f_k\|_{\mathcal{Z}} &\leqslant \sup_{z\in\mathbb{D}}\big|(C_\varphi I_g f_k)''(z)\big|\big(1-|z|^2\big)+\big|(C_\varphi I_g f_k)'(0)\big|+\big|(C_\varphi I_g f_k)(0)\big|\\ &=\sup_{z\in\mathbb{D}}\big(1-|z|^2\big)\big|f_k''\big(\varphi(z)\big)\varphi'^2(z)g\big(\varphi(z)\big)+f_k'\big(\varphi(z)\big)\big[\varphi''(z)g\big(\varphi(z)\big)+\varphi'^2(z)g'\big(\varphi(z)\big)\big]\big|\\ &\quad+\big|\varphi'(0)\big|\big|f_k'\big(\varphi(0)\big)\big|\big|g\big(\varphi(0)\big)\big|+\left|\int_0^{\varphi(0)}f_k'(\zeta)g(\zeta)\,d\zeta\right|=I_1+I_2+I_3.\end{aligned} \tag{29}$$

By the boundedness of $C_\varphi I_g : H^\infty$ (or $\mathcal{B}$) $\to\mathcal{Z}$, we see that (17) and (18) hold. From this, (27), (28), and by using Lemmas 3 and 4, it follows that

$$\begin{aligned}I_1 &\leqslant \sup_{z\in K}\big(1-|z|^2\big)\big|f_k''\big(\varphi(z)\big)\varphi'^2(z)g\big(\varphi(z)\big)\big|+\sup_{z\in\mathbb{D}\setminus K}\big(1-|z|^2\big)\big|f_k''\big(\varphi(z)\big)\varphi'^2(z)g\big(\varphi(z)\big)\big|\\ &\quad+\sup_{z\in K}\big(1-|z|^2\big)\big|f_k'\big(\varphi(z)\big)\big[\varphi''(z)g\big(\varphi(z)\big)+\varphi'^2(z)g'\big(\varphi(z)\big)\big]\big|\\ &\quad+\sup_{z\in\mathbb{D}\setminus K}\big(1-|z|^2\big)\big|f_k'\big(\varphi(z)\big)\big[\varphi''(z)g\big(\varphi(z)\big)+\varphi'^2(z)g'\big(\varphi(z)\big)\big]\big|\\ &\leqslant M_2\sup_{|\zeta|\leqslant\delta}\big|f_k''(\zeta)\big|+M_1\sup_{|\zeta|\leqslant\delta}\big|f_k'(\zeta)\big|+C\varepsilon\|f_k\|_{\mathcal{B}} \qquad (30)\\ &\leqslant M_2\sup_{|\zeta|\leqslant\delta}\big|f_k''(\zeta)\big|+M_1\sup_{|\zeta|\leqslant\delta}\big|f_k'(\zeta)\big|+C\varepsilon\|f_k\|_\infty. \qquad (31)\end{aligned}$$

Further, we have that

$$I_2=\big|\varphi'(0)\big|\big|f_k'\big(\varphi(0)\big)\big|\big|g\big(\varphi(0)\big)\big|\leqslant\big|\varphi'(0)\big|M_g\max_{|\zeta|\leqslant|\varphi(0)|}\big|f_k'(\zeta)\big| \tag{32}$$

and

$$I_3\leqslant\big|\varphi(0)\big|M_g\max_{|\zeta|\leqslant|\varphi(0)|}\big|f_k'(\zeta)\big|, \tag{33}$$

where $M_g=\max_{|\zeta|\leqslant|\varphi(0)|}|g(\zeta)|<\infty$.

Since f_k converges to 0 uniformly on compact subsets of $\mathbb{D}$ as $k\to\infty$, Cauchy's estimate gives that f_k' and f_k'' converges to 0 uniformly on compact subsets of $\mathbb{D}$ as $k\to\infty$, in particular on the compact $|\zeta|\leqslant|\varphi(0)|$. Hence, letting $k\to\infty$ in (31)–(33) (or (30), (32) and (33)), and since ε is an arbitrary positive number, we have that

$$\lim_{k\to\infty}\|C_\varphi I_g f_k\|_{\mathcal{Z}}=0.$$

From this and Lemma 2 the compactness of $C_\varphi I_g : H^\infty$ (or $\mathcal{B}$) $\to\mathcal{Z}$ follows.

(i) $\Rightarrow$ (iv). We assume that $C_\varphi I_g : H^\infty\to\mathcal{Z}$ is compact. Then it is clear that $C_\varphi I_g : H^\infty\to\mathcal{Z}$ is bounded. We prove that (25) and (26) hold. Let $(z_k)_{k\in\mathbb{N}}$ be a sequence in $\mathbb{D}$ such that $|\varphi(z_k)|\to 1$ as $k\to\infty$. Let

$$h_k(z)=2\frac{1-|\varphi(z_k)|^2}{1-\overline{\varphi(z_k)}z}-\frac{(1-|\varphi(z_k)|^2)^2}{(1-\overline{\varphi(z_k)}z)^2},\quad k\in\mathbb{N}. \tag{34}$$

According to the proof of Theorem 1 we know that $\sup_{k\in\mathbb{N}}\|h_k\|_\infty\leqslant 8$. Clearly, h_k converges to 0 uniformly on compact subsets of $\mathbb{D}$ as $k\to\infty$. In view of the compactness of $C_\varphi I_g : H^\infty\to\mathcal{Z}$, we have $\lim_{k\to\infty}\|C_\varphi I_g h_k\|_{\mathcal{Z}}=0$. Therefore, we have

$$\begin{aligned}&\frac{2(1-|z_k|^2)|\varphi(z_k)|^2|\varphi'^2(z_k)g(\varphi(z_k))|}{(1-|\varphi(z_k)|^2)^2}\\ &\quad\leqslant\big(1-|z_k|^2\big)\big|h_k''\big(\varphi(z_k)\big)\varphi'^2(z_k)g\big(\varphi(z_k)\big)+h_k'\big(\varphi(z_k)\big)\big[\varphi''(z_k)g\big(\varphi(z_k)\big)+\varphi'^2(z_k)g'\big(\varphi(z_k)\big)\big]\big|\\ &\quad\leqslant\|C_\varphi I_g h_k\|_{\mathcal{Z}}\to 0,\quad\text{as } k\to\infty.\end{aligned} \tag{35}$$

From the above inequality, and $|\varphi(z_k)|\to 1$ as $k\to\infty$, it follows that

$$\lim_{k\to\infty}\frac{(1-|z_k|^2)|\varphi'(z_k)|^2|g(\varphi(z_k))|}{(1-|\varphi(z_k)|^2)^2}=0, \tag{36}$$

i.e. we obtain (25).

Let

$$f_k(z) = \frac{1-|\varphi(z_k)|^2}{1-\overline{\varphi(z_k)}z}, \quad k \in \mathbb{N}. \tag{37}$$

Then $\sup_{k\in\mathbb{N}} \|f_k\|_\infty \leqslant 2$ and f_k converges to 0 uniformly on compact subsets of $\mathbb{D}$ as $k \to \infty$. By means of the compactness of $C_\varphi I_g : H^\infty \to \mathcal{Z}$, we have $\lim_{k\to\infty} \|C_\varphi I_g f_k\|_{\mathcal{Z}} = 0$. Therefore, we have

$$\begin{aligned}
&\left|\frac{(1-|z_k|^2)|\varphi(z_k)||\varphi''(z_k)g(\varphi(z_k)) + \varphi'^2(z_k)g'(\varphi(z_k))|}{1-|\varphi(z_k)|^2} - \left(1-|z_k|^2\right)\frac{2|\varphi(z_k)|^2|\varphi'^2(z_k)||g(\varphi(z_k))|}{(1-|\varphi(z_k)|^2)^2}\right| \\
&\quad \leqslant \left(1-|z_k|^2\right)\left|f_k''\big(\varphi(z_k)\big)\varphi'^2(z_k)g\big(\varphi(z_k)\big) + f_k'\big(\varphi(z_k)\big)\left[\varphi''(z_k)g\big(\varphi(z_k)\big) + \varphi'^2(z_k)g'\big(\varphi(z_k)\big)\right]\right| \\
&\quad \leqslant \|C_\varphi I_g f_k\|_{\mathcal{Z}} \to 0, \quad \text{as } k \to \infty.
\end{aligned}$$

From this and (36) it follows that

$$\lim_{k\to\infty} \frac{(1-|z_k|^2)|\varphi''(z_k)g(\varphi(z_k)) + \varphi'^2(z_k)g'(\varphi(z_k))|}{1-|\varphi(z_k)|^2} = 0,$$

which means that (26) holds.

(ii) $\Rightarrow$ (i) and (ii) $\Rightarrow$ (iii) are obvious.

(iii) $\Rightarrow$ (iv). Since the test functions h_k and f_k also belong to the little Bloch space, similar to the proof of (i) $\Rightarrow$ (iv) we can get the desired result. The proof of the theorem is completed. □

Theorem 3. *Suppose that φ is an analytic self-map of the unit disk and $g \in H(\mathbb{D})$. Then the following statements are equivalent.*

(i) $C_\varphi I_g : H^\infty \to \mathcal{Z}_0$ *is compact*;
(ii) $C_\varphi I_g : \mathcal{B} \to \mathcal{Z}_0$ *is compact*;
(iii) $C_\varphi I_g : \mathcal{B} \to \mathcal{Z}_0$ *is bounded*;
(iv) $C_\varphi I_g : \mathcal{B}_0 \to \mathcal{Z}_0$ *is compact*;
(v)

$$\lim_{|z|\to 1} \frac{(1-|z|^2)|\varphi'(z)|^2|g(\varphi(z))|}{(1-|\varphi(z)|^2)^2} = 0 \tag{38}$$

and

$$\lim_{|z|\to 1} \frac{(1-|z|^2)|\varphi''(z)g(\varphi(z)) + \varphi'^2(z)g'(\varphi(z))|}{1-|\varphi(z)|^2} = 0. \tag{39}$$

Proof. (ii) $\Rightarrow$ (iii). This is obvious.

(iii) $\Rightarrow$ (iv). Clearly $C_\varphi I_g : \mathcal{B}_0 \to \mathcal{Z}_0$ is bounded. By using standard duality arguments and the fact that $\mathcal{B}_0$ is weak-star dense in $\mathcal{B}$, it follows that $C_\varphi I_g = (C_\varphi I_g)^{**}$ on $\mathcal{B}$. Hence $(C_\varphi I_g)^{**}(\mathcal{B}_0^{**}) = C_\varphi I_g(\mathcal{B}) \subset \mathcal{Z}_0$. By Gantmacher's theorem it follows that $C_\varphi I_g : \mathcal{B}_0 \to \mathcal{Z}_0$ is weakly compact. It follows from Lemma 5 that $C_\varphi I_g : \mathcal{B}_0 \to \mathcal{Z}_0$ is compact.

(v) $\Rightarrow$ (i) and (ii). Taking the supremum in inequality (12) (or (11)) over all $f \in H^\infty$ (or $f \in \mathcal{B}$) such that $\|f\|_\infty \leqslant 1$ (corresponding $\|f\|_{\mathcal{B}} \leqslant 1$), and letting $|z| \to 1$, yields

$$\lim_{|z|\to 1} \sup_{\|f\|_\infty \leqslant 1 \ (\text{or } \|f\|_{\mathcal{B}} \leqslant 1)} \left(1-|z|^2\right)\left|\big(C_\varphi I_g(f)\big)''(z)\right| = 0.$$

Hence, by Lemma 1 we see that the operator $C_\varphi I_g : H^\infty$ (or $\mathcal{B}$) $\to \mathcal{Z}_0$ is compact.

(i) or (iv) $\Rightarrow$ (v). Now suppose that $C_\varphi I_g : H^\infty$ (or $\mathcal{B}_0$) $\to \mathcal{Z}_0$ is compact. Then $C_\varphi I_g : H^\infty$ (or $\mathcal{B}_0$) $\to \mathcal{Z}_0$ is bounded and $C_\varphi I_g : H^\infty$ (or $\mathcal{B}_0$) $\to \mathcal{Z}$ is compact. By the boundedness of $C_\varphi I_g : H^\infty$ (or $\mathcal{B}_0$) $\to \mathcal{Z}_0$ and by taking the function $f(z) = z$ it follows that

$$\lim_{|z|\to 1}\left(1-|z|^2\right)\left|\varphi''(z)g\big(\varphi(z)\big) + \varphi'^2(z)g'\big(\varphi(z)\big)\right| = 0. \tag{40}$$

From (40) and by taking the function $f(z) = z^2$ we obtain that

$$\lim_{|z|\to 1}\left(1-|z|^2\right)\left|\varphi'(z)\right|^2\left|g\big(\varphi(z)\big)\right| = 0. \tag{41}$$

By the proof of Theorem 2 we know that

$$\lim_{|\varphi(z)|\to 1} \frac{(1-|z|^2)|\varphi'(z)|^2|g(\varphi(z))|}{(1-|\varphi(z)|^2)^2} = 0 \tag{42}$$

and

$$\lim_{|\varphi(z)|\to 1} \frac{(1-|z|^2)|\varphi''(z)g(\varphi(z))+\varphi'^2(z)g'(\varphi(z))|}{1-|\varphi(z)|^2}=0. \tag{43}$$

We prove that (41) and (42) imply (38). The proof of the fact that (40) and (43) imply (39) is similar, hence it will be omitted.

From (42) it follows that for every $\varepsilon > 0$, there exists $r \in (0,1)$ such that

$$\frac{(1-|z|^2)|\varphi'(z)|^2|g(\varphi(z))|}{(1-|\varphi(z)|^2)^2} < \varepsilon$$

when $r < |\varphi(z)| < 1$. Employing (41) we see that there exists $\sigma \in (0,1)$ such that

$$\left(1-|z|^2\right)\left|\varphi'(z)\right|^2\left|g\left(\varphi(z)\right)\right| \leqslant \varepsilon\left(1-r^2\right)^2,$$

when $\sigma < |z| < 1$.

Therefore, when $\sigma < |z| < 1$ and $r < |\varphi(z)| < 1$, we have that

$$\frac{(1-|z|^2)|\varphi'(z)|^2|g(\varphi(z))|}{(1-|\varphi(z)|^2)^2} < \varepsilon. \tag{44}$$

On the other hand, if $\sigma < |z| < 1$ and $|\varphi(z)| \leqslant r$, we obtain

$$\frac{(1-|z|^2)|\varphi'(z)|^2|g(\varphi(z))|}{(1-|\varphi(z)|^2)^2} < \frac{1}{(1-r^2)^2}\left(1-|z|^2\right)\left|\varphi'(z)\right|^2\left|g\left(\varphi(z)\right)\right| < \varepsilon. \tag{45}$$

From (44) and (45), we obtain (38), as desired. The proof is completed. □

Theorem 4. *Let $g \in H(\mathbb{D})$ and φ be an analytic self-map of $\mathbb{D}$. Then $C_\varphi I_g : \mathcal{B}_0 \to \mathcal{Z}_0$ is bounded if and only if $C_\varphi I_g : \mathcal{B}_0 \to \mathcal{Z}$ is bounded and*

$$\lim_{|z|\to 1}\left(1-|z|^2\right)\left|\varphi''(z)g\left(\varphi(z)\right)+\varphi'^2(z)g'\left(\varphi(z)\right)\right|=0 \tag{46}$$

and

$$\lim_{|z|\to 1}\left(1-|z|^2\right)\left|\varphi'^2(z)g\left(\varphi(z)\right)\right|=0. \tag{47}$$

Proof. Suppose that $C_\varphi I_g : \mathcal{B}_0 \to \mathcal{Z}_0$ is bounded, then $C_\varphi I_g : \mathcal{B}_0 \to \mathcal{Z}$ is bounded. By taking the functions given by $f(z)=z$ and $f(z)=z^2$, and employing the boundedness of $C_\varphi I_g : \mathcal{B}_0 \to \mathcal{Z}_0$, (46) and (47) follow.

Conversely, assume that $C_\varphi I_g : \mathcal{B}_0 \to \mathcal{Z}$ is bounded and (46), (47) hold. Since for each polynomial p we have

$$\begin{aligned}\left(1-|z|^2\right)\left|(C_\varphi I_g p)''(z)\right| &\leqslant \left(1-|z|^2\right)\left|p''\left(\varphi(z)\right)\varphi'^2(z)g\left(\varphi(z)\right)+p'\left(\varphi(z)\right)\left[\varphi''(z)g\left(\varphi(z)\right)+\varphi'^2(z)g'\left(\varphi(z)\right)\right]\right| \\ &\leqslant \|p'\|_\infty\left(1-|z|^2\right)\left|\varphi''(z)g\left(\varphi(z)\right)+\varphi'^2(z)g'\left(\varphi(z)\right)\right|+\|p''\|_\infty\left(1-|z|^2\right)\left|\varphi'^2(z)g\left(\varphi(z)\right)\right|,\end{aligned}$$

thus from (46) and (47) we see that for each polynomial p, $C_\varphi I_g(p) \in \mathcal{Z}_0$. The set of all polynomials is dense in $\mathcal{B}_0$, thus for every $f \in \mathcal{B}_0$ there is a sequence of polynomials $(p_k)_{k\in\mathbb{N}}$ such that $\|p_k - f\|_{\mathcal{B}} \to 0$ as $k \to \infty$. In view of Theorem 1, the operator $C_\varphi I_g : \mathcal{B} \to \mathcal{Z}$ is bounded, thus it follows that

$$\|C_\varphi I_g p_k - C_\varphi I_g f\|_{\mathcal{Z}} \leqslant \|C_\varphi I_g\|\|p_k - f\|_{\mathcal{B}} \to 0, \quad \text{as } k \to \infty.$$

Hence $C_\varphi I_g(\mathcal{B}_0) \subset \mathcal{Z}_0$, since $\mathcal{Z}_0$ is a closed subset of $\mathcal{Z}$. □

The following theorem concerning the boundedness and compactness of the operators $I_g C_\varphi : H^\infty$ (or $\mathcal{B}$) $\to \mathcal{Z}$ (or $\mathcal{Z}_0$) can be proved similar to Theorems 1–4. Hence, we omit its proof.

Theorem 5. *Suppose that φ is an analytic self-map of the unit disk and $g \in H(\mathbb{D})$. Then*

(a) *$I_g C_\varphi : H^\infty \to \mathcal{Z}$ is bounded if and only if $I_g C_\varphi : \mathcal{B} \to \mathcal{Z}$ is bounded if and only if $I_g C_\varphi : \mathcal{B}_0 \to \mathcal{Z}$ is bounded if and only if*

$$\sup_{z\in\mathbb{D}} \frac{(1-|z|^2)|\varphi'(z)|^2|g(z)|}{(1-|\varphi(z)|^2)^2} < \infty$$

and

$$\sup_{z\in\mathbb{D}} \frac{(1-|z|^2)|\varphi''(z)g(z)+\varphi'(z)g'(z)|}{1-|\varphi(z)|^2} < \infty.$$

(b) $I_gC_\varphi : H^\infty \to \mathcal{Z}$ *is compact if and only if* $I_gC_\varphi : \mathcal{B} \to \mathcal{Z}$ *is compact if and only if* $I_gC_\varphi : \mathcal{B}_0 \to \mathcal{Z}$ *is compact if and only if* $I_gC_\varphi : H^\infty \to \mathcal{Z}$ *is bounded (or* $I_gC_\varphi : \mathcal{B} \to \mathcal{Z}$ *is bounded),*

$$\lim_{|\varphi(z)|\to 1} \frac{(1-|z|^2)|\varphi'(z)|^2|g(z)|}{(1-|\varphi(z)|^2)^2} = 0$$

and

$$\lim_{|\varphi(z)|\to 1} \frac{(1-|z|^2)|\varphi''(z)g(z)+\varphi'(z)g'(z)|}{1-|\varphi(z)|^2} = 0.$$

(c) $I_gC_\varphi : H^\infty \to \mathcal{Z}_0$ *is compact if and only if* $I_gC_\varphi : \mathcal{B} \to \mathcal{Z}_0$ *is bounded if and only if* $I_gC_\varphi : \mathcal{B} \to \mathcal{Z}_0$ *is compact if and only if* $I_gC_\varphi : \mathcal{B}_0 \to \mathcal{Z}_0$ *is compact if and only if*

$$\lim_{|z|\to 1} \frac{(1-|z|^2)|\varphi'(z)|^2|g(z)|}{(1-|\varphi(z)|^2)^2} = 0$$

and

$$\lim_{|z|\to 1} \frac{(1-|z|^2)|\varphi''(z)g(z)+\varphi'(z)g'(z)|}{1-|\varphi(z)|^2} = 0.$$

(d) $I_gC_\varphi : \mathcal{B}_0 \to \mathcal{Z}_0$ *is bounded if and only if* $I_gC_\varphi : \mathcal{B}_0 \to \mathcal{Z}$ *is bounded,*

$$\lim_{|z|\to 1}\big(1-|z|^2\big)\big|\varphi'(z)\big|^2\big|g(z)\big| = 0 \quad \text{and} \quad \lim_{|z|\to 1}\big(1-|z|^2\big)\big|\varphi''(z)g(z)+\varphi'(z)g'(z)\big| = 0.$$

From Theorems 1–4 with $g(z)=1$, we obtain the following results about the characterization of the boundedness and compactness of the composition operator $C_\varphi : H^\infty$ (or $\mathcal{B}$) $\to \mathcal{Z}$ (or $\mathcal{Z}_0$).

Corollary 1. *Suppose that* φ *is an analytic self-map of the unit disk. Then*

(a) $C_\varphi : H^\infty \to \mathcal{Z}$ *is bounded if and only if* $C_\varphi : \mathcal{B} \to \mathcal{Z}$ *is bounded if and only if* $C_\varphi : \mathcal{B}_0 \to \mathcal{Z}$ *is bounded if and only if*

$$\sup_{z\in\mathbb{D}} \frac{(1-|z|^2)|\varphi'(z)|^2}{(1-|\varphi(z)|^2)^2} < \infty \quad \text{and} \quad \sup_{z\in\mathbb{D}} \frac{(1-|z|^2)|\varphi''(z)|}{1-|\varphi(z)|^2} < \infty.$$

(b) $C_\varphi : H^\infty \to \mathcal{Z}$ *is compact if and only* $C_\varphi : \mathcal{B} \to \mathcal{Z}$ *is compact if and only* $C_\varphi : \mathcal{B}_0 \to \mathcal{Z}$ *is compact if and only if* $C_\varphi : H^\infty \to \mathcal{Z}$ *is bounded (or* $C_\varphi : \mathcal{B} \to \mathcal{Z}$ *is bounded),*

$$\lim_{|\varphi(z)|\to 1} \frac{(1-|z|^2)|\varphi'(z)|^2}{(1-|\varphi(z)|^2)^2} = 0 \quad \text{and} \quad \lim_{|\varphi(z)|\to 1} \frac{(1-|z|^2)|\varphi''(z)|}{1-|\varphi(z)|^2} = 0.$$

(c) $C_\varphi : H^\infty \to \mathcal{Z}_0$ *is compact if and only if* $C_\varphi : \mathcal{B} \to \mathcal{Z}_0$ *is bounded if and only if* $C_\varphi : \mathcal{B} \to \mathcal{Z}_0$ *is compact if and only if* $C_\varphi : \mathcal{B}_0 \to \mathcal{Z}_0$ *is compact if and only if*

$$\lim_{|z|\to 1} \frac{(1-|z|^2)|\varphi'(z)|^2}{(1-|\varphi(z)|^2)^2} = 0 \quad \text{and} \quad \lim_{|z|\to 1} \frac{(1-|z|^2)|\varphi''(z)|}{1-|\varphi(z)|^2} = 0.$$

(d) $C_\varphi : \mathcal{B}_0 \to \mathcal{Z}_0$ *is bounded if and only if* $C_\varphi : \mathcal{B}_0 \to \mathcal{Z}$ *is bounded,*

$$\lim_{|z|\to 1}\big(1-|z|^2\big)\big|\varphi'(z)\big|^2 = 0 \quad \text{and} \quad \lim_{|z|\to 1}\big(1-|z|^2\big)\big|\varphi''(z)\big| = 0.$$

Remark 1. From Theorem 1 we see that $I_g : H^\infty \to \mathcal{Z}$ is bounded if and only if $I_g : \mathcal{B} \to \mathcal{Z}$ is bounded if and only if $g=0$.

4. The boundedness and compactness of $C_\varphi J_g : H^\infty$ (or $\mathcal{B}$) $\to \mathcal{Z}$ (or $\mathcal{Z}_0$)

In this section, we characterize the boundedness and compactness of the operator $C_\varphi J_g : H^\infty$ (or $\mathcal{B}$) $\to \mathcal{Z}$ (or $\mathcal{Z}_0$).

Theorem 6. *Suppose that* φ *is an analytic self-map of the unit disk and* $g \in H(\mathbb{D})$*. Then* $C_\varphi J_g : H^\infty \to \mathcal{Z}$ *is bounded if and only if*

$$\sup_{z\in\mathbb{D}} \frac{(1-|z|^2)|g'(\varphi(z))||\varphi'(z)|^2}{(1-|\varphi(z)|^2)} < \infty \tag{48}$$

and

$$\sup_{z\in\mathbb{D}}\big(1-|z|^2\big)\big|g''\big(\varphi(z)\big)\varphi'^2(z)+g'\big(\varphi(z)\big)\varphi''(z)\big| < \infty. \tag{49}$$

Proof. Suppose that (48) and (49) hold. For a function $f \in H^\infty$, we have that

$$\left|(C_\varphi J_g f)(0)\right| = \left|\int_0^{\varphi(0)} f(\zeta) g'(\zeta)\,d\zeta\right| \leqslant \|f\|_\infty \max_{|\zeta| \leqslant |\varphi(0)|} \left|g'(\zeta)\right| \tag{50}$$

and

$$\left|(C_\varphi J_g f)'(0)\right| = \left|f(\varphi(0))\right|\left|g'(\varphi(0))\varphi'(0)\right| \leqslant \|f\|_\infty \left|g'(\varphi(0))\varphi'(0)\right|. \tag{51}$$

On the other hand,

$$\left(1-|z|^2\right)\left|(C_\varphi J_g f)''(z)\right| = \left(1-|z|^2\right)\left|f(\varphi(z))\left[g''(\varphi(z))\varphi'^2(z) + g'(\varphi(z))\varphi''(z)\right] + f'(\varphi(z))g'(\varphi(z))\varphi'^2(z)\right| \tag{52}$$

$$\leqslant \left(\left(1-|z|^2\right)\left|g''(\varphi(z))\varphi'^2(z) + g'(\varphi(z))\varphi''(z)\right| + \frac{C(1-|z|^2)|g'(\varphi(z))||\varphi'(z)|^2}{(1-|\varphi(z)|^2)}\right)\|f\|_\infty. \tag{53}$$

From (50)–(53) and by conditions (48) and (49) it follows that the operator $C_\varphi J_g : H^\infty \to \mathcal{Z}$ is bounded.

Conversely, suppose that $C_\varphi J_g : H^\infty \to \mathcal{Z}$ is bounded, i.e. there exists a constant C such that

$$\|C_\varphi J_g f\|_{\mathcal{Z}} \leqslant C\|f\|_\infty$$

for all $f \in H^\infty$. By taking $f(z) = 1$, it follows that

$$\sup_{z\in\mathbb{D}}\left(1-|z|^2\right)\left|g''(\varphi(z))\varphi'^2(z) + g'(\varphi(z))\varphi''(z)\right| < \infty. \tag{54}$$

From (54), the fact $\|\varphi\|_\infty \leqslant 1$, and by taking the function $f(z) = z$, we obtain

$$\sup_{z\in\mathbb{D}}\left(1-|z|^2\right)\left|g'(\varphi(z))\varphi'^2(z)\right| < \infty. \tag{55}$$

Then by taking the functions defined in (23), using the method in the proof of Theorem 1, (54) and (55), we obtain the desired result. □

The proofs of the following theorem is similar to the proof of Theorem 2, hence will be omitted.

Theorem 7. *Suppose that φ is an analytic self-map of the unit disk and $g \in H(\mathbb{D})$. Then $C_\varphi J_g : H^\infty \to \mathcal{Z}$ is compact if and only if $C_\varphi J_g : H^\infty \to \mathcal{Z}$ is bounded,*

$$\lim_{|\varphi(z)|\to 1} \frac{(1-|z|^2)|g'(\varphi(z))||\varphi'(z)|^2}{(1-|\varphi(z)|^2)} = 0$$

and

$$\lim_{|\varphi(z)|\to 1}\left(1-|z|^2\right)\left|g''(\varphi(z))\varphi'^2(z) + g'(\varphi(z))\varphi''(z)\right| = 0.$$

Theorem 8. *Suppose that φ is an analytic self-map of the unit disk and $g \in H(\mathbb{D})$. Then the following statements are equivalent.*

(i) $C_\varphi J_g : H^\infty \to \mathcal{Z}_0$ *is bounded*;
(ii) $C_\varphi J_g : H^\infty \to \mathcal{Z}_0$ *is compact*;
(iii)

$$\lim_{|z|\to 1}\left(1-|z|^2\right)\left|g''(\varphi(z))\varphi'^2(z) + g'(\varphi(z))\varphi''(z)\right| = 0 \tag{56}$$

and

$$\lim_{|z|\to 1} \frac{(1-|z|^2)|g'(\varphi(z))||\varphi'(z)|^2}{(1-|\varphi(z)|^2)} = 0. \tag{57}$$

Proof. (ii) ⇒ (i) is obvious.

(iii) ⇒ (ii). Taking the supremum in (53) over all $f \in H^\infty$ such that $\|f\|_\infty \leqslant 1$, then letting $|z| \to 1$, using (56) and (57), and applying Lemma 1 the implication follows.

(i) ⇒ (iii). Suppose that $C_\varphi J_g : H^\infty \to \mathcal{Z}_0$ is bounded. Then, taking $f(z) = 1$ and $f(z) = z$ we see that

$$\lim_{|z|\to 1}\left(1-|z|^2\right)\left|g''(\varphi(z))\varphi'^2(z) + g'(\varphi(z))\varphi''(z)\right| = 0 \tag{58}$$

and

$$\lim_{|z|\to 1}\big(1-|z|^2\big)\big|g'\big(\varphi(z)\big)\big|\big|\varphi'(z)\big|^2=0. \tag{59}$$

Hence, we only need to prove that (57) holds. If $\|\varphi\|_\infty<1$, then the last equality implies the result.

Hence, suppose that $\|\varphi\|_\infty=1$. Assume to the contrary that there is a positive number ε_0 and a sequence $(z_k)_{k\in\mathbb{N}}$ such that $|z_k|\to 1$ as $k\to\infty$, $\lim_{k\to\infty}|\varphi(z_k)|=1$, and

$$\frac{(1-|z_k|^2)|\varphi'(z_k)|^2}{1-|\varphi(z_k)|^2}\big|g'\big(\varphi(z_k)\big)\big|\geqslant\varepsilon_0.$$

Without loss of generality we may assume that $(\varphi(z_k))_{k\in\mathbb{N}}$ is an interpolating sequence in $\mathbb{D}$. Let f be an interpolating Blaschke product with zeros $(\varphi(z_k))_{k\in\mathbb{N}}$ (see [14]). Then

$$\big(1-\big|\varphi(z_k)\big|^2\big)\big|f'\big(\varphi(z_k)\big)\big|\geqslant\delta$$

for every $k\in\mathbb{N}$ and for some $\delta>0$.

Since $C_\varphi J_g f\in\mathcal{Z}_0$, we have

$$\begin{aligned}0&=\lim_{|z|\to 1}\big(1-|z|^2\big)\big|(C_\varphi J_g f)''(z)\big|\\&\geqslant\Big|\lim_{|z|\to 1}\big(1-|z|^2\big)\big|f'\big(\varphi(z)\big)g'\big(\varphi(z)\big)\big|\big|\varphi'(z)\big|^2-\lim_{|z|\to 1}\big(1-|z|^2\big)\big|f\big(\varphi(z)\big)\big[g''\big(\varphi(z)\big)\varphi'^2(z)+g'\big(\varphi(z)\big)\varphi''(z)\big]\big|\Big|.\end{aligned} \tag{60}$$

In view of the boundedness of f and from (56), it follows that

$$\lim_{|z|\to 1}\big(1-|z|^2\big)\big|f\big(\varphi(z)\big)\big|\big|g''\big(\varphi(z)\big)\varphi'^2(z)+g'\big(\varphi(z)\big)\varphi''(z)\big|=0. \tag{61}$$

From (60) and (61) we get

$$\lim_{|z|\to 1}\big(1-|z|^2\big)\big|f'\big(\varphi(z)\big)g'\big(\varphi(z)\big)\big|\big|\varphi'(z)\big|^2=0. \tag{62}$$

On the other hand, for every $k\in\mathbb{N}$ we have

$$\begin{aligned}\big(1-|z_k|^2\big)\big|\varphi'(z_k)\big|^2\big|g'\big(\varphi(z_k)\big)\big|\big|f'\big(\varphi(z_k)\big)\big|&=\frac{(1-|z_k|^2)|\varphi'(z_k)|^2}{1-|\varphi(z_k)|^2}\big|g'\big(\varphi(z_k)\big)\big|\big|f'\big(\varphi(z_k)\big)\big|\big(1-\big|\varphi(z_k)\big|^2\big)\\&\geqslant\frac{(1-|z_k|^2)|\varphi'(z_k)|^2}{1-|\varphi(z_k)|^2}\big|g'\big(\varphi(z_k)\big)\big|\delta\geqslant\delta\varepsilon_0>0,\end{aligned}$$

which is a contradiction with (62). □

Corollary 2. *Suppose that $g\in H(\mathbb{D})$. Then*

(a) *$J_g: H^\infty\to\mathcal{Z}$ is bounded if and only if $g\in\mathcal{Z}$ and $\sup_{z\in\mathbb{D}}|g'(z)|<\infty$.*
(b) *$J_g: H^\infty\to\mathcal{Z}$ is compact if and only if $J_g: H^\infty\to\mathcal{Z}_0$ is bounded if and only if $J_g: H^\infty\to\mathcal{Z}_0$ is compact if and only if g is a constant.*

By using the same methods as in the proofs of Theorems 6–8 we can prove the next theorem.

Theorem 9. *Suppose that φ is an analytic self-map of the unit disk and $g\in H(\mathbb{D})$. Then*

(a) *$J_gC_\varphi: H^\infty\to\mathcal{Z}$ is bounded if and only if $g\in\mathcal{Z}$ and*

$$\sup_{z\in\mathbb{D}}\frac{(1-|z|^2)|\varphi'(z)||g'(z)|}{1-|\varphi(z)|^2}<\infty.$$

(b) *$J_gC_\varphi: H^\infty\to\mathcal{Z}$ is compact if and only if $J_gC_\varphi: H^\infty\to\mathcal{Z}$ is bounded,*

$$\lim_{|\varphi(z)|\to 1}\big(1-|z|^2\big)\big|g''(z)\big|=0\quad\text{and}\quad\lim_{|\varphi(z)|\to 1}\frac{(1-|z|^2)|\varphi'(z)||g'(z)|}{1-|\varphi(z)|^2}=0.$$

(c) *$J_gC_\varphi: H^\infty\to\mathcal{Z}_0$ is bounded if and only if $J_gC_\varphi: H^\infty\to\mathcal{Z}_0$ is compact if and only if $g\in\mathcal{Z}_0$ and*

$$\lim_{|z|\to 1}\frac{(1-|z|^2)|\varphi'(z)||g'(z)|}{1-|\varphi(z)|^2}=0.$$

Next we consider the boundedness and compactness of $C_\varphi J_g : \mathcal{B} \to \mathcal{Z}$. Similar to the proofs of Theorems 6, 7 and 3, we obtain the following theorems. We will only sketch the proof of the next theorem for the benefit of the reader.

Theorem 10. *Suppose that φ is an analytic self-map of the unit disk and $g \in H(\mathbb{D})$. Then the following statements are equivalent.*

(i) $C_\varphi J_g : \mathcal{B} \to \mathcal{Z}$ *is bounded*;
(ii) $C_\varphi J_g : \mathcal{B}_0 \to \mathcal{Z}$ *is bounded*;
(iii)

$$\sup_{z\in\mathbb{D}} \frac{(1-|z|^2)|g'(\varphi(z))||\varphi'(z)|^2}{1-|\varphi(z)|^2} < \infty \tag{63}$$

and

$$\sup_{z\in\mathbb{D}} \left(1-|z|^2\right)\left|g''\left(\varphi(z)\right)\varphi'^2(z) + g'\left(\varphi(z)\right)\varphi''(z)\right| \ln \frac{2}{1-|\varphi(z)|^2} < \infty. \tag{64}$$

Proof. Assume that (63) and (64) hold. From (52), using the fact that there is a positive constant C independent of $f \in \mathcal{B}$ such that

$$\left|f(z)\right| \leqslant \left|f(0)\right| + C \ln \frac{1}{1-|z|}$$

and conditions (63) and (64), we obtain that the operator $C_\varphi J_g : \mathcal{B}$ (or $\mathcal{B}$) $\to \mathcal{Z}$ is bounded.

Now, assume that operator $C_\varphi J_g : \mathcal{B}$ (or $\mathcal{B}$) $\to \mathcal{Z}$ is bounded. Then using the test functions

$$F_w(z) = \ln \frac{1}{1-\bar{w}z} - \frac{1-|w|^2}{1-\bar{w}z}, \quad w \in \mathbb{D},$$

and

$$G_w(z) = \ln \frac{1}{1-\bar{w}z}, \quad w \in \mathbb{D},$$

similar to the proof of Theorem 1, it can be proved that (63) and (64) hold. □

Theorem 11. *Suppose that φ is an analytic self-map of the unit disk and $g \in H(\mathbb{D})$. Then the following statements are equivalent.*

(i) $C_\varphi J_g : \mathcal{B} \to \mathcal{Z}$ *is compact*;
(ii) $C_\varphi J_g : \mathcal{B}_0 \to \mathcal{Z}$ *is compact*;
(iii) $C_\varphi J_g : \mathcal{B} \to \mathcal{Z}$ *is bounded,*

$$\lim_{|\varphi(z)|\to 1} \frac{(1-|z|^2)|g'(\varphi(z))||\varphi'(z)|^2}{1-|\varphi(z)|^2} = 0$$

and

$$\lim_{|\varphi(z)|\to 1} \left(1-|z|^2\right)\left|g''\left(\varphi(z)\right)\varphi'^2(z) + g'\left(\varphi(z)\right)\varphi''(z)\right| \ln \frac{2}{1-|\varphi(z)|^2} = 0.$$

Theorem 12. *Suppose that φ is an analytic self-map of the unit disk and $g \in H(\mathbb{D})$. Then the following statements are equivalent.*

(i) $C_\varphi J_g : \mathcal{B} \to \mathcal{Z}_0$ *is compact*;
(ii) $C_\varphi J_g : \mathcal{B} \to \mathcal{Z}_0$ *is bounded*;
(iii) $C_\varphi J_g : \mathcal{B}_0 \to \mathcal{Z}_0$ *is compact*;
(iv)

$$\lim_{|z|\to 1} \left(1-|z|^2\right)\left|g''\left(\varphi(z)\right)\varphi'^2(z) + g'\left(\varphi(z)\right)\varphi''(z)\right| \ln \frac{2}{1-|\varphi(z)|^2} = 0 \tag{65}$$

and

$$\lim_{|z|\to 1} \frac{(1-|z|^2)|g'(\varphi(z))||\varphi'(z)|^2}{1-|\varphi(z)|^2} = 0. \tag{66}$$

Proof. The proof is similar to the proof of Theorem 3, hence will be omitted. □

Theorem 13. *Suppose that φ is an analytic self-map of the unit disk and $g \in H(\mathbb{D})$. Then $C_\varphi J_g : \mathcal{B}_0 \to \mathcal{Z}_0$ is bounded if and only if $C_\varphi J_g : \mathcal{B}_0 \to \mathcal{Z}$ is bounded and*

$$\lim_{|z|\to 1}\left(1-|z|^2\right)\left|g''\left(\varphi(z)\right)\varphi'^2(z)+g'\left(\varphi(z)\right)\varphi''(z)\right|=0$$

and

$$\lim_{|z|\to 1}\left(1-|z|^2\right)\left|g'\left(\varphi(z)\right)\right|\left|\varphi'(z)\right|^2=0.$$

Proof. The proof is similar to the proof of Theorem 4, hence is omitted. □

Corollary 3. *Suppose that $g \in H(\mathbb{D})$. Then*

(a) *$J_g : \mathcal{B} \to \mathcal{Z}$ is bounded if and only if $J_g : \mathcal{B}_0 \to \mathcal{Z}$ is bounded if and only if $J_g : \mathcal{B}_0 \to \mathcal{Z}_0$ is bounded if and only if $\sup_{z\in\mathbb{D}}(1-|z|^2)|g''(z)|\ln\frac{2}{1-|z|^2} < \infty$ and $\sup_{z\in\mathbb{D}}|g'(z)| < \infty$.*
(b) *$J_g : \mathcal{B}$ (or $\mathcal{B}_0$) $\to \mathcal{Z}$ is compact if and only if $J_g : \mathcal{B}$ (or $\mathcal{B}_0$) $\to \mathcal{Z}_0$ is compact if and only if $J_g : \mathcal{B} \to \mathcal{Z}_0$ is bounded if and only if g is a constant.*

The following result can be proved similarly as above theorems.

Theorem 14. *Suppose that φ is an analytic self-map of the unit disk and $g \in H(\mathbb{D})$. Then*

(i) *$J_g C_\varphi : \mathcal{B} \to \mathcal{Z}$ is bounded if and only if $J_g C_\varphi : \mathcal{B}_0 \to \mathcal{Z}$ is bounded if and only if*

$$\sup_{z\in\mathbb{D}}\frac{(1-|z|^2)|g'(z)||\varphi'(z)|}{1-|\varphi(z)|^2} < \infty \quad \text{and} \quad \sup_{z\in\mathbb{D}}\left(1-|z|^2\right)\left|g''(z)\right|\ln\frac{2}{1-|\varphi(z)|^2} < \infty.$$

(ii) *$J_g C_\varphi : \mathcal{B} \to \mathcal{Z}$ is compact if and only if $J_g C_\varphi : \mathcal{B}_0 \to \mathcal{Z}$ is compact if and only if $J_g C_\varphi : \mathcal{B} \to \mathcal{Z}$ is bounded,*

$$\lim_{|\varphi(z)|\to 1}\frac{(1-|z|^2)|g'(z)||\varphi'(z)|}{1-|\varphi(z)|^2} = 0 \quad \text{and} \quad \lim_{|\varphi(z)|\to 1}\left(1-|z|^2\right)\left|g''(z)\right|\ln\frac{2}{1-|\varphi(z)|^2} = 0.$$

(iii) *$J_g C_\varphi : \mathcal{B} \to \mathcal{Z}_0$ is compact if and only if $J_g C_\varphi : \mathcal{B} \to \mathcal{Z}_0$ is bounded if and only if*

$$\lim_{|z|\to 1}\left(1-|z|^2\right)\left|g''(z)\right|\ln\frac{2}{1-|\varphi(z)|^2} = 0 \quad \text{and} \quad \lim_{|z|\to 1}\frac{(1-|z|^2)|g'(z)||\varphi'(z)|}{1-|\varphi(z)|^2} = 0.$$

(iv) *$J_g C_\varphi : \mathcal{B}_0 \to \mathcal{Z}_0$ is bounded if and only if $J_g C_\varphi : \mathcal{B}_0 \to \mathcal{Z}$ is bounded, $g \in \mathcal{Z}_0$ and*

$$\lim_{|z|\to 1}\left(1-|z|^2\right)\left|g'(z)\right|\left|\varphi'(z)\right| = 0.$$

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