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# Generalized composition operators on Zygmund spaces and Bloch type spaces

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**Abstract**

The boundedness and compactness of the generalized composition operator on Zygmund spaces and Bloch type spaces are investigated in this paper.


*Keywords:* Zygmund space; Bloch type space; Composition operator

## 1. Introduction

Throughout the paper, $\varphi$ denotes a nonconstant analytic self-map of the unit disk $\mathbb{D}$ in the complex plane $\mathbb{C}$. Associated with $\varphi$ is the composition operator $C_\varphi$ defined by $C_\varphi f = f \circ \varphi$ for $f \in H(\mathbb{D})$, the class of all analytic functions on $\mathbb{D}$. It is well known that the composition operator $C_\varphi$ is bounded on the classical Hardy, Bergman and Bloch spaces. The main subject in the study of composition operators is to describe operator theoretic properties of $C_\varphi$ in terms of function theoretic properties of $\varphi$, see, for example, [3,12,17].

A function $f \in H(\mathbb{D})$ is said to belong to the Bloch type space (or $\alpha$-Bloch space), denoted by $\mathcal{B}^\alpha = \mathcal{B}^\alpha(\mathbb{D})$, if

$$B_\alpha(f) = \sup_{z\in\mathbb{D}} \left(1-|z|^2\right)^\alpha \left|f'(z)\right| < \infty.$$

The space $\mathcal{B}^\alpha$ becomes a Banach space with the norm $\|f\|_{\mathcal{B}^\alpha} = |f(0)| + B_\alpha(f)$. Let $\mathcal{B}_0^\alpha$ denote the subspace of $\mathcal{B}^\alpha$ consisting of those $f \in \mathcal{B}^\alpha$ such that

$$\lim_{|z|\to 1} \left(1-|z|^2\right)^\alpha \left|f'(z)\right| = 0.$$

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This space is called the little Bloch type space. For $\alpha = 1$, we obtain the well-known classical Bloch space, simply denoted by $\mathcal{B}$. For $0 < \alpha < 1$, $\mathcal{B}^\alpha$ can be identified with the analytic Lipschitz space $\Lambda_{1-\alpha}$ (see, for example, [5, Theorem 5.1] or [16]).

Denote by $\mathcal{Z}$ the class of all $f \in H(\mathbb{D}) \cap C(\overline{\mathbb{D}})$ such that

$$\|f\| = \sup \frac{|f(e^{i(\theta+h)}) + f(e^{i(\theta-h)}) - 2f(e^{i\theta})|}{h} < \infty, \tag{1}$$

where the supremum is taken over all $e^{i\theta} \in \partial\mathbb{D}$ and $h > 0$. From a theorem of Zygmund (see [5, Theorem 5.3]) and the Closed Graph Theorem we see that $f \in \mathcal{Z}$ if and only if $\sup_{z\in\mathbb{D}}(1-|z|^2)|f''(z)| < \infty$. It is easy to see that $\mathcal{Z}$ is a Banach space under the norm $\|\cdot\|_{\mathcal{Z}}$, where

$$\|f\|_{\mathcal{Z}} = \left|f(0)\right| + \left|f'(0)\right| + \sup_{z\in\mathbb{D}}\left(1-|z|^2\right)\left|f''(z)\right|. \tag{2}$$

We call $\mathcal{Z}$ the Zygmund space.

The little Zygmund space of $\mathbb{D}$, denoted by $\mathcal{Z}_0$, is the closed subspace of $\mathcal{Z}$ consisting of functions $f$ with

$$\lim_{|z|\to 1}\left(1-|z|^2\right)\left|f''(z)\right| = 0.$$

From (2) it is easy to obtain

$$\left|f'(z) - f'(0)\right| \leqslant C\|f\|_{\mathcal{Z}} \log\frac{1}{1-|z|^2}. \tag{3}$$

The Bergman space, denoted by $A^p = A^p(\mathbb{D})$, is the space of all analytic functions $f$ on $\mathbb{D}$ such that $\int_{\mathbb{D}} |f(z)|^p\,dA(z) < \infty$. The Dirichlet space $\mathcal{D}^p = \mathcal{D}^p(\mathbb{D})$ is the space of all analytic functions on $\mathbb{D}$ such that $f' \in A^p$ with the norm

$$\|f\|_{\mathcal{D}^p}^p = \left|f(0)\right|^p + \int_{\mathbb{D}} \left|f'(z)\right|^p\,dA(z).$$

Let $1 \leqslant p < \infty$, $p \neq 2$. From [7] we know that an operator $T$ is a surjective isometry of $\mathcal{D}^p$ with respect to the norm $\|\cdot\|_{\mathcal{D}^p}$ if and only if there is an automorphism $\phi$ of $\mathbb{D}$ and unimodular constants $\lambda_1$ and $\lambda_2$ such that

$$(Tf)(z) = \lambda_1 f(0) + \lambda_2 \int_0^z \left(\phi'(\xi)\right)^{2/p} f'\left(\phi(\xi)\right)d\xi \tag{4}$$

for every $f \in \mathcal{D}^p$. Let $S^p$ be the space of analytic functions $f$ on $\mathbb{D}$ such that $f' \in H^p$. An operator $T$ is a surjective isometry of $S^p$ under the norm

$$\|f\|_{S^p}^p = \left|f(0)\right|^p + \|f'\|_{H^p}^p$$

if and only if there is an automorphism $\phi$ of $\mathbb{D}$ and unimodular constants $\lambda_1$ and $\lambda_2$ such that

$$(Tf)(z) = \lambda_1 f(0) + \lambda_2 \int_0^z \left(\phi'(\xi)\right)^{1/p} f'\left(\phi(\xi)\right)d\xi \tag{5}$$

for every $f \in S^p$.

In view of (4) and (5) we see that composition operators and weighted composition operators naturally come from the isometry of some function spaces in some senses. Motivated by (4) and (5), for $g \in H(\mathbb{D})$, we define a linear operator as follows

$$\left(C_\varphi^g f\right)(z) = \int_0^z f'\left(\varphi(\xi)\right)g(\xi)\,d\xi.$$

The operator $C_\varphi^g$ is called the generalized composition operator. When $g = \varphi'$, we see that this operator is essentially composition operator, since the following difference $C_\varphi^g - C_\varphi$ is a constant. Therefore, $C_\varphi^g$ is a generalization of the composition operator. To the best of our knowledge, the operator $C_\varphi^g$ is introduced in the present paper for the first time.

Recall that if $X$ and $Y$ are Banach spaces, $L : X \to Y$ is a linear operator, then $L$ is said to be compact if for every bounded sequence $(x_n)$ in $X$, the sequence $(L(x_n))$ has a convergent subsequence. The operator $L$ is said to be weakly compact if for every bounded sequence $(x_n)$ in $X$, $(L(x_n))$ has a weakly convergent subsequence, i.e., there is a subsequence $(x_{n_m})$ such that for every $\Lambda \in Y^*$, $\Lambda(L(x_{n_m}))$ converges. A useful characterization for an operator to be weakly compact is the following Gantmacher's theorem: $L$ is weakly compact if and only if $L^{**}(X^{**}) \subset Y$, where $L^{**}$ is the second adjoint of $L$ (see, for example, [4]).

Some characterizations of the boundedness and compactness of the composition operator, as well as Volterra type operator, on the Bloch type space and the Zygmund space can be found in [2,6,8–10,13–15].

The purpose of this paper is to study the boundedness and compactness of the generalized composition operator on the Zygmund space and the Bloch type space and the little Bloch type space. We also study the weak compactness of the generalized composition operator on the little Bloch type space.

Constants are denoted by $C$ in this paper, they are positive and not necessarily the same in each occurrence. The notation $a \preccurlyeq b$ means that there is a positive constant $C$ such that $a \leqslant Cb$. We say that $a \asymp b$ if both $a \preccurlyeq b$ and $b \preccurlyeq a$ hold.

## 2. The boundedness and compactness of $C_\varphi^g : \mathcal{Z} \to \mathcal{B}^\alpha$

In this section, we characterize the boundedness and compactness of the generalized composition operator from Zygmund spaces to Bloch type spaces. For this purpose, we start this section by stating some useful lemmas. By standard arguments (see, for example, [3, Proposition 3.11]) the following lemma follows.

**Lemma 1.** *Let $0 < \alpha < \infty$, $g \in H(\mathbb{D})$ and $\varphi$ be an analytic self-map of $\mathbb{D}$. Let $X = \mathcal{Z}$ or $\mathcal{B}^\alpha$; $Y = \mathcal{Z}$ or $\mathcal{B}^\alpha$. Then $C_\varphi^g : X \to Y$ is compact if and only if $C_\varphi^g : X \to Y$ is bounded and for any bounded sequence $(f_k)_{k\in\mathbb{N}}$ in $X$ which converges to zero uniformly on $\mathbb{D}$ as $k \to \infty$, we have $\|C_\varphi^g f_k\|_Y \to 0$ as $k \to \infty$.*

**Lemma 2.** *Assume that $\alpha > 0$. A closed set $K$ in $\mathcal{B}_0^\alpha$ is compact if and only if it is bounded and satisfies*

$$\lim_{|z|\to 1} \sup_{f\in K} \left(1 - |z|^2\right)^\alpha \left|f'(z)\right| = 0.$$

**Remark.** For $\alpha = 1$, this lemma was proved in [10]. For the case of $\alpha \neq 1$, see [11].

Now we are ready to state and prove the main results in this section.

**Theorem 1.** *Let $0 < \alpha < \infty$, $g \in H(\mathbb{D})$ and $\varphi$ be an analytic self-map of $\mathbb{D}$. Then $C_\varphi^g : \mathcal{Z} \to \mathcal{B}^\alpha$ is bounded if and only if*

$$K := \sup_{z\in\mathbb{D}} \left(1 - |z|^2\right)^\alpha \left|g(z)\right| \log \frac{1}{1 - |\varphi(z)|^2} < \infty. \tag{6}$$

**Proof.** Suppose that (6) holds. Then for arbitrary $z \in \mathbb{D}$ and $f \in \mathcal{Z}$, by (3) we have

$$\begin{aligned}\left(1 - |z|^2\right)^\alpha \left|\left(C_\varphi^g f\right)'(z)\right| &= \left(1 - |z|^2\right)^\alpha \left|f'\left(\varphi(z)\right)\right| \left|g(z)\right| \\ &\leqslant C\|f\|_{\mathcal{Z}} \left(1 - |z|^2\right)^\alpha \left|g(z)\right| \log \frac{1}{1 - |\varphi(z)|^2}.\end{aligned}$$

From this, (6) and since $C_\varphi^g f(0) = 0$, it follows that $C_\varphi^g : \mathcal{Z} \to \mathcal{B}^\alpha$ is bounded.

Conversely, assume that $C_\varphi^g : \mathcal{Z} \to \mathcal{B}^\alpha$ is bounded. Let

$$h(z) = (z - 1)\left[\left(1 + \log \frac{1}{1 - z}\right)^2 + 1\right]$$

and put

$$f_a(z)=\frac{h(\bar a z)}{\bar a}\left(\log\frac{1}{1-|a|^2}\right)^{-1} \tag{7}$$

for any $a\in\mathbb{D}$ such that $1/\sqrt{2}<|a|<1$. Then we have

$$f_a'(z)=\left(\log\frac{1}{1-\bar a z}\right)^2\left(\log\frac{1}{1-|a|^2}\right)^{-1}$$

and

$$f_a''(z)=\frac{2\bar a}{1-\bar a z}\left(\log\frac{1}{1-\bar a z}\right)\left(\log\frac{1}{1-|a|^2}\right)^{-1},$$

which implies that

$$\left|f_a''(z)\right|\leqslant\frac{2}{1-|z|}\left(C+\log\frac{1}{1-|a|}\right)\left(\log\frac{1}{1-|a|^2}\right)^{-1}\leqslant\frac{C}{1-|z|}$$

for $1/\sqrt{2}<|a|<1$ and $\sup_{1/\sqrt{2}<|a|<1}\|f_a\|_{\mathcal{Z}}<\infty$. Therefore we have

$$\left\|C_\varphi^g f_{\varphi(a)}\right\|_{\mathcal{B}^\alpha}=\sup_{z\in\mathbb{D}}\left(1-|z|^2\right)^\alpha\left|\left(C_\varphi^g f_{\varphi(a)}\right)'(z)\right|\geqslant\left(1-|a|^2\right)^\alpha\left|g(a)\right|\log\frac{1}{1-|\varphi(a)|^2}.$$

This together with the Maximum Modulus Principle imply (6), completing the proof of the theorem. □

**Theorem 2.** *Let $0<\alpha<\infty$, $g\in H(\mathbb{D})$ and $\varphi$ be an analytic self-map of $\mathbb{D}$. Then $C_\varphi^g:\mathcal{Z}\to\mathcal{B}^\alpha$ is compact if and only if $C_\varphi^g:\mathcal{Z}\to\mathcal{B}^\alpha$ is bounded and*

$$\lim_{|\varphi(z)|\to1}\left(1-|z|^2\right)^\alpha\left|g(z)\right|\log\frac{1}{1-|\varphi(z)|^2}=0. \tag{8}$$

**Proof.** First assume that $C_\varphi^g:\mathcal{Z}\to\mathcal{B}^\alpha$ is bounded and (8) holds. From the boundedness of $C_\varphi^g$ with $f(z)=z$, we see that

$$L:=\sup_{z\in\mathbb{D}}\left(1-|z|^2\right)^\alpha\left|g(z)\right|<\infty.$$

Let $(f_k)_{k\in\mathbb{N}}$ be a sequence in $\mathcal{Z}$ such that $\sup_{k\in\mathbb{N}}\|f_k\|_{\mathcal{Z}}\leqslant M$ and $f_k\to0$ uniformly on compact subsets of $\mathbb{D}$ as $k\to\infty$. By (8) we have that for every $\varepsilon>0$, there is a constant $\delta\in(0,1)$, such that $\delta<|\varphi(z)|<1$ implies

$$\left(1-|z|^2\right)^\alpha\left|g(z)\right|\log\frac{1}{1-|\varphi(z)|^2}<\varepsilon/M.$$

Let $K=\{w\in\mathbb{D}\colon |w|\leqslant\delta\}$. By (3), we have

$$\begin{aligned}\left\|C_\varphi^g f_k\right\|_{\mathcal{B}^\alpha}&=\sup_{z\in\mathbb{D}}\left(1-|z|^2\right)^\alpha\left|f_k'\left(\varphi(z)\right)g(z)\right|\\&\leqslant\sup_{\{z\in\mathbb{D}:\ |\varphi(z)|\leqslant\delta\}}\left(1-|z|^2\right)^\alpha\left|g(z)\right|\left|f_k'\left(\varphi(z)\right)\right|+\sup_{\{z\in\mathbb{D}:\ \delta<|\varphi(z)|<1\}}\left(1-|z|^2\right)^\alpha\left|g(z)\right|\left|f_k'\left(\varphi(z)\right)\right|\\&\leqslant L\sup_{w\in K}\left|f_k'(w)\right|+C\|f_k\|_{\mathcal{Z}}\sup_{\{z\in\mathbb{D}:\ \delta<|\varphi(z)|<1\}}\left(1-|z|^2\right)^\alpha\left|g(z)\right|\log\frac{1}{1-|\varphi(z)|^2}\\&\leqslant L\sup_{w\in K}\left|f_k'(w)\right|+C\varepsilon.\end{aligned}$$

By the Cauchy estimate, if $(f_k)_{k\in\mathbb{N}}$ is a sequence converging to zero on compact subsets of $\mathbb{D}$, then the sequence $(f_k')_{k\in\mathbb{N}}$ also converges to zero on compact subsets of $\mathbb{D}$ as $k\to\infty$. In particular, since $K$ is compact it follows that $\lim_{k\to\infty}\sup_{w\in K}|f_k'(w)|=0$. Using these facts and letting $k\to\infty$ in the last inequality, we obtain that $\limsup_{k\to\infty}\|C_\varphi^g f_k\|_{\mathcal{B}^\alpha}\leqslant C\varepsilon$. Since $\varepsilon$ is an arbitrary positive number it follows that the last limit is equal to zero. Employing Lemma 1, the implication follows.

Conversely, suppose that $C_\varphi^g : \mathcal{Z} \to \mathcal{B}^\alpha$ is compact. Note that $f_a$ defined by (7) converges to zero uniformly on compact subsets of $\mathbb{D}$ as $|a| \to 1$ and $f_a'(a) = \log \frac{1}{1-|a|^2}$ for each $a \in \mathbb{D} \setminus \{0\}$. Let $(z_k)_{k\in\mathbb{N}}$ be a sequence in $\mathbb{D}$ such that $|\varphi(z_k)| \to 1$ as $k \to \infty$. We choose test functions $(f_k)_{k\in\mathbb{N}}$ defined by

$$f_k(z) = \frac{\overline{\varphi(z_k)}z - 1}{\overline{\varphi(z_k)}}\left[\left(1 + \log \frac{1}{1 - \overline{\varphi(z_k)}z}\right)^2 + 1\right]\left(\log \frac{1}{1 - |\varphi(z_k)|^2}\right)^{-1}. \tag{9}$$

From the proof of Theorem 1 we see that $\sup_{k\in\mathbb{N}} \|f_k\|_{\mathcal{Z}} \leqslant C$. Moreover, $f_k$ converges to zero uniformly on compact subsets of $\mathbb{D}$. Hence, in view of Lemma 1 it follows that $\|C_\varphi^g f_k\|_{\mathcal{B}^\alpha} \to 0$, as $k \to \infty$. Since

$$\begin{aligned}\|C_\varphi^g f_k\|_{\mathcal{B}^\alpha} &= \sup_{z\in\mathbb{D}} (1 - |z|^2)^\alpha \left|(C_\varphi^g f_k)'(z)\right| \geqslant (1 - |z_k|^2)^\alpha \left|g(z_k)\right| \left|f_k'(\varphi(z_k))\right| \\ &= (1 - |z_k|^2)^\alpha \left|g(z_k)\right| \log \frac{1}{1 - |\varphi(z_k)|^2},\end{aligned}$$

we obtain $\lim_{k\to\infty} (1 - |z_k|^2)^\alpha |g(z_k)| \log \frac{1}{1-|\varphi(z_k)|^2} = 0$, from which the result follows. □

**Theorem 3.** *Let $0 < \alpha < \infty$, $g \in H(\mathbb{D})$ and $\varphi$ be an analytic self-map of $\mathbb{D}$. Then $C_\varphi^g : \mathcal{Z} \to \mathcal{B}_0^\alpha$ is bounded if and only if*

$$\lim_{|z|\to 1} (1 - |z|^2)^\alpha \left|g(z)\right| = 0 \tag{10}$$

*and*

$$\lim_{|\varphi(z)|\to 1} (1 - |z|^2)^\alpha \left|g(z)\right| \log \frac{1}{1 - |\varphi(z)|^2} = 0. \tag{11}$$

**Proof.** Assume that (10) and (11) hold. By (11) we have that for every $\varepsilon > 0$ there exists $r \in (0, 1)$ such that

$$(1 - |z|^2)^\alpha \left|g(z)\right| \log \frac{1}{1 - |\varphi(z)|^2} < \varepsilon$$

when $r < |\varphi(z)| < 1$. From (10), there exists $\rho \in (0, 1)$ such that

$$(1 - |z|^2)^\alpha \left|g(z)\right| < \varepsilon / \log \frac{1}{1 - r^2}$$

when $\rho < |z| < 1$.

Therefore, when $\rho < |z| < 1$ and $r < |\varphi(z)| < 1$, we have that

$$(1 - |z|^2)^\alpha \left|g(z)\right| \log \frac{1}{1 - |\varphi(z)|^2} < \varepsilon. \tag{12}$$

If $\rho < |z| < 1$ and $|\varphi(z)| \leqslant r$, then

$$(1 - |z|^2)^\alpha \left|g(z)\right| \log \frac{1}{1 - |\varphi(z)|^2} < (1 - |z|^2)^\alpha \left|g(z)\right| \log \frac{1}{1 - r^2} < \varepsilon. \tag{13}$$

Combining (12) and (13), we obtain

$$\lim_{|z|\to 1} (1 - |z|^2)^\alpha \left|g(z)\right| \log \frac{1}{1 - |\varphi(z)|^2} = 0. \tag{14}$$

From this, by the Maximum Modulus Theorem and Theorem 1 the boundedness of $C_\varphi^g : \mathcal{Z} \to \mathcal{B}^\alpha$ follows. For any $f \in \mathcal{Z}$, in view of (3), we have

$$(1 - |z|^2)^\alpha \left|(C_\varphi^g f)'(z)\right| \leqslant C\|f\|_{\mathcal{Z}} (1 - |z|^2)^\alpha \left|g(z)\right| \log \frac{1}{1 - |\varphi(z)|^2}.$$

By (14), it follows that $C_\varphi^g f \in \mathcal{B}_0^\alpha$, for each $f \in \mathcal{Z}$. Since $\mathcal{B}_0^\alpha$ is a closed subset of $\mathcal{B}^\alpha$, we obtain $C_\varphi^g(\mathcal{Z}) \subseteq \mathcal{B}_0^\alpha$. Therefore $C_\varphi^g : \mathcal{Z} \to \mathcal{B}_0^\alpha$ is bounded.

Conversely, suppose that $C_\varphi^g : \mathcal{Z} \to \mathcal{B}_0^\alpha$ is bounded, then for $f(z) = z$ we obtain that (10) holds. Now assume that condition (11) does not hold. If it were, then it would exist $\varepsilon_0 > 0$ and a sequence $(z_k)_{k\in\mathbb{N}} \in \mathbb{D}$, such that $\lim_{k\to\infty}|\varphi(z_k)| = 1$ and

$$\left(1 - |z_k|^2\right)^\alpha \left|g(z_k)\right| \log \frac{1}{1 - |\varphi(z_k)|^2} \geqslant \varepsilon_0 > 0$$

for sufficiently large $k$. We may also assume that

$$\frac{1 - |\varphi(z_{k-1})|}{2} > 1 - \left|\varphi(z_k)\right|, \quad k \in \mathbb{N}.$$

Then, for every nonnegative integer $s$ there is at most one $z_k$ such that $1 - \frac{1}{2^s} \leqslant |\varphi(z_k)| < 1 - \frac{1}{2^{(s+1)}}$. Hence, there is $m_0 \in \mathbb{N}$ such that for any Carleson window

$$Q = \left\{re^{i\theta}\colon\ 0 < 1 - r < l(Q),\ |\theta - \theta_0| < l(Q)\right\}$$

and $s \in \mathbb{N}$, there are at most $m_0$ elements in

$$\left\{\varphi(z_k) \in Q\colon\ 2^{-(s+1)} l(Q) < 1 - \left|\varphi(z_k)\right| < 2^{-s} l(Q)\right\}.$$

Therefore, $(\varphi(z_k))_{k\in\mathbb{N}}$ is an interpolating sequence for $\mathcal{B}$, in sense of [1].

By [1] we have some $p \in \mathcal{B}$ such that

$$p\left(\varphi(z_k)\right) = \log \frac{1}{1 - |\varphi(z_k)|^2}, \quad k \in \mathbb{N}.$$

Let $f(z) = \int_0^z p(\xi)\, d\xi$. Then from the definition of Bloch functions and Zygmund functions, we see that $f \in \mathcal{Z}$. We obtain

$$\begin{aligned}
\left(1 - |z_k|^2\right)^\alpha \left|\left(C_\varphi^g f\right)'(z_k)\right| &= \left(1 - |z_k|^2\right)^\alpha \left|g(z_k)\right| \left|f'\left(\varphi(z_k)\right)\right| \\
&= \left(1 - |z_k|^2\right)^\alpha \left|g(z_k)\right| \left|p\left(\varphi(z_k)\right)\right| \\
&= \left(1 - |z_k|^2\right)^\alpha \left|g(z_k)\right| \log \frac{1}{1 - |\varphi(z_k)|^2} \\
&\geqslant \varepsilon_0 > 0.
\end{aligned}$$

Since $\lim_{k\to\infty}|\varphi(z_k)| = 1$ implies that $\lim_{k\to\infty}|z_k| = 1$, from the above inequality we obtain that $C_\varphi^g f \notin \mathcal{B}_0^\alpha$, which is a contradiction. □

**Theorem 4.** *Let $0 < \alpha < \infty$, $g \in H(\mathbb{D})$ and $\varphi$ be an analytic self-map of $\mathbb{D}$. Then $C_\varphi^g : \mathcal{Z} \to \mathcal{B}_0^\alpha$ is compact if and only if*

$$\lim_{|z|\to 1} \left(1 - |z|^2\right)^\alpha \left|g(z)\right| \log \frac{1}{1 - |\varphi(z)|^2} = 0. \tag{15}$$

**Proof.** Suppose that $C_\varphi^g : \mathcal{Z} \to \mathcal{B}_0^\alpha$ is compact. Then $C_\varphi^g : \mathcal{Z} \to \mathcal{B}_0^\alpha$ is bounded, therefore by Theorem 3 we see that (10) and (11) hold. By the proof of Theorem 3 we have that (15) holds.

Conversely, assume that (15) holds. From Lemma 2, we see that $C_\varphi^g : \mathcal{Z} \to \mathcal{B}_0^\alpha$ is compact if and only if

$$\lim_{|z|\to 1} \sup_{\|f\|_{\mathcal{Z}} \leqslant 1} \left(1 - |z|^2\right)^\alpha \left|\left(C_\varphi^g f\right)'(z)\right| = 0. \tag{16}$$

By (3) we have that

$$\left(1 - |z|^2\right)^\alpha \left|\left(C_\varphi^g f\right)'(z)\right| \leqslant C\|f\|_{\mathcal{Z}} \left(1 - |z|^2\right)^\alpha \left|g(z)\right| \log \frac{1}{1 - |\varphi(z)|^2}. \tag{17}$$

Taking the supremum in (17) over the unit ball of the space $\mathcal{Z}$, then letting $|z| \to 1$, we obtain (16), from which the compactness of $C_\varphi^g : \mathcal{Z} \to \mathcal{B}_0^\alpha$ follows. □

## 3. The boundedness and compactness of $C_\varphi^g : \mathcal{Z} \to \mathcal{Z}$

In this section, we characterize the boundedness and compactness of the operator $C_\varphi^g$ on Zygmund spaces.

**Theorem 5.** *Let $\varphi$ be an analytic self-map of $\mathbb{D}$ and $g \in H(\mathbb{D})$. Then $C_\varphi^g : \mathcal{Z} \to \mathcal{Z}$ is bounded if and only if*

$$\sup_{z\in\mathbb{D}} \frac{(1-|z|^2)|\varphi'(z)||g(z)|}{1-|\varphi(z)|^2} < \infty \quad \text{and} \quad \sup_{z\in\mathbb{D}} (1-|z|^2)\left|g'(z)\right| \log\frac{1}{1-|\varphi(z)|^2} < \infty. \tag{18}$$

**Proof.** Assume that $C_\varphi^g : \mathcal{Z} \to \mathcal{Z}$ is bounded, i.e., there exists a constant $C$ such that $\|C_\varphi^g f\|_{\mathcal{Z}} \leqslant C\|f\|_{\mathcal{Z}}$ for all $f \in \mathcal{Z}$. Taking the functions $f(z) = z$ and $f(z) = z^2$, respectively, we obtain

$$\sup_{z\in\mathbb{D}} (1-|z|^2)\left|g'(z)\right| < \infty \tag{19}$$

and

$$\sup_{z\in\mathbb{D}} (1-|z|^2)\left|\varphi'(z)g(z) + g'(z)\varphi(z)\right| < \infty. \tag{20}$$

Using these facts and the boundedness of the function $\varphi(z)$, we get

$$\sup_{z\in\mathbb{D}} (1-|z|^2)\left|\varphi'(z)\right|\left|g(z)\right| < \infty. \tag{21}$$

Let the function $f_a$ be defined by (7) for $a \in \mathbb{D}$ such that $|a| > 1/2$, we have

$$\begin{aligned} C\left\|C_\varphi^g\right\|_{\mathcal{Z}\to\mathcal{Z}} &\geqslant \left\|C_\varphi^g f_{\varphi(\lambda)}\right\|_{\mathcal{Z}} \\ &\geqslant (1-|\lambda|^2)\left|g'(\lambda)\right| \log\frac{1}{1-|\varphi(\lambda)|^2} - 2\frac{(1-|\lambda|^2)|\varphi'(\lambda)||g(\lambda)||\varphi(\lambda)|}{1-|\varphi(\lambda)|^2}, \end{aligned}$$

that is

$$(1-|\lambda|^2)\left|g'(\lambda)\right| \log\frac{1}{1-|\varphi(\lambda)|^2} \leqslant C\left\|C_\varphi^g\right\|_{\mathcal{Z}\to\mathcal{Z}} + 2\frac{(1-|\lambda|^2)|\varphi'(\lambda)||g(\lambda)||\varphi(\lambda)|}{1-|\varphi(\lambda)|^2}. \tag{22}$$

Set

$$h_a(z) = \frac{h(\bar{a}z)}{\bar{a}}\left(\log\frac{1}{1-|a|^2}\right)^{-1} - \int_0^z \log\frac{1}{1-\bar{a}w}\,dw \tag{23}$$

for $a \in \mathbb{D}$ such that $|a| > 1/2$. Then,

$$h_a'(z) = \left(\log\frac{1}{1-\bar{a}z}\right)^2\left(\log\frac{1}{1-|a|^2}\right)^{-1} - \log\frac{1}{1-\bar{a}z}$$

and

$$h_a''(z) = \frac{2\bar{a}}{1-\bar{a}z}\left(\log\frac{1}{1-\bar{a}z}\right)\left(\log\frac{1}{1-|a|^2}\right)^{-1} - \frac{\bar{a}}{1-\bar{a}z}.$$

Similar to the case of $f_a$, we have $h_a \in \mathcal{Z}$ and $M_1 = \sup_{1/2<|a|<1}\|h_a\|_{\mathcal{Z}} < \infty$. From this and by using the facts that $h_a'(a) = 0$, $h_a''(a) = \bar{a}/(1-|a|^2)$, it follows that

$$C\left\|C_\varphi^g\right\|_{\mathcal{Z}\to\mathcal{Z}} \geqslant \left\|C_\varphi^g h_{\varphi(\lambda)}\right\|_{\mathcal{Z}} \geqslant \frac{(1-|\lambda|^2)|\varphi'(\lambda)||g(\lambda)||\varphi(\lambda)|}{1-|\varphi(\lambda)|^2}. \tag{24}$$

From (24), we have

$$\sup_{|\varphi(\lambda)|>\frac{1}{2}} \frac{(1-|\lambda|^2)|\varphi'(\lambda)||g(\lambda)|}{1-|\varphi(\lambda)|^2} < \sup_{|\varphi(\lambda)|>\frac{1}{2}} 2\frac{(1-|\lambda|^2)|\varphi'(\lambda)||g(\lambda)||\varphi(\lambda)|}{1-|\varphi(\lambda)|^2} \leqslant \sup_{|\varphi(\lambda)|>\frac{1}{2}} C\left\|C_\varphi^g\right\|_{\mathcal{Z}\to\mathcal{Z}} < \infty. \tag{25}$$

By (21), we see that

$$\sup_{|\varphi(\lambda)|\leqslant\frac{1}{2}} \frac{(1-|\lambda|^2)|\varphi'(\lambda)||g(\lambda)|}{1-|\varphi(\lambda)|^2} \leqslant \sup_{|\varphi(\lambda)|\leqslant\frac{1}{2}} \frac{4}{3}\left(1-|\lambda|^2\right)\left|\varphi'(\lambda)\right|\left|g(\lambda)\right| < \infty. \tag{26}$$

From (25) and (26) we obtain the first inequality of (18). Similarly, from (19) and (22), the second inequality in (18) follows, as desired.

Conversely, assume that (18) holds. For a function $f \in \mathcal{Z}$, from (2) and (3) it follows that

$$\begin{aligned}\left(1-|z|^2\right)\left|\left(C_\varphi^g f\right)''(z)\right| &= \left(1-|z|^2\right)\left|\left(f'(\varphi)g\right)'(z)\right| \\ &\leqslant \left(1-|z|^2\right)\left|\varphi'(z)\right|\left|g(z)\right|\left|f''\left(\varphi(z)\right)\right| + \left(1-|z|^2\right)\left|g'(z)\right|\left|f'\left(\varphi(z)\right)\right| \\ &\leqslant C\frac{(1-|z|^2)|\varphi'(z)||g(z)|}{1-|\varphi(z)|^2}\|f\|_{\mathcal{Z}} + C\left(1-|z|^2\right)\left|g'(z)\right|\left(\log\frac{1}{1-|\varphi(z)|^2}\right)\|f\|_{\mathcal{Z}}.\end{aligned}$$

In addition,

$$\left(C_\varphi^g f\right)(0) = 0 \quad \text{and} \quad \left|\left(C_\varphi^g f\right)'(0)\right| \leqslant C\left|g(0)\right|\left(\log\frac{1}{1-|\varphi(0)|^2}\right)\|f\|_{\mathcal{Z}}.$$

Using these facts and (18) it follows that $C_\varphi^g : \mathcal{Z} \to \mathcal{Z}$ is bounded. □

**Theorem 6.** *Let $\varphi$ be an analytic self-map of $\mathbb{D}$ and $g \in H(\mathbb{D})$. Then $C_\varphi^g : \mathcal{Z} \to \mathcal{Z}$ is compact if and only if $C_\varphi^g : \mathcal{Z} \to \mathcal{Z}$ is bounded and*

$$\lim_{|\varphi(z)|\to 1} \frac{(1-|z|^2)|\varphi'(z)||g(z)|}{1-|\varphi(z)|^2} = 0 \quad \textit{and} \quad \lim_{|\varphi(z)|\to 1}\left(1-|z|^2\right)\left|g'(z)\right|\log\frac{1}{1-|\varphi(z)|^2} = 0. \tag{27}$$

**Proof.** Suppose that $C_\varphi^g : \mathcal{Z} \to \mathcal{Z}$ is compact. Then it is clear that $C_\varphi^g : \mathcal{Z} \to \mathcal{Z}$ is bounded. Let $(z_k)_{k\in\mathbb{N}}$ be a sequence in $\mathbb{D}$ such that $|\varphi(z_k)| \to 1$ as $k \to \infty$. Let $f_k$ be defined by (9), then $\sup_{k\in\mathbb{N}}\|f_k\|_{\mathcal{Z}} < \infty$ and $f_k \to 0$ uniformly on compact subsets of $\mathbb{D}$ as $k \to \infty$. Since $C_\varphi^g : \mathcal{Z} \to \mathcal{Z}$ is compact, it gives $\lim_{k\to\infty}\|C_\varphi^g f_k\|_{\mathcal{Z}} = 0$. Note that

$$f_k'\left(\varphi(z_k)\right) = \log\frac{1}{1-|\varphi(z_k)|^2}, \qquad f_k''\left(\varphi(z_k)\right) = \frac{2\overline{\varphi(z_k)}}{1-|\varphi(z_k)|^2}.$$

We have

$$\left\|C_\varphi^g f_k\right\|_{\mathcal{Z}} \geqslant \left|\frac{2(1-|z_k|^2)|\varphi'(z_k)||g(z_k)||\varphi(z_k)|}{1-|\varphi(z_k)|^2} - \left(1-|z_k|^2\right)\left|g'(z_k)\right|\log\frac{1}{1-|\varphi(z_k)|^2}\right|,$$

and consequently

$$\lim_{|\varphi(z_k)|\to 1} \frac{2(1-|z_k|^2)|\varphi'(z_k)||g(z_k)||\varphi(z_k)|}{1-|\varphi(z_k)|^2} = \lim_{|\varphi(z_k)|\to 1}\left(1-|z_k|^2\right)\left|g'(z_k)\right|\log\frac{1}{1-|\varphi(z_k)|^2} \tag{28}$$

if one of these two limits exists.

Next, set

$$h_k(z) = \frac{h(\overline{\varphi(z_k)}z)}{\overline{\varphi(z_k)}}\left(\log\frac{1}{1-|\varphi(z_k)|^2}\right)^{-1} - \int_0^z \log^3\frac{1}{1-\overline{\varphi(z_k)}w}\,dw\left(\log\frac{1}{1-|\varphi(z_k)|^2}\right)^{-2}.$$

Then $h_k'(\varphi(z_k))=0$, $\sup_{k\in\mathbb{N}}\|h_k\|_{\mathcal{Z}}\leqslant C$ and $h_k$ converges to 0 uniformly on compact subsets of $\mathbb{D}$ as $k\to\infty$. Since $C_\varphi^g:\mathcal{Z}\to\mathcal{Z}$ is compact, we have $\lim_{k\to\infty}\|C_\varphi^g h_k\|_{\mathcal{Z}}=0$. On the other hand,

$$\frac{(1-|z_k|^2)|\varphi'(z_k)||g(z_k)||\varphi(z_k)|}{1-|\varphi(z_k)|^2}\leqslant\big\|C_\varphi^g h_k\big\|_{\mathcal{Z}}.$$

Hence

$$\lim_{k\to\infty}\frac{(1-|z_k|^2)|\varphi'(z_k)||g(z_k)||\varphi(z_k)|}{1-|\varphi(z_k)|^2}=0.$$

Therefore

$$\lim_{|\varphi(z_k)|\to 1}\frac{(1-|z_k|^2)|\varphi'(z_k)||g(z_k)|}{1-|\varphi(z_k)|^2}=\lim_{k\to\infty}\frac{(1-|z_k|^2)|\varphi'(z_k)||g(z_k)||\varphi(z_k)|}{1-|\varphi(z_k)|^2}=0. \tag{29}$$

This together with (28) imply

$$\lim_{|\varphi(z_k)|\to 1}\big(1-|z_k|^2\big)\big|g'(z_k)\big|\log\frac{1}{1-|\varphi(z_k)|^2}=0.$$

The implication follows from the last two equalities.

Conversely, assume that $C_\varphi^g:\mathcal{Z}\to\mathcal{Z}$ is bounded and (27) holds. From the proof of Theorem 5 we have

$$C_1=\sup_{z\in\mathbb{D}}\big(1-|z|^2\big)\big|g'(z)\big|<\infty,\qquad C_2=\sup_{z\in\mathbb{D}}\big(1-|z|^2\big)\big|\varphi'(z)\big|\big|g(z)\big|<\infty. \tag{30}$$

On the other hand, from (27) we have that for every $\varepsilon>0$, there is a $\delta\in(0,1)$, such that

$$\frac{(1-|z|^2)|\varphi'(z)||g(z)|}{1-|\varphi(z)|^2}<\varepsilon\quad\text{and}\quad\big(1-|z|^2\big)\big|g'(z)\big|\log\frac{1}{1-|\varphi(z)|^2}<\varepsilon, \tag{31}$$

whenever $\delta<|\varphi(z)|<1$.

Assume that $(f_k)_{k\in\mathbb{N}}$ is a sequence in $\mathcal{Z}$ such that $\sup_{k\in\mathbb{N}}\|f_k\|_{\mathcal{Z}}\leqslant L$ and $f_k$ converges to 0 uniformly on compact subsets of $\mathbb{D}$ as $k\to\infty$. Let $U=\{z\in\mathbb{D}\colon\ |\varphi(z)|\leqslant\delta\}$. Then by (30) and (31), it follows that

$$\begin{aligned}
&\sup_{z\in\mathbb{D}}\big(1-|z|^2\big)\big|\big(C_\varphi^g f_k\big)''(z)\big|\\
&\quad\leqslant\sup_{z\in U}\big(1-|z|^2\big)\big|\varphi'(z)\big|\big|g(z)\big|\big|f_k''\big(\varphi(z)\big)\big|+\sup_{z\in U}\big(1-|z|^2\big)\big|g'(z)\big|\big|f_k'\big(\varphi(z)\big)\big|\\
&\qquad+\sup_{z\in\mathbb{D}\setminus U}\big(1-|z|^2\big)\big|\varphi'(z)\big|\big|g(z)\big|\big|f_k''\big(\varphi(z)\big)\big|+\sup_{z\in\mathbb{D}\setminus U}\big(1-|z|^2\big)\big|g'(z)\big|\big|f_k'\big(\varphi(z)\big)\big|\\
&\quad\leqslant C_2\sup_{z\in U}\big|f_k''\big(\varphi(z)\big)\big|+C\sup_{z\in\mathbb{D}\setminus U}\frac{(1-|z|^2)|\varphi'(z)||g(z)|}{1-|\varphi(z)|^2}\|f_k\|_{\mathcal{Z}}\\
&\qquad+C_1\sup_{z\in U}\big|f_k'\big(\varphi(z)\big)\big|+C\sup_{z\in\mathbb{D}\setminus U}\big(1-|z|^2\big)\big|g'(z)\big|\left(\log\frac{1}{1-|\varphi(z)|^2}\right)\|f_k\|_{\mathcal{Z}}\\
&\quad\leqslant C_2\sup_{|w|\leqslant\delta}\big|f_k''(w)\big|+C_1\sup_{|w|\leqslant\delta}\big|f_k'(w)\big|+2C\varepsilon\|f_k\|_{\mathcal{Z}},
\end{aligned}$$

i.e. we obtain

$$\begin{aligned}
\big\|C_\varphi^g f_k\big\|_{\mathcal{Z}}&=\big|f_k'\big(\varphi(0)\big)\big|\big|g(0)\big|+\sup_{z\in\mathbb{D}}\big(1-|z|^2\big)\big|\big(C_\varphi^g f_k\big)''(z)\big|\\
&\leqslant C_2\sup_{|w|\leqslant\delta}\big|f_k''(w)\big|+C_1\sup_{|w|\leqslant\delta}\big|f_k'(w)\big|+2C\varepsilon\|f_k\|_{\mathcal{Z}}+\big|f_k'\big(\varphi(0)\big)\big|\big|g(0)\big|.
\end{aligned}$$

The result follows by an argument analogous to that in the proof of Theorem 2. □

## 4. The boundedness and compactness of $C_\varphi^g : \mathcal{B}^\alpha \to \mathcal{Z}$

In this section, we consider the boundedness and compactness of $C_\varphi^g : \mathcal{B}^\alpha(\mathcal{B}_0^\alpha) \to \mathcal{Z}(\mathcal{Z}_0)$. Before this, we state some useful lemmas.

**Lemma 3.** *A closed set $K$ in $\mathcal{Z}_0$ is compact if and only if it is bounded and satisfies*

$$\lim_{|z|\to 1} \sup_{f\in K} \left(1-|z|^2\right)\left|f''(z)\right| = 0.$$

**Proof.** The proof is similar to the proof of Lemma 2 in [11], hence will be omitted. □

**Lemma 4.** *Let $0<\alpha<\infty$, $g \in H(\mathbb{D})$ and $\varphi$ be an analytic self-map of $\mathbb{D}$. Suppose that $X$ is a Banach space. Then $C_\varphi^g : \mathcal{B}_0^\alpha \to X$ is compact if and only if $C_\varphi^g : \mathcal{B}_0^\alpha \to X$ is weakly compact.*

**Proof.** It is well known that the dual space of $\mathcal{B}_0^\alpha$ is isomorphic to the Bergman space $A^1$, i.e. $(\mathcal{B}_0^\alpha)^* \cong A^1$ (see [16]). Assume that $C_\varphi^g : \mathcal{B}_0^\alpha \to X$ is compact. By a well-known theorem then this is equivalent to $(C_\varphi^g)^* : X^* \to A^1$ is compact. Now recall that $A^1$ has the Schur property, that is, every weakly convergent sequence in $A^1$ is norm-convergent (see, for example, [4, Theorem 12, p. 295]). Hence, this is equivalent to $(C_\varphi^g)^* : X^* \to A^1$ is weakly compact, which is equivalent to $C_\varphi^g : \mathcal{B}_0^\alpha \to X$ is weakly compact. □

**Theorem 7.** *Let $0<\alpha<\infty$, $g \in H(\mathbb{D})$ and $\varphi$ be an analytic self-map of $\mathbb{D}$. Then the following statements are equivalent.*

(i) *$C_\varphi^g : \mathcal{B}^\alpha \to \mathcal{Z}$ is bounded*;
(ii) *$C_\varphi^g : \mathcal{B}_0^\alpha \to \mathcal{Z}$ is bounded*;
(iii)

$$\sup_{z\in\mathbb{D}} \frac{(1-|z|^2)|\varphi'(z)||g(z)|}{(1-|\varphi(z)|^2)^{\alpha+1}} < \infty \quad \textit{and} \quad \sup_{z\in\mathbb{D}} \frac{(1-|z|^2)|g'(z)|}{(1-|\varphi(z)|^2)^{\alpha}} < \infty. \tag{32}$$

**Proof.** (i) ⇒ (ii). This implication is obvious.

(ii) ⇒ (iii). Assume that $C_\varphi^g : \mathcal{B}_0^\alpha \to \mathcal{Z}$ is bounded. As in the proof of Theorem 5 we have that (19) and (21) hold. For $w \in \mathbb{D}$, set $f_w(z) = \frac{1-|w|^2}{(1-\overline{w}z)^\alpha}$. It is easy to check that $f_w \in \mathcal{B}_0^\alpha$ and $C = \sup_{w\in\mathbb{D}} \|f_w\|_{\mathcal{B}^\alpha} < \infty$. Hence, we have

$$\begin{aligned} C\left\|C_\varphi^g\right\|_{\mathcal{B}_0^\alpha\to\mathcal{Z}} &\geqslant \left\|C_\varphi^g f_{\varphi(\lambda)}\right\|_{\mathcal{Z}} \\ &\geqslant \frac{\alpha(1-|\lambda|^2)|g'(\lambda)||\varphi(\lambda)|}{(1-|\varphi(\lambda)|^2)^\alpha} - \frac{\alpha(\alpha+1)(1-|\lambda|^2)|\varphi'(\lambda)||g(\lambda)||\varphi(\lambda)|^2}{(1-|\varphi(\lambda)|^2)^{\alpha+1}}, \end{aligned}$$

that is

$$\frac{\alpha(1-|\lambda|^2)|g'(\lambda)||\varphi(\lambda)|}{(1-|\varphi(\lambda)|^2)^\alpha} \leqslant C\left\|C_\varphi^g\right\|_{\mathcal{B}_0^\alpha\to\mathcal{Z}} + \frac{\alpha(\alpha+1)(1-|\lambda|^2)|\varphi'(\lambda)||g(\lambda)||\varphi(\lambda)|^2}{(1-|\varphi(\lambda)|^2)^{\alpha+1}}. \tag{33}$$

Set

$$h_w(z) = \frac{1-|w|^2}{(1-\overline{w}z)^\alpha} - \frac{\alpha}{\alpha+1}\frac{(1-|w|^2)^2}{(1-\overline{w}z)^{\alpha+1}}.$$

Then $h_w \in \mathcal{B}_0^\alpha$, for each $w \in \mathbb{D}$ and $C := \sup_{|w|<1} \|h_w\|_{\mathcal{B}^\alpha} < \infty$. Since $h_w'(w) = 0$ and $|h_w''(w)| = (\alpha|w|^2)/(1-|w|^2)^{\alpha+1}$, we have that

$$C\left\|C_\varphi^g\right\|_{\mathcal{B}_0^\alpha\to\mathcal{Z}} \geqslant \left\|C_\varphi^g h_{\varphi(\lambda)}\right\|_{\mathcal{Z}} \geqslant \frac{\alpha(1-|\lambda|^2)|\varphi'(\lambda)||g(\lambda)||\varphi(\lambda)|^2}{(1-|\varphi(\lambda)|^2)^{\alpha+1}}. \tag{34}$$

By (34), we have

$$\sup_{|\varphi(\lambda)|>\frac{1}{2}} \frac{(1-|\lambda|^2)|\varphi'(\lambda)||g(\lambda)|}{(1-|\varphi(\lambda)|^2)^{\alpha+1}} < \sup_{|\varphi(\lambda)|>\frac{1}{2}} 4\frac{(1-|\lambda|^2)|\varphi'(\lambda)||g(\lambda)||\varphi(\lambda)|^2}{(1-|\varphi(\lambda)|^2)^{\alpha+1}} \leqslant \sup_{|\varphi(\lambda)|>\frac{1}{2}} C\left\|C_\varphi^g\right\|_{\mathcal{B}_0^\alpha\to\mathcal{Z}} < \infty. \tag{35}$$

By (21), it follows that

$$\sup_{|\varphi(\lambda)|\leqslant\frac{1}{2}} \frac{(1-|\lambda|^2)|\varphi'(\lambda)||g(\lambda)|}{(1-|\varphi(\lambda)|^2)^{\alpha+1}} \leqslant \sup_{|\varphi(\lambda)|\leqslant\frac{1}{2}} \left(\frac{4}{3}\right)^{\alpha+1}\left(1-|\lambda|^2\right)\left|\varphi'(\lambda)\right|\left|g(\lambda)\right| < \infty. \tag{36}$$

From (35) and (36) we obtain the first inequality in (32). Similarly, the second inequality in (32) follows from (19) and (33), as desired.

(iii) ⇒ (i). Assume that (32) holds. For any $f \in \mathcal{B}^\alpha$,

$$\left(1-|z|^2\right)\left|\left(C_\varphi^g f\right)''(z)\right| \leqslant \left(1-|z|^2\right)\left|\varphi'(z)\right|\left|g(z)\right|\left|f''\left(\varphi(z)\right)\right| + \left(1-|z|^2\right)\left|g'(z)\right|\left|f'\left(\varphi(z)\right)\right| \leqslant C\frac{(1-|z|^2)|\varphi'(z)||g(z)|}{(1-|\varphi(z)|^2)^{\alpha+1}}\|f\|_{\mathcal{B}^\alpha} + \frac{(1-|z|^2)|g'(z)|}{(1-|\varphi(z)|^2)^{\alpha}}\|f\|_{\mathcal{B}^\alpha}. \tag{37}$$

In addition, $|(C_\varphi^g f)'(0)| \leqslant |g(0)|(1-|\varphi(0)|^2)^{-\alpha}\|f\|_{\mathcal{B}^\alpha}$. From this and (32) it follows that $C_\varphi^g : \mathcal{B}^\alpha \to \mathcal{Z}$ is bounded. □

**Theorem 8.** *Let $0 < \alpha < \infty$, $g \in H(\mathbb{D})$ and $\varphi$ be an analytic self-map of $\mathbb{D}$. Then the following statements are equivalent.*

(i) $C_\varphi^g : \mathcal{B}^\alpha \to \mathcal{Z}$ *is compact*;
(ii) $C_\varphi^g : \mathcal{B}_0^\alpha \to \mathcal{Z}$ *is compact*;
(iii) $C_\varphi^g : \mathcal{B}^\alpha \to \mathcal{Z}$ *is bounded and*

$$\lim_{|\varphi(z)|\to 1} \frac{(1-|z|^2)|\varphi'(z)||g(z)|}{(1-|\varphi(z)|^2)^{\alpha+1}} = 0 \quad \textit{and} \quad \lim_{|\varphi(z)|\to 1} \frac{(1-|z|^2)|g'(z)|}{(1-|\varphi(z)|^2)^{\alpha}} = 0. \tag{38}$$

**Proof.** (i) ⇒ (ii). This is clear.

(ii) ⇒ (iii). From the assumption, it is clear that $C_\varphi^g : \mathcal{B}_0^\alpha \to \mathcal{Z}$ is bounded, which in view of Theorem 7 is equivalent to $C_\varphi^g : \mathcal{B}^\alpha \to \mathcal{Z}$ is bounded. Let $(z_k)_{k\in\mathbb{N}}$ be a sequence in $\mathbb{D}$ such that $|\varphi(z_k)| \to 1$ as $k \to \infty$. Let $f_k(z) = \frac{1-|\varphi(z_k)|^2}{(1-\overline{\varphi(z_k)}z)^\alpha}$, $k \in \mathbb{N}$. Then $f_k \in \mathcal{B}_0^\alpha$, $\sup_{k\in\mathbb{N}}\|f_k\|_{\mathcal{B}^\alpha} < \infty$ and $f_k \to 0$ uniformly on compact subsets of $\mathbb{D}$ as $k \to \infty$. Since $C_\varphi^g : \mathcal{B}_0^\alpha \to \mathcal{Z}$ is compact, it gives $\lim_{k\to\infty}\|C_\varphi^g f_k\|_{\mathcal{Z}} = 0$. We have

$$\left\|C_\varphi^g f_k\right\|_{\mathcal{Z}} \geqslant \left|\frac{\alpha(\alpha+1)(1-|z_k|^2)|\varphi'(z_k)||g(z_k)||\varphi(z_k)|^2}{(1-|\varphi(z_k)|^2)^{\alpha+1}} - \frac{\alpha(1-|z_k|^2)|g'(z_k)||\varphi(z_k)|}{(1-|\varphi(z_k)|^2)^{\alpha}}\right|,$$

and consequently

$$\lim_{|\varphi(z_k)|\to 1} \frac{(\alpha+1)(1-|z_k|^2)|\varphi'(z_k)||g(z_k)||\varphi(z_k)|}{(1-|\varphi(z_k)|^2)^{\alpha+1}} = \lim_{|\varphi(z_k)|\to 1} \frac{(1-|z_k|^2)|g'(z_k)|}{(1-|\varphi(z_k)|^2)^{\alpha}} \tag{39}$$

if one of these two limits exists.

Next, set

$$h_k(z) = \frac{1-|\varphi(z_k)|^2}{(1-\overline{\varphi(z_k)}z)^\alpha} - \frac{\alpha}{\alpha+1}\frac{(1-|\varphi(z_k)|^2)^2}{(1-\overline{\varphi(z_k)}z)^{\alpha+1}}. \tag{40}$$

Then $h_k \in \mathcal{B}_0^\alpha$, $\sup_{k\in\mathbb{N}}\|h_k\|_{\mathcal{B}^\alpha} \leqslant C$ and $h_k$ converges to 0 uniformly on compact subsets of $\mathbb{D}$ as $k \to \infty$. Since $C_\varphi^g : \mathcal{B}_0^\alpha \to \mathcal{Z}$ is compact, we have $\lim_{k\to\infty}\|C_\varphi^g h_k\|_{\mathcal{Z}} = 0$. On the other hand, we have that

$$\frac{\alpha(1-|z_k|^2)|\varphi'(z_k)||g(z_k)||\varphi(z_k)|^2}{(1-|\varphi(z_k)|^2)^{\alpha+1}} \leqslant \left\|C_\varphi^g h_k\right\|_{\mathcal{Z}}.$$

Therefore

$$\lim_{|\varphi(z_k)|\to 1}\frac{(1-|z_k|^2)|\varphi'(z_k)||g(z_k)|}{(1-|\varphi(z_k)|^2)^{\alpha+1}} = \lim_{k\to\infty}\frac{(1-|z_k|^2)|\varphi'(z_k)||g(z_k)||\varphi(z_k)|^2}{(1-|\varphi(z_k)|^2)^{\alpha+1}} = 0. \tag{41}$$

This together with (39) imply

$$\lim_{|\varphi(z_k)|\to 1}\frac{(1-|z_k|^2)|g'(z_k)|}{(1-|\varphi(z_k)|^2)^{\alpha}} = 0.$$

The result follows from the last two equalities.

(iii) ⇒ (i). The proof is similar to the proof of Theorem 6, hence we omit it. □

**Theorem 9.** *Let $0 < \alpha < \infty$, $g \in H(\mathbb{D})$ and $\varphi$ be an analytic self-map of $\mathbb{D}$. Then $C_\varphi^g : \mathcal{B}_0^\alpha \to \mathcal{Z}_0$ is bounded if and only if $C_\varphi^g : \mathcal{B}_0^\alpha \to \mathcal{Z}$ is bounded and*

$$\lim_{|z|\to 1}\left(1-|z|^2\right)\left|\varphi'(z)g(z)\right| = 0 \quad \textit{and} \quad \lim_{|z|\to 1}\left(1-|z|^2\right)\left|g'(z)\right| = 0. \tag{42}$$

**Proof.** Suppose that $C_\varphi^g : \mathcal{B}_0^\alpha \to \mathcal{Z}_0$ is bounded, then $C_\varphi^g : \mathcal{B}_0^\alpha \to \mathcal{Z}$ is bounded. By taking the functions given by $f(z) = z$ and $f(z) = z^2$, and employing the boundedness of $C_\varphi^g : \mathcal{B}_0^\alpha \to \mathcal{Z}_0$, (42) follows.

Conversely, assume that $C_\varphi^g : \mathcal{B}_0^\alpha \to \mathcal{Z}$ is bounded and (42) holds. Then for each polynomial $p(z)$, we have

$$\begin{aligned}\left(1-|z|^2\right)\left|\left(C_\varphi^g p\right)''(z)\right| &\leqslant \left(1-|z|^2\right)\left|p'\left(\varphi(z)\right)\right|\left|g'(z)\right| + \left(1-|z|^2\right)\left|p''\left(\varphi(z)\right)\right|\left|\varphi'(z)g(z)\right| \\ &\leqslant \|p'\|_\infty\left(1-|z|^2\right)\left|g'(z)\right| + \|p''\|_\infty\left(1-|z|^2\right)\left|\varphi'(z)g(z)\right|.\end{aligned}$$

From (42) it follows that for each polynomial $p$, $C_\varphi^g(p) \in \mathcal{Z}_0$. The set of all polynomials is dense in $\mathcal{B}_0^\alpha$, thus for every $f \in \mathcal{B}_0^\alpha$ there is a sequence of polynomials $(p_k)_{k\in\mathbb{N}}$ such that $\|p_k - f\|_{\mathcal{B}^\alpha} \to 0$ as $k \to \infty$. In view of Theorem 7, the operator $C_\varphi^g : \mathcal{B}^\alpha \to \mathcal{Z}$ is bounded, thus it follows that

$$\left\|C_\varphi^g p_k - C_\varphi^g f\right\|_{\mathcal{Z}} \leqslant \left\|C_\varphi^g\right\|\|p_k - f\|_{\mathcal{B}^\alpha} \to 0, \quad \text{as } k \to \infty.$$

Hence $C_\varphi^g(\mathcal{B}_0^\alpha) \subset \mathcal{Z}_0$, since $\mathcal{Z}_0$ is a closed subset of $\mathcal{Z}$. □

**Theorem 10.** *Let $0 < \alpha < \infty$, $g \in H(\mathbb{D})$ and $\varphi$ be an analytic self-map of $\mathbb{D}$. Then the following statements are equivalent.*

(i) $C_\varphi^g : \mathcal{B}^\alpha \to \mathcal{Z}_0$ *is bounded*;

(ii) $C_\varphi^g : \mathcal{B}_0^\alpha \to \mathcal{Z}_0$ *is compact*;

(iii) $C_\varphi^g : \mathcal{B}_0^\alpha \to \mathcal{Z}_0$ *is weakly compact*;

(iv) $C_\varphi^g : \mathcal{B}^\alpha \to \mathcal{Z}_0$ *is compact*;

(v)

$$\lim_{|z|\to 1}\frac{(1-|z|^2)|\varphi'(z)||g(z)|}{(1-|\varphi(z)|^2)^{\alpha+1}} = 0 \quad \textit{and} \quad \lim_{|z|\to 1}\frac{(1-|z|^2)|g'(z)|}{(1-|\varphi(z)|^2)^{\alpha}} = 0. \tag{43}$$

**Proof.** (i) ⇒ (ii). Clearly $C_\varphi^g : \mathcal{B}_0^\alpha \to \mathcal{Z}_0$ is bounded. By using standard duality arguments and the fact that $\mathcal{B}_0^\alpha$ is weak-star dense in $\mathcal{B}^\alpha$, it follows that $C_\varphi^g = (C_\varphi^g)^{**}$ on $\mathcal{B}^\alpha$. Hence $(C_\varphi^g)^{**}(\mathcal{B}_0^{**}) = C_\varphi^g(\mathcal{B}^\alpha) \subset \mathcal{Z}_0$. By Gantmacher's Theorem it follows that $C_\varphi^g : \mathcal{B}_0^\alpha \to \mathcal{Z}_0$ is compact.

(iv) ⇒ (i). This implication is obvious.

(ii) ⇔ (iii). It follows from Lemma 4.

(v) ⇒ (iv). Taking the supremum in (37) over all $f \in \mathcal{B}^\alpha$ such that $\|f\|_{\mathcal{B}^\alpha} \leqslant 1$, then letting $|z| \to 1$, we obtain

$$\lim_{|z|\to 1} \sup_{\|f\|_{\mathcal{B}^\alpha}\leqslant 1} \big(1-|z|^2\big)\big|\big(C_\varphi^g f\big)''(z)\big| = 0.$$

From this and Lemma 3 it follows that $C_\varphi^g : \mathcal{B}^\alpha \to \mathcal{Z}_0$ is compact.

(iv) ⇒ (ii). This is obvious.

(ii) ⇒ (v). Clearly $C_\varphi^g : \mathcal{B}_0^\alpha \to \mathcal{Z}$ is compact and hence (38) holds. Since $C_\varphi^g : \mathcal{B}_0^\alpha \to \mathcal{Z}_0$ is compact, then it is clear that $C_\varphi^g : \mathcal{B}_0^\alpha \to \mathcal{Z}_0$ is bounded. By taking $f(z) = z$ and $f(z) = z^2$, we obtain (42). From (42) and (38), similar to the proof of Theorem 3, we obtain (43), as desired. □

## 5. The boundedness and compactness of $C_\varphi^g : \mathcal{B}^\alpha \to \mathcal{B}^\beta$

By modifying the proofs of Theorems 7–10, we can obtain the following results. The proofs of the following theorems will be omitted.

**Theorem 11.** *Let $0 < \alpha, \beta < \infty$, $g \in H(\mathbb{D})$ and $\varphi$ be an analytic self-map of $\mathbb{D}$. Then the following statements are equivalent.*

(i) $C_\varphi^g : \mathcal{B}^\alpha \to \mathcal{B}^\beta$ *is bounded*;

(ii) $C_\varphi^g : \mathcal{B}_0^\alpha \to \mathcal{B}^\beta$ *is bounded*;

(iii)

$$\sup_{z\in\mathbb{D}} \frac{(1-|z|^2)^\beta |g(z)|}{(1-|\varphi(z)|^2)^\alpha} < \infty. \tag{44}$$

**Theorem 12.** *Let $0 < \alpha, \beta < \infty$, $g \in H(\mathbb{D})$ and $\varphi$ be an analytic self-map of $\mathbb{D}$. Then the following statements are equivalent.*

(i) $C_\varphi^g : \mathcal{B}^\alpha \to \mathcal{B}^\beta$ *is compact*;

(ii) $C_\varphi^g : \mathcal{B}_0^\alpha \to \mathcal{B}^\beta$ *is compact*;

(iii) $C_\varphi^g : \mathcal{B}^\alpha \to \mathcal{B}^\beta$ *is bounded and*

$$\lim_{|\varphi(z)|\to 1} \frac{(1-|z|^2)^\beta |g(z)|}{(1-|\varphi(z)|^2)^\alpha} = 0. \tag{45}$$

**Theorem 13.** *Let $0 < \alpha, \beta < \infty$, $g \in H(\mathbb{D})$ and $\varphi$ be an analytic self-map of $\mathbb{D}$. Then $C_\varphi^g : \mathcal{B}_0^\alpha \to \mathcal{B}_0^\beta$ is bounded if and only if $C_\varphi^g : \mathcal{B}_0^\alpha \to \mathcal{B}^\beta$ is bounded and*

$$\lim_{|z|\to 1} \big(1-|z|^2\big)^\beta \big|g(z)\big| = 0. \tag{46}$$

**Theorem 14.** *Let $0 < \alpha, \beta < \infty$, $g \in H(\mathbb{D})$ and $\varphi$ be an analytic self-map of $\mathbb{D}$. Then the following statements are equivalent.*

(i) $C_\varphi^g : \mathcal{B}^\alpha \to \mathcal{B}_0^\beta$ *is bounded*;

(ii) $C_\varphi^g : \mathcal{B}_0^\alpha \to \mathcal{B}_0^\beta$ *is compact*;

(iii) $C_\varphi^g : \mathcal{B}_0^\alpha \to \mathcal{B}_0^\beta$ *is weakly compact*;

(iv) $C_\varphi^g : \mathcal{B}^\alpha \to \mathcal{B}_0^\beta$ *is compact*;

(v)

$$\lim_{|z|\to 1} \frac{(1-|z|^2)^\beta |g(z)|}{(1-|\varphi(z)|^2)^\alpha} = 0. \tag{47}$$

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## References

[1] B. Boe, A. Nikolau, Interpolation by functions in the Bloch space, J. Anal. Math. 94 (2004) 171–194.
[2] B. Choe, H. Koo, W. Smith, Composition operators on small spaces, Integral Equations Operator Theory 56 (2006) 357–380.
[3] C. Cowen, B. MacCluer, Composition Operators on Spaces of Analytic Functions, Stud. Adv. Math., CRC Press, Boca Raton, 1995.
[4] N. Danford, J.T. Schwartz, Linear Operators I, Interscience Publishers, John Willey and Sons, New York, 1958.
[5] P. Duren, Theory of $H^p$ Spaces, Academic Press, New York, 1973.
[6] Z. Hu, Extended Cesàro operators on the Bloch space in the unit ball of $\mathbb{C}^n$, Acta Math. Sci. Ser. B Engl. Ed. 23 (4) (2003) 561–566.
[7] W. Hornor, J.E. Jamison, Isometries of some Banach spaces of analytic functions, Integral Equations Operator Theory 41 (2001) 401–425.
[8] S. Li, S. Stević, Volterra-type operators on Zygmund spaces, J. Ineq. Appl. 2007 (2007), Article ID 32124, 10 pp., doi:10.1155/2007/32124.
[9] Z. Lou, Composition operators on Bloch type spaces, Analysis (Munich) 23 (1) (2003) 81–95.
[10] K. Madigan, A. Matheson, Compact composition operators on the Bloch space, Trans. Amer. Math. Soc. 347 (7) (1995) 2679–2687.
[11] S. Ohno, K. Stroethoff, R. Zhao, Weithted composition operators between Bloch type spaces, Rocky Mountain J. Math. 33 (2003) 191–215.
[12] J. Shapiro, Composition Operators and Classical Function Theory, Springer-Verlag, New York, 1993.
[13] S. Stević, On an integral operator on the unit ball in $\mathbb{C}^n$, J. Inequal. Appl. 2005 (1) (2005) 81–88.
[14] J. Xiao, Riemann–Stieltjes operators on weighted Bloch and Bergman spaces of the unit ball, J. London Math. Soc. 70 (2) (2004) 199–214.
[15] J. Xiao, Composition operators associated with Bloch-type spaces, Complex Var. Theory Appl. 46 (2001) 109–121.
[16] K. Zhu, Bloch type spaces of analytic functions, Rocky Mountain J. Math. 23 (3) (1993) 1143–1177.
[17] K. Zhu, Operator Theory in Function Spaces, Marcel Dekker, New York, 1990.