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Disk of convexity of sections of univalent harmonic functions

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ABSTRACT

One of the classical results from Szegö shows that if $h(z) = z + \sum_{n=2}^{\infty} a_n z^n$ is analytic and univalent in the unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$, then the section $s_n(h)(z) = \sum_{k=1}^{n} a_k z^k$ of h is univalent in $|z| < 1/4$. The exact (largest) radius of the univalence r_n of $s_n(h)$ remains an open problem. On the other hand, not much is known in the case of harmonic univalent functions. It is then natural to consider the class $\mathcal{P}_H^0$ of normalized harmonic mappings $f = h + \overline{g}$ in the unit disk $\mathbb{D}$ satisfying the condition $\mathrm{Re}\, h'(z) > |g'(z)|$ for $z \in \mathbb{D}$, where $g'(0) = 0$. Functions in $\mathcal{P}_H^0$ are known to be univalent and close-to-convex in $\mathbb{D}$. In this paper, we first show that each $f \in \mathcal{P}_H^0$ is convex in the disk $|z| < \sqrt{2} - 1$, and then determine the value of r so that the partial sums of $f \in \mathcal{P}_H^0$ are convex in $|z| < r$.

1. Introduction

For $r > 0$, let $\mathbb{D}_r := \{z \in \mathbb{C} : |z| < r\}$ and $\mathbb{D} := \mathbb{D}_1$, the open unit disk. One of the classical results of Szegö [25] (see also [6, Theorem 8.5]) shows that if

$$h(z) = \sum_{n=1}^{\infty} a_n z^n \tag{1}$$

is analytic and univalent in $\mathbb{D}$ (with $a_1 = 1$ so that h is called normalized), denoted by $h \in \mathcal{S}$, then the n-th *partial sums*

$$s_n(h)(z) := \sum_{k=1}^{n} a_k z^k$$

of h is univalent in $|z| < 1/4$ and the number $1/4$ cannot be replaced by a larger one as the section $s_2(k)(z) = z + 2z^2$ of the Koebe function $k(z) = z/(1-z)^2$ shows. A function $h \in \mathcal{S}$ is called starlike in $\mathbb{D}$ if $h(\mathbb{D})$ is a domain that is starlike (with respect to the origin). Similar comments apply in a general context whenever we say a univalent function is convex (resp. close-to-convex) in $\mathbb{D}$, see [6]. The largest radius of univalence r_n of $s_n(h)(z)$ $(h \in \mathcal{S})$ is not yet known. However, Jenkins [10] (see also [6, Section 8.2]) observed that

$$r_n \geq 1 - (4 + \epsilon) n^{-1} \log n$$

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for each $\epsilon > 0$ and for large n. Although the Koebe function $k(z)$ is univalent in the disk $|z| < 1 - 3n^{-1}\log n$ and constant 3 cannot be made smaller, Bshouty and Hengartner [2, p. 408] observed that the Koebe function is not extremal for the problem of determining the radius of univalency of the partial sums of $f \in \mathcal{S}$. On the other hand, the radius of starlikeness of $s_n(h)(z)$ for starlike function h was proved by Robertson [20]. Later, Ruscheweyh [22] proved a stronger result by showing that the partial sums $s_n(h)(z)$ are indeed starlike in $|z| < 1/4$ not only for functions h in the class $\mathcal{S}$ but also for the closed convex hull of $\mathcal{S}$. The reader is referred to [16,17,20,22,23] for many interesting results of this type.

Let $\mathcal{P}$ be the class of all analytic functions h in the unit disk $\mathbb{D}$ such that $h(0) = h'(0) - 1 = 0$ and $\operatorname{Re} h'(z) > 0$ in $\mathbb{D}$. It is well-known that $\mathcal{P} \subsetneq \mathcal{S}$.

Theorem A. *Let $h \in \mathcal{P}$. Then we have the following:*

(a) *each partial sum $s_n(h)$ is univalent in $|z| < 1/2$,*
(b) *$h(z)$ maps the disk $|z| < \sqrt{2} - 1$ onto a convex domain,*
(c) *each partial sum $s_n(h)$ is convex in $|z| < 1/4$.*

The numbers $1/2$, $\sqrt{2}-1$ and $1/4$ are best possible constants, and each of these numbers cannot be replaced by a greater number.

The items (a) and (b) are due to MacGregor [15] whereas (c) is obtained by Ram Singh [24]. Our aim of this article is to discuss a harmonic analog of Theorem A.

Let $\mathcal{H}$ consist of all complex-valued harmonic functions $f = h + \overline{g}$ defined on $\mathbb{D}$, where h and g are analytic on $\mathbb{D}$ such that $h(0) = 0 = h'(0) - 1$ and $g(0) = 0$. A necessary and sufficient condition for $f = h + \overline{g} \in \mathcal{H}$ to be locally univalent and sense-preserving in $\mathbb{D}$ is that the Jacobian $J_f(z)$ is positive in $\mathbb{D}$, where

$$J_f(z) = |f_z(z)|^2 - |f_{\bar{z}}(z)|^2 = |h'(z)|^2 - |g'(z)|^2.$$

Also, let $\mathcal{H}_0 = \{f = h + \overline{g} \in \mathcal{H} : g'(0) = 0\}$. Following the results of Clunie and Sheil-Small [3], let $\mathcal{S}_H$ denote the subclass of functions from $\mathcal{H}$ that are sense-preserving and univalent in $\mathbb{D}$, and further set

$$\mathcal{S}_H^0 = \mathcal{S}_H \cap \mathcal{H}_0.$$

Clearly, each $f = h + \overline{g} \in \mathcal{S}_H^0$ has the form

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad \text{and} \quad g(z) = \sum_{n=2}^{\infty} b_n z^n. \tag{2}$$

Recall that both $\mathcal{S}_H$ and $\mathcal{S}_H^0$ coincide with $\mathcal{S}$ whenever the co-analytic part $g(z)$ is identically zero. Thus, both $\mathcal{S}_H$ and $\mathcal{S}_H^0$ are natural harmonic generalizations of $\mathcal{S}$, but only $\mathcal{S}_H^0$ is known to be compact although both $\mathcal{S}_H$ and $\mathcal{S}_H^0$ are normal. Important geometric subclasses of $\mathcal{S}_H$ and $\mathcal{S}_H^0$ such as convex, close-to-convex, starlikeness and typically real harmonic functions are discussed by Clunie and Sheil-Small [3]. For many interesting results and expositions on planar harmonic univalent mappings, we refer to the book by Duren [7]. However, the class $\mathcal{S}_H^0$ is the central object in the study of harmonic univalent mappings. On the other hand, not much is known in the literature concerning a harmonic analog of the results on $\mathcal{S}$ of Szegö [25]. Only recently the present authors in [12,13] have initiated some discussion on this topic. Therefore, it is natural to consider for example the harmonic analog of the class $\mathcal{P}$ and discuss corresponding results for their sections. As a harmonic analog of $\mathcal{P}$, the authors in [11,18] considered the classes

$$\mathcal{P}_H = \{f = h + \overline{g} \in \mathcal{H} : \operatorname{Re} h'(z) > |g'(z)| \text{ for } z \in \mathbb{D}\} \quad \text{and} \quad \mathcal{P}_H^0 = \mathcal{P}_H \cap \mathcal{H}_0.$$

Note that $\mathcal{P} = \{f = h + \overline{g} \in \mathcal{P}_H : g(z) \equiv 0\}$. In [18], it was proved that functions in $\mathcal{P}_H$ are close-to-convex in $\mathbb{D}$, see also [1].

For $f = h + \overline{g} \in \mathcal{H}_0$, where h and g are in the form (2), we define the sequences of partial sums of h, g and f as follows:

$$s_p(h)(z) = \sum_{k=1}^{p} a_k z^k, \qquad s_q(g)(z) = \sum_{k=2}^{q} b_k z^k, \qquad s_{p,q}(f) = s_p(h) + \overline{s_q(g)},$$

where $a_1 = 1$, $p \geq 1$ and $q \geq 2$. In [12], the present authors discussed the radius of univalency of sections of functions $f \in \mathcal{P}_H^0$.

Theorem B ([12, Theorems 4–6]). *Let $h \in \mathcal{P}_H^0$. Suppose that p and q satisfy any one of the following conditions:*

(a) *$p = 1$ and $q \geq 2$,*
(b) *$3 \leq p < q$,*
(c) *$p = q \geq 2$,*
(d) *$p > q \geq 3$,*
(e) *$p = 3$ and $q = 2$.*

Then $s_{p,q}(f)$ is univalent in $|z| < 1/2$. Moreover, we have

(f) *for $2 < q$, $s_{2,q}(f)$ is univalent in $|z| < \frac{3-\sqrt{5}}{2} \approx 0.381966$.*
(g) *for $p \geq 4$, $s_{p,2}(f)$ is univalent in $|z| < 0.433797$.*

In this paper, we first show that each $f \in \mathcal{P}_H^0$ is convex in the disk $|z| < \sqrt{2}-1$ and then determine the value of r so that the partial sums of $f \in \mathcal{P}_H^0$ are convex in $|z| < r$. Finally, we give upper bounds for the area of the images of $|z| < r$ under $f \in \mathcal{P}_H^0$, and for the lengths of the images of the circles $|z| = r$ under $f \in \mathcal{P}_H^0$, where $0 < r < 1$.

2. The radius of convexity

Definition 1. A domain $D \subset \mathbb{C}$ is called convex in the direction α $(0 \le \alpha < \pi)$ if every line parallel to the line through 0 and $e^{i\alpha}$ has a connected intersection with D. A univalent harmonic function f in $\mathbb{D}$ is said to be *convex in the direction* α if $f(\mathbb{D})$ is convex in the direction α.

Clearly, a convex function is convex in every direction. The class of functions convex in one direction has been studied by many mathematicians (see, for example, [4,5,9,21]) as a subclass of functions introduced by Robertson [19]. The following result helps to construct univalent harmonic functions, each of which maps $\mathbb{D}$ onto a domain that is convex in the direction α.

Lemma C (*Method of Shearing*)**.** *A harmonic function $f = h+\overline{g}$ locally univalent in $\mathbb{D}$ is a univalent mapping of $\mathbb{D}$ onto a domain convex in the direction α $(0 \le \alpha < \pi)$ if and only if $h - e^{2i\alpha}g$ is a conformal univalent mapping of $\mathbb{D}$ onto a domain convex in the direction α.*

Theorem 1. *The radius of convexity for the class $\mathcal{P}_H^0$ is $\sqrt{2}-1$. Moreover, the bound $\sqrt{2}-1$ is sharp.*

Proof. Let $f = h+\overline{g} \in \mathcal{P}_H^0$. Then for each $0 \le \theta < 2\pi$, we have

$$\mathrm{Re}\left(h'(z) - e^{-2i\theta}g'(z)\right) \ge \mathrm{Re}\, h'(z) - |g'(z)| > 0, \quad z \in \mathbb{D},$$

and thus, $h - e^{-2i\theta}g \in \mathcal{P}$. It follows from Theorem A(b) that $h - e^{-2i\theta}g$ is convex in the disk $|z| < \sqrt{2}-1$ and hence $e^{i\theta}h - e^{-i\theta}g$ is convex in $|z| < \sqrt{2}-1$ for each $\theta \in [0, 2\pi)$. Therefore, $e^{i\theta}h - e^{-i\theta}g$ is convex in the real axis in $|z| < \sqrt{2}-1$ for each $0 \le \theta < 2\pi$. By Lemma C, we conclude that f is convex in $|z| < \sqrt{2}-1$. The extremal is attained by $f(z) = -z - 2\log(1-z)$. □

3. The convexity of sections

Throughout this section we consider $f = h+\overline{g} \in \mathcal{P}_H^0$, where h and g are in the form (2).

Lemma D (*[12]*)**.** *Let $f = h+\overline{g} \in \mathcal{P}_H^0$. Then for each $n \ge 2$, we have*

(1) $|a_n| + |b_n| \le \frac{2}{n}$;
(2) $||a_n| - |b_n|| \le \frac{2}{n}$;
(3) $|a_n| \le \frac{2}{n}$;
(4) $|b_n| \le \frac{1}{n}$.

Note that Lemma D(1) implies both Lemma D(2) and (3).

Theorem 2. *Let $f = h+\overline{g} \in \mathcal{P}_H^0$. Then for each $q \ge 2$, $s_{1,q}(f)$ is convex in $|z| < 1/4$.*

Proof. By assumption, we know that

$$s_{1,q}(f)(z) = z + \overline{s_q(g)(z)} = z + \sum_{k=2}^{q} \overline{b_k z^k}.$$

Since

$$\mathrm{Re}\frac{z + z\overline{\left(zs_q'(g_0)(z)\right)'}}{z - \overline{zs_q'(g_0)(z)}} = \mathrm{Re}\frac{z + \sum_{k=2}^{q}\overline{k^2 b_k z^k}}{z - \sum_{k=2}^{q}\overline{kb_k z^k}} \quad \text{and} \quad \lim_{z\to 0}\frac{z + \sum_{k=2}^{q}\overline{k^2 b_k z^k}}{z - \sum_{k=2}^{q}\overline{kb_k z^k}} = 1,$$

it suffices to prove

$$A =: \mathrm{Re}\left\{\left(z + \sum_{k=2}^{q}\overline{k^2 b_k z^k}\right)\left(\bar{z} - \sum_{k=2}^{q}kb_k z^k\right)\right\} > 0 \quad \text{for } |z| = \frac{1}{4}.$$

Now, we find that

$$\begin{aligned} A &= |z|^2 + \mathrm{Re}\left(\sum_{k=2}^{q} \overline{k^2 b_k z^{k+1}} - \sum_{k=2}^{q} k b_k z^{k+1}\right) - \mathrm{Re}\left\{\left(\sum_{k=2}^{q} \overline{k^2 b_k z^k}\right)\left(\sum_{k=2}^{q} k b_k z^k\right)\right\} \\ &\geq |z|^2 - \sum_{k=2}^{q}(k^2-k)\left|b_k z^{k+1}\right| - \left(\sum_{k=2}^{q} k^2 \left|b_k z^k\right|\right)\left(\sum_{k=2}^{q} k\left|b_k z^k\right|\right) \\ &\geq |z|^2 - \sum_{k=2}^{q}(k-1)|z|^{k+1} - \left(\sum_{k=2}^{q} k|z|^k\right)\left(\sum_{k=2}^{q} |z|^k\right) \quad \text{(by Lemma D)} \\ &= |z|^2 - |z|^3 \frac{1-q|z|^{q-1}+(q-1)|z|^q}{(1-|z|)^2} - |z|^4 \frac{(2-|z|-(q+1)|z|^{q-1}+q|z|^q)(1-|z|^{q-1})}{(1-|z|)^3}. \end{aligned}$$

Thus, for $|z| = 1/4$, we have

$$\begin{aligned} \frac{A(1-|z|)^3}{|z|^2} &\geq (1-|z|)^3 - |z|(1-|z|)\left(1-q|z|^{q-1}+(q-1)|z|^q\right) \\ &\quad - |z|^2\left(2-|z|-(q+3)|z|^{q-1}+(q+1)|z|^q+(q+1)|z|^{2q-2}-q|z|^{2q-1}\right) \\ &= \frac{27}{64} - \frac{3}{16}\left(1-\frac{q}{4^{q-1}}+\frac{q-1}{4^q}\right) - \frac{1}{16}\left(\frac{7}{4}-\frac{q+3}{4^{q-1}}+\frac{q+1}{4^q}+\frac{q+1}{4^{2q-2}}-\frac{q}{4^{2q-1}}\right) \\ &= \frac{1}{8} + \frac{4q+3}{4^{q+1}} - \frac{4q-2}{4^{q+2}} - \frac{q+1}{4^{2q}} + \frac{q}{4^{2q+1}} \\ &= \frac{1}{8} + \frac{12q+14}{4^{q+2}} - \frac{3q+4}{4^{2q+1}} \\ &= \frac{1}{8} + \frac{12q(4^q-1)+(14\times 4^q-16)}{4^{2q+2}} > 0. \end{aligned}$$

The result now follows. □

We need the following well-known result for functions with positive real part, [15] (see also [8, Theorem 4 on pp. 81–82 of Vol. 1]).

Lemma E. *Let $p \in \mathcal{P}$. Then we have the following:*

$$|p'(z)| \geq \frac{1-|z|}{1+|z|} \quad \text{and} \quad \left|\frac{p''(z)}{p'(z)}\right| \leq \frac{2}{1-|z|^2}, \quad z \in \mathbb{D}.$$

These inequalities are sharp. Equality occurs for suitable z if and only if

$$p(z) = -z - 2e^{-i\alpha}\log(1-ze^{i\alpha}).$$

Theorem 3. *Let $f = h + \overline{g} \in \mathcal{P}_H^0$, and suppose that p and q satisfy one of the following conditions:*

(1) $3 \leq p < q$,
(2) $p = q \geq 2$,
(3) $p > q \geq 3$.

Then $s_{p,q}(f)$ is convex in $|z| < 1/4$.

Proof. Let $\sigma_p(h)(z) = \sum_{k=p+1}^{\infty} a_k z^k$ and $\sigma_q(g) = \sum_{k=q+1}^{\infty} b_k z^k$ so that

$$h(z) = s_p(h)(z) + \sigma_p(h)(z) \quad \text{and} \quad g(z) = s_q(g)(z) + \sigma_q(g)(z).$$

Thus, for each $|\varepsilon| = 1$, we may write

$$1 + z\frac{s_p''(h)(z) + \varepsilon s_q''(g)(z)}{s_p'(h)(z) + \varepsilon s_q'(g)(z)} = 1 + B + C, \tag{3}$$

where

$$B = z\frac{h''(z) + \varepsilon g''(z)}{h'(z) + \varepsilon g'(z)},$$

and

$$C=\frac{B\left(\sigma_p'(h)(z)+\varepsilon\sigma_q'(g)(z)\right)-z\left(\sigma_p''(h)(z)+\varepsilon\sigma_q''(g)(z)\right)}{h'(z)+\varepsilon g'(z)-\left(\sigma_p'(h)(z)+\varepsilon\sigma_q'(g)(z)\right)}.$$

Since $h+\varepsilon g\in\mathcal{P}$, Lemma E shows that

$$|B|\leq\frac{2|z|}{1-|z|^2}\quad\text{and}\quad\left|h'(z)+\varepsilon g'(z)\right|\geq\frac{1-|z|}{1+|z|}.\tag{4}$$

If $p\leq q$, then Lemma D yields that

$$\begin{aligned}\left|\sigma_p'(h)(z)+\varepsilon\sigma_q'(g)(z)\right|&=\left|\sum_{k=p+1}^{q}ka_kz^{k-1}+\sum_{k=q+1}^{\infty}k(a_k+\varepsilon b_k)z^{k-1}\right|\\&\leq 2\sum_{k=p+1}^{\infty}|z|^{k-1}\leq 2\frac{|z|^p}{1-|z|}.\end{aligned}\tag{5}$$

Similarly,

$$\begin{aligned}\left|z\left(\sigma_p''(h)(z)+\varepsilon\sigma_q''(g)(z)\right)\right|&\leq\left|\sum_{k=p+1}^{q}k(k-1)a_kz^{k-1}+\sum_{k=q+1}^{\infty}k(k-1)(a_k+\varepsilon b_k)z^{k-1}\right|\\&\leq 2\sum_{k=p+1}^{q}(k-1)|z|^{k-1}+2\sum_{k=q+1}^{\infty}(k-1)|z|^{k-1}\\&=2\sum_{k=p+1}^{\infty}(k-1)|z|^{k-1}\\&\leq 2\left(\frac{p|z|^p}{1-|z|}+\frac{|z|^{p+1}}{(1-|z|)^2}\right).\end{aligned}\tag{6}$$

Using the estimates (4)–(6), we deduce by the triangle inequality that

$$\begin{aligned}|C|&\leq\frac{\frac{4|z|^{p+1}}{(1-|z|)^2(1+|z|)}+2\frac{p|z|^p}{1-|z|}+2\frac{|z|^{p+1}}{(1-|z|)^2}}{\frac{1-|z|}{1+|z|}-2\frac{|z|^p}{1-|z|}}\\&\leq\frac{1}{1-|z|}\left(\frac{2|z|^p\left(3|z|+p(1-|z|^2)+|z|^2\right)}{(1-|z|)^2-2|z|^p(1+|z|)}\right).\end{aligned}$$

Thus, by (4) and the last inequality, we see that

$$\begin{aligned}\operatorname{Re}(1+B+C)&\geq 1-|B|-|C|\\&\geq\frac{1}{1-|z|}\left(\frac{1-|z|^2-2|z|}{1+|z|}-\frac{2|z|^p\left(3|z|+p(1-|z|^2)+|z|^2\right)}{1-2|z|+|z|^2-2|z|^p-2|z|^{p+1}}\right)\end{aligned}$$

which for $|z|=1/4$ gives that

$$\operatorname{Re}(1+B+C)\geq\frac{4}{3}\left(\frac{7}{20}-\frac{15p+13}{18\times 4^{p-1}-20}\right)=:A(p).$$

Since the function $A(p)$ monotonically increases with respect to p for $p\geq 3$, the last estimate shows that

$$\operatorname{Re}(1+B+C)\geq A(p)\geq A(3)\geq\frac{4}{3}\left(\frac{7}{20}-\frac{45+13}{18\times 4^2-20}\right)>0.$$

The last relation and (3) imply that for each $|\varepsilon|=1$, the section $s_p(h)+\varepsilon s_q(g)$ is convex in $|z|<1/4$ when $3\leq p\leq q$. The arbitrariness of ε shows that each $s_{p,q}(f)$ is convex in $|z|<1/4$ whenever $3\leq p\leq q$.

In the case $p=q\geq 2$, it follows from the proof of Theorem 1 that for each $0\leq\theta<2\pi$, $h-e^{-2i\theta}g\in\mathcal{P}$ and hence, Theorem A(c) implies that the function

$$s_p(h-e^{-2i\theta}g)=s_p(h)-e^{-2i\theta}s_p(g)$$

is convex in $|z|<1/4$. The arbitrariness of θ and Lemma C yield that $s_{p,p}(f)$ is convex in $|z|<1/4$ for $p\geq 2$.

If $p > q$, then Lemma D yields that

$$\begin{aligned}\left|\sigma_p'(h)(z) + \varepsilon\sigma_q'(g)(z)\right| &= \left|\sum_{k=q+1}^{p} \varepsilon k b_k z^{k-1} + \sum_{k=p+1}^{\infty} k(a_k + \varepsilon b_k) z^{k-1}\right| \\ &\le \sum_{k=q+1}^{p} |z|^{k-1} + 2\sum_{k=p+1}^{\infty} |z|^{k-1} \\ &\le \frac{|z|^p + |z|^q}{1-|z|}, \end{aligned} \tag{7}$$

and

$$\begin{aligned}\left|z\sigma_p''(h)(z) + \varepsilon z\sigma_q''(g)(z)\right| &= \left|\sum_{k=q+1}^{p} \varepsilon k(k-1) b_k z^{k-1} + \sum_{k=p+1}^{\infty} k(k-1)(a_k + \varepsilon b_k) z^{k-1}\right| \\ &\le \sum_{k=q+1}^{p} (k-1)|z|^{k-1} + 2\sum_{k=p+1}^{\infty} (k-1)|z|^{k-1} \\ &= \sum_{k=q+1}^{\infty} (k-1)|z|^{k-1} + \sum_{k=p+1}^{\infty} (k-1)|z|^{k-1} \\ &= \left(\frac{q|z|^q}{1-|z|} + \frac{|z|^{q+1}}{(1-|z|)^2}\right) + \left(\frac{p|z|^p}{1-|z|} + \frac{|z|^{p+1}}{(1-|z|)^2}\right) \\ &\le \frac{p|z|^p + q|z|^q - (p-1)|z|^{p+1} - (q-1)|z|^{q+1}}{(1-|z|)^2} \end{aligned} \tag{8}$$

so that using the estimates (4), (7) and (8), we obtain that

$$\begin{aligned}|C| &\le \frac{\frac{2|z|(|z|^p+|z|^q)}{(1-|z|)^2(1+|z|)} + \frac{p|z|^p+q|z|^q-(p-1)|z|^{p+1}-(q-1)|z|^{q+1}}{(1-|z|)^2}}{\frac{(1-|z|)^2-(|z|^p+|z|^q)(1+|z|)}{(1-|z|)(1+|z|)}} \\ &= \frac{1}{1-|z|}\left(\frac{p|z|^p + 3|z|^{p+1} - (p-1)|z|^{p+2} + q|z|^q + 3|z|^{q+1} - (q-1)|z|^{q+2}}{(1-|z|)^2 - (|z|^p+|z|^q)(1+|z|)}\right).\end{aligned}$$

Thus, by (4) and the last inequality, we see that $\mathrm{Re}\,(1+B+C) \ge 1-|B|-|C|$, which for $|z| = 1/4$ reduces to the inequality

$$\mathrm{Re}\,(1+B+C) \ge \frac{4}{3}\left(\frac{7}{20} - \frac{\frac{15p+13}{4^{p+2}} + \frac{15q+13}{4^{q+2}}}{\frac{9}{16} - \frac{5}{4}\left(\frac{1}{4^p} + \frac{1}{4^q}\right)}\right).$$

Moreover, for $p > q \ge 3$, one has

$$\frac{7}{20} - \frac{\frac{15p+13}{4^{p+2}} + \frac{15q+13}{4^{q+2}}}{\frac{9}{16} - \frac{5}{4}\left(\frac{1}{4^p} + \frac{1}{4^q}\right)} \ge \frac{7}{20} - \frac{305}{2204} > 0$$

which implies that for each ε with $|\varepsilon| = 1$, $s_p(h) + \varepsilon s_q(g)$ is convex in $|z| < 1/4$ whenever $3 \le q \le p$ and thus, each section $s_{p,q}(f)$ is convex in $|z| < 1/4$ whenever $3 \le q \le p$. □

Theorem 4. *If $p = 2 < q$, then $s_{2,q}(f)$ is convex in $|z| < 0.210222$. If $q = 2 < p$, then $s_{p,2}(f)$ is convex in $|z| < 0.234906$.*

Proof. We first suppose that $p = 2 < q$. Then, for each $|\varepsilon| = 1$, it suffices to show that

$$S = \mathrm{Re}\left(1 + z\frac{s_2''(h)(z) + \varepsilon s_q''(g)(z)}{s_2'(h)(z) + \varepsilon s_q'(g)(z)}\right) > 0$$

in the disk $|z| < 0.210222$.

In the case $p = 2 < q$, (4)–(6) continue to hold. Therefore, we can easily deduce that

$$\begin{aligned}(1-|z|)S &\ge \frac{1-|z|^2-2|z|}{1+|z|} - \frac{2|z|^2\left(2+3|z|-|z|^2\right)}{1-2|z|-|z|^2-2|z|^3} \\ &= \frac{1-4|z|-2|z|^2-8|z|^3+|z|^4+4|z|^5}{(1+|z|)(1-2|z|-|z|^2-2|z|^3)},\end{aligned}$$

which is larger than 0 if

$$1-4|z|-2|z|^2-8|z|^3+|z|^4+4|z|^5>0.$$

This gives the disk of convexity as $|z|<0.210222$. Thus, we have proved that $s_{2,q}(f)$ is convex in the disk $|z|<0.210222$ if $q>2$.

Next, we consider the case $q=2<p$. Then (4), (7) and (8) continue to hold. Therefore, if we let

$$T=\mathrm{Re}\left(1+z\frac{s_p''(h)(z)+\varepsilon s_2''(g)(z)}{s_p'(h)(z)+\varepsilon s_2'(g)(z)}\right)$$

then we obtain that

$$\begin{aligned}(1-|z|)T &\geq \frac{1-|z|^2-2|z|}{1+|z|}-\frac{|z|^p(p+3|z|-(p-1)|z|^2)+|z|^2(2+3|z|-|z|^2)}{(1-|z|)^2-(|z|^p+|z|^2)(1+|z|)}\\ &\geq \frac{1-|z|^2-2|z|}{1+|z|}-\frac{|z|^3(3+3|z|-2|z|^2)+|z|^2(2+3|z|-|z|^2)}{(1-|z|)^2-(|z|^3+|z|^2)(1+|z|)}\\ &= \frac{1-4|z|+|z|^2-8|z|^3-5|z|^4+4|z|^5+3|z|^6}{(1+|z|)(1-2|z|-2|z|^3-|z|^4)},\end{aligned}$$

which is larger than 0 if

$$1-4|z|+|z|^2-8|z|^3-5|z|^4+4|z|^5+3|z|^6>0,$$

i.e. if $|z|<0.234906$. It follows that $s_{p,2}(f)$ is convex in $|z|<0.234906$ if $p>2$. □

4. Areas and arc lengths

Theorem 5. *Suppose* $0<r<1$. *The area of the image of* $|z|<r$ *for functions in* $\mathcal{P}_H^0$ *is maximal for* $f_0(z)=-z-2\log(1-z)$. *This function also maximizes the length of the image of* $C_r=\{z:|z|=r\}$ *for functions in the class* $\mathcal{P}_H^0$.

Proof. Suppose $f=h+\overline{g}\in\mathcal{P}_H^0$, where h and g are in the form (2). Let $A_r(f)$ and $L_r(f)$ denote the area of the image of $|z|<r$ under f and the length of the image of $|z|=r$ under f, respectively. Then

$$\begin{aligned}A_r(f) &= \iint_{\mathbb{D}_r}\left(|h'(z)|^2-|g'(z)|^2\right)dxdy\\ &= \int_0^r rdr\int_0^{2\pi}\left(|h'(re^{i\theta})|^2-|g'(re^{i\theta})|^2\right)d\theta\\ &= \pi\left(r^2+\sum_{n=2}^{\infty}n(|a_n|^2-|b_n|^2)r^{2n}\right)\\ &\leq \pi\left(r^2+\sum_{n=2}^{\infty}n\left(\frac{2}{n}\right)^2r^{2n}\right)\quad\text{(by Lemma D)}\\ &= A_r(f_0)\end{aligned}$$

and the first part of the proof follows. Next we compute,

$$\begin{aligned}L_r(f) &= \int_{C_r}|f_z(z)\,dz+f_{\bar z}(z)\,d\bar z|\\ &= \int_{C_r}|h'(z)\,dz+\overline{g'(z)}\,d\bar z|\\ &= r\int_0^{2\pi}\left|h'(re^{i\theta})-\overline{g'(re^{i\theta})e^{2i\theta}}\right|d\theta.\end{aligned}\tag{9}$$

There exists a $\phi=\phi(\theta)$ such that $-\overline{g'(re^{i\theta})e^{2i\theta}}=g'(re^{i\theta})e^{i\phi(\theta)}$. Moreover, since $f\in\mathcal{P}_H^0$, we have $\mathrm{Re}\,h'(z)>|g'(z)|$ and hence, as pointed out earlier, it follows that

$$\mathrm{Re}\left(h'(z)+\varepsilon g'(z)\right)>0\quad\text{for } z\in\mathbb{D}.$$

That is,

$$h'(z) + \varepsilon g'(z) \prec \frac{1+z}{1-z} \quad \text{for } z \in \mathbb{D}$$

and in particular, we write

$$h'(re^{i\theta}) + g'(re^{i\theta})e^{i\phi(\theta)} \prec f_0'(re^{i\theta}),$$

where $\prec$ denotes the usual subordination [6] and $f_0(z) = -z - 2\log(1-z)$. Using the last subordination relation and Theorem 2 in [14], (9) reduces to

$$L_r(f) \le r \int_0^{2\pi} |f_0'(re^{i\theta})|\, d\theta = L_r(f_0).$$

The result follows. $\square$

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