International Mathematical Forum, 2, 2007, no. 29, 1399 - 1414

Exponential Stability of Reaction-Diffusion Generalized Cohen-Grossberg Neural Networks with both Variable and Distributed Delays[1]

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Abstract

In this paper, a generalized reaction-diffusion model of Cohen - Grossberg neural networks with time-varying and distributed delays is investigated. By employing analytic methods, inequality technique and M-matrix theory, some sufficient conditions ensuring the existence, uniqueness and global exponential stability of equilibrium point for Cohen-Grossberg neural networks with time-varying and distributed delays are obtained. Several examples are given to show the effectiveness of the obtained results.

Mathematics Subject Classification: 92B20, 34K20

Keywords: Cohen-Grossberg neural networks; delays; global exponential stability; reaction-diffusion

1 Introduction

Cohen-Grossberg neural network model was initially proposed by Cohen and Grossberg [18] in 1983 and soon has attracted considerable attention in theoretical research and engineering applications. In reality, time delays inevitably exist in biological and artificial neural networks due to the finite switching speed of neurons and amplifiers. It is also important to incorporate time delay in various neural networks. In recent years, there exist some results on global asymptotical stability, global exponential stability and periodic solutions for

[1]This work is a project supported by Scientific Research Fund of Sichuan Provincial Education Department under Grant 2006A109.

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the neural networks with constant delays or time-varying delays (see [1]-[3],[8]-[12],[19]-[22], [24, 37]). Although the use of finite delays in models with delayed feedback provides a good approximation to simple circuits consisting of a small number of neurons, neural networks usually should have a spatial extent due to the presence of a multitude of parallel pathways with a variety of axon sizes and lengths. Thus, there will be a distributed of propagation delays in finite or/and infinite time (see [6],[13]-[16], [25]-[29],[38]). In the case, the signal propagation is no longer instantaneous and cannot be modeled with finite delays or infinite delays. A more appropriate and ideal way is to incorporate finite delays and infinite delays, e.g.,Refs. [27]-[29]. However, strictly speaking, diffusion effects cannot be avoided in the neural networks when electrons are moving in asymmetric electromagnetic fields. So we must consider that the activations vary in space as well as in time. In [4]-[7],[14]-[17],[23],[29, 30],[34, 36], the authors have considered the stability of neural networks with diffusion terms, which are expressed by partial differential equations. It is also common to consider the diffusion effect in biological systems such as immigration.

Motivated by the above discussions, the objective of this paper is to study the global exponential stability of a class of generalized reaction-diffusion Cohen-Grossberg neural networks with time-varying and distributed delays. Our methods, which does not make use of Lyapunov functional, is simple and valid for stability analysis of neural networks with variable and/or unbounded delays, without assuming the boundedness of the activation functions and the differentiability of time-varying delays, as needed in most other papers.

The rest of this paper is organized as follows. Model description and preliminaries are given in section 2. In section 3, We give out main results and their proof. Remarks and examples are given to illustrate our theory in section 4. Finally, in section 5 we give the conclusion.

2 Model description and preliminaries

Consider a class of generalized reaction-diffusion Cohen-Grossberg neural networks with variable and distributed delays described by the following system:

$$\begin{cases} \frac{\partial u_i(t,x)}{\partial t} = \sum\limits_{k=1}^{m} \left(\frac{\partial}{\partial x_k} D_{ik} \frac{\partial u_i(t,x)}{\partial x_k} \right) - \alpha_i(u_i(t,x)) \Big[\beta_i(u_i(t,x)) \\ \qquad - \sum\limits_{j=1}^{n} a_{ij} f_j(u_j(t,x)) - \sum\limits_{j=1}^{n} b_{ij} g_j(u_j(t-\tau_{ij}(t),x)) \\ \qquad - \sum\limits_{j=1}^{n} c_{ij} \int_{-\infty}^{t} K_{ij}(t-s) h_j(u_j(s,x)) ds + J_i \Big], \ i = 1, 2 \cdots, n, \ x \in \Omega, \\ \frac{\partial u_i}{\partial \tilde{n}} = \left(\frac{\partial u_i}{\partial x_1}, \frac{\partial u_i}{\partial x_2}, \cdots, \frac{\partial u_i}{\partial x_m} \right)^T = 0, \quad t \geq 0, \quad x \in \partial\Omega, \\ u_i(s,x) = \phi_i(s,x), \quad -\infty < s \leq 0, \end{cases} \tag{1}$$

where $n \geq 2$ is the number of neurons in the network, $u_i(t,x)$ corresponds to the state of the ith neuron at time t and in space x; the smooth function $D_{ik} = D_{ik}(t,x,u) \geq 0$ corresponds to the transmission diffusion operator along the ith neuron; $\alpha_i(u_i(t,x))$ presents an amplification function; $\beta_i(u_i(t,x))$ is an appropriately behaved function; J_i denotes external input to the $i$the neuron; a_{ij}, b_{ij}, c_{ij} denote the connection strengths of the jth neuron on the ith neuron, respectively; $f_j(u_j(t,x)), g_j(u_j(t,x)), h_j(u_j(t,x))$ denote the activation functions of jth neuron at time t and in space x; Ω is a bounded compact set in space R^m with smooth boundary $\partial\Omega$ and measure $mes\Omega > 0$; $\phi_i(s,x)$ is the initial conditions, $-\infty < s \leq 0$, ϕ_i is continuous on $(-\infty, 0] \times R^m$; $\tau_{ij}(t)$ corresponds to the transmission delay and satisfies $0 \leq \tau_{ij}(t) \leq \tau$ (τ is a constant); the delay kernel $K_{ij} : [0,+\infty) \to [0,+\infty)$ is real valued nonnegative continuous function and satisfies [27]

$$\int_0^{+\infty} e^{\delta s} K_{ij}(s) ds = p_{ij}(\delta),$$

where $p_{ij}(\delta)$ is continuous function in $[0,\gamma)$, $\gamma > 0$, and $p_{ij}(0) = 1,\ i,j = 1,2,\cdots,n$.

Throughout this paper we assume that:

(**A1**) Each function $\alpha_i(u)$ is bounded, positive and continuous, i.e. there exist constants $\underline{\alpha}_i,\ \bar{\alpha}_i$ such that

$$0 < \underline{\alpha}_i \leq \alpha_i(u) \leq \bar{\alpha}_i < +\infty,\ for\ u \in R,\ i = 1,2,\cdots,n.$$

(**A2**) $\beta_i(u)$ is monotone increasing, i.e., there exists a positive diagonal matrix $\beta = diag(\beta_1, \beta_2, \cdots, \beta_n)$ such that

$$\frac{\beta_i(u) - \beta_i(v)}{u - v} \geq \beta_i,$$

for all $u, v \in R\ (u \neq v),\ i = 1,2,\cdots,n$.

(**A3**) For the activation functions $f_i(u),\ g_i(u)$ and $h_i(u)$, there exist positive diagonal matrices $F = diag(F_1, F_2, \cdots, F_n)$, $G = diag(G_1, G_2, \cdots, G_n)$ and $H_i = diag(H_1, H_2, \cdots, H_n)$ such that

$$F_i = \sup_{u \neq v} \left| \frac{f_i(u) - f_i(v)}{u - v} \right|,\ G_i = \sup_{u \neq v} \left| \frac{g_i(u) - g_i(v)}{u - v} \right|,\ H_i = \sup_{u \neq v} \left| \frac{h_i(u) - h_i(v)}{u - v} \right|,$$

for all $u, v \in R\ (u \neq v),\ i = 1,2,\cdots,n$.

To begin with, we introduce some notation and recall some basic definitions.

For an $n \times n$ matrix A, $|A|$ denotes the absolute value matrix given by $|A| = (|a_{ij}|)_{n \times n}$. For $u(t,x) = (u_1(t,x), u_2(t,x), \cdots, u_n(t,x))^T \in R^n$, define $\|u_i(t,x)\|_2 = \left[\int_\Omega |u_i(t,x)|^2 dx \right]^{\frac{1}{2}}, \quad i = 1,2,\cdots,n.$

Let $\Phi = C((-\infty, 0] \times R^m, R^n)$ be the linear space of bounded and continuous functions which map $(-\infty, 0] \times R^m$ into R^n. The norm on Φ is defined by

$$\|\phi\| = \sup_{-\infty < s \leq 0} \sum_{i=1}^n \|\phi_i(s,x)\|_2$$

for any $\phi(s,x) = (\phi_1(s,x), \phi_2(s,x), \cdots, \phi_n(s,x))^T \in \Phi$. It can be proved that Φ is a Banach space.

Definition 1 *An equilibrium point $u^* = (u_1^*, u_2^*, \cdots, u_n^*)^T$ of system (1) is said to be globally exponentially stable, if there exist positive constants $\lambda > 0$ and $M \geq 1$ such that*

$$\sum_{i=1}^{n} \|u_i(t,x) - u_i^*\|_2 \leq M\|\phi - u^*\|e^{-\lambda t} \qquad \text{for all } t \geq 0,$$

where $\|\phi - u^*\| = \sup\limits_{-\infty < s \leq 0} \sum\limits_{i=1}^{n} \|\phi_i(s,x) - u_i^*\|_2$

Definition 2 *[32] A real matrix $D = (d_{ij})_{n\times n}$ is said to be a nonsingular M-matrix if $d_{ij} \leq 0$, $i,j = 1,2,\cdots,n$, $i \neq j$, and all successive principal minors of D are positive.*

To the nonsingular M-matrix, we have

Lemma 1 *[32] Each of the following conditions is equivalent:*

(i) D is a nonsingular M-matrix.

(ii) D^{-1} exists and D^{-1} is a nonnegative matrix.

(iii) The diagonal elements of D are all positive and there exists a positive vector d such that $Dd > 0$ or $D^T d > 0$.

3 Main results

Theorem 1 *Under assumptions* (A1), (A2) *and* (A3), *the system* (1) *has has a unique equilibrium point, which is globally exponentially stable if*

$$\underline{\alpha}\beta - \overline{\alpha}(|A|F + |B|G + |C|H)$$

is a nonsingular M-matrix. Where

$$\underline{\alpha} = diag(\underline{\alpha}_1, \underline{\alpha}_2, \cdots, \underline{\alpha}_n),\ \overline{\alpha} = diag(\overline{\alpha}_1, \overline{\alpha}_2, \cdots, \overline{\alpha}_n),\ \beta = diag(\beta_1, \beta_2, \cdots, \beta_n),$$

$$|A| = (|a_{ij}|)_{n\times n}, \quad F = diag(F_1, F_2, \cdots, F_n),$$

$$|B| = (|b_{ij}|)_{n\times n}, \quad G = diag(G_1, G_2, \cdots, G_n),$$

$$|C| = (|c_{ij}|)_{n\times n}, \quad H = diag(H_1, H_2, \cdots, H_n).$$

Proof. Since $\underline{\alpha}\beta-\overline{\alpha}(|A|F+|B|G+|C|H)$ is a nonsingular M-matrix, and $\overline{\alpha}^{-1}\underline{\alpha}\leq E$ (E is the identical matrix), $\beta-(|A|F+|B|G+|C|H)$ is a nonsingular M-matrix. From Theorem 1 in [27], we know that the following model

$$\begin{aligned}\frac{du_i(t)}{dt} &= -\alpha_i(u_i(t))\Big[\beta_i(u_i(t))-\sum_{j=1}^{n}a_{ij}f_j(u_j(t))-\sum_{j=1}^{n}b_{ij}g_j(u_j(t-\tau_{ij}(t))\\ &\quad -\sum_{j=1}^{n}c_{ij}\int_{-\infty}^{t}K_{ij}(t-s)h_j(u_j(s))ds+J_i\Big],\quad i=1,2\cdots,n, \qquad (2)\end{aligned}$$

has one unique equilibrium point. Obviously, the equilibrium point of model (2) is also of model (1). Let $u^*=(u_1^*,u_2^*,\cdots,u_n^*)^T$ be the equilibrium point of (1), $u(t,x)=(u_1(t,x),u_2(t,x),\cdots,u_n(t,x))^T$ is any solution of the system (1), then

$$\begin{aligned}\frac{\partial(u_i(t,x)-u_i^*)}{\partial t} &= \sum_{k=1}^{m}\frac{\partial}{\partial x_k}\Big(D_{ik}\frac{\partial(u_i(t,x)-u_i^*)}{\partial x_k}\Big)\\ &\quad -\alpha_i(u_i(t,x))\Big[\beta_i(u_i(t,x))-\beta_i(u_i^*)-\sum_{j=1}^{n}a_{ij}[f_j(u_j(t,x))-f_j(u_j^*)]\\ &\quad -\sum_{j=1}^{n}b_{ij}[g_j(u_j(t-\tau_{ij}(t),x)-g_j(u_j^*)]\\ &\quad -\sum_{j=1}^{n}c_{ij}\int_{-\infty}^{t}K_{ij}(t-s)[h_j(u_j(s,x))-h_j(u_j^*)]ds\Big] \qquad (3)\end{aligned}$$

for $t\geq 0,\ i=1,2,\cdots,n$.

Multiply both sides of (3) by $u_i(t,x)-u_i^*$ and integrate it, we get

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_\Omega(u_i(t,x)-u_i^*)^2dx\\ =\ &\sum_{k=1}^{m}\int_\Omega(u_i(t,x)-u_i^*)\frac{\partial}{\partial x_k}\Big(D_{ik}\frac{\partial(u_i(t,x)-u_i^*)}{\partial x_k}\Big)dx\\ &-\int_\Omega(u_i(t,x)-u_i^*)\alpha_i(u_i(t,x))[\beta_i(u_i(t,x))-\beta_i(u_i^*)]dx\\ &+\int_\Omega\alpha_i(u_i(t,x))(u_i(t,x)-u_i^*)\sum_{j=1}^{n}a_{ij}[f_j(u_j(t,x))-f_j(u_j^*)]dx\\ &+\int_\Omega\alpha_i(u_i(t,x))(u_i(t,x)-u_i^*)\sum_{j=1}^{n}b_{ij}[g_j(u_j(t-\tau_{ij}(t),x)-g_j(u_j^*)]dx\\ &+\int_\Omega\alpha_i(u_i(t,x))(u_i(t,x)-u_i^*)\sum_{j=1}^{n}c_{ij}\Big[\int_{-\infty}^{t}K_{ij}(t-s)[h_j(u_j(s,x))-h_j(u_j^*)]ds\Big]dx \qquad (4)\end{aligned}$$

By the boundary condition of model (1),we have

$$\begin{aligned}
&\sum_{k=1}^{m}\int_{\Omega}(u_i(t,x)-u_i^*)\frac{\partial}{\partial x_k}\Big(D_{ik}\frac{\partial(u_i(t,x)-u_i^*)}{\partial x_k}\Big)_d x\\
=\ &\int_{\Omega}(u_i(t,x)-u_i^*)\Big(D_{ik}\frac{\partial(u_i(t,x)-u_i^*)}{\partial x_k}\Big)_{k=1}^{m}dx\\
=\ &\int_{\Omega}\nabla\cdot\Big((u_i(t,x)-u_i^*)D_{ik}\frac{\partial(u_i(t,x)-u_i^*)}{\partial x_k}\Big)_{k=1}^{m}dx\\
&-\int_{\Omega}\Big(D_{ik}\frac{\partial(u_i(t,x)-u_i^*)}{\partial x_k}\Big)_{k=1}^{m}\nabla\cdot(u_i(t,x)-u_i^*)dx\\
=\ &\int_{\partial\Omega}\Big((u_i(t,x)-u_i^*)D_{ik}\frac{\partial(u_i(t,x)-u_i^*)}{\partial x_k}\Big)_{k=1}^{m}d\rho-\sum_{k=1}^{m}\int_{\Omega}D_{ik}\Big(\frac{\partial(u_i(t,x)-u_i^*)}{\partial x_k}\Big)^2dx\\
=\ &-\sum_{k=1}^{m}\int_{\Omega}D_{ik}\Big(\frac{\partial(u_i(t,x)-u_i^*)}{\partial x_k}\Big)^2dx, \qquad (5)
\end{aligned}$$

in which $\nabla=(\frac{\partial}{\partial x_1},\frac{\partial}{\partial x_2},\cdots,\frac{\partial}{\partial x_n})^T$ is the gradient operator, and

$$\Big(D_{ik}\frac{\partial(u_i(t,x)-u_i^*)}{\partial x_k}\Big)_{k=1}^{m}=\Big(D_{ik}\frac{\partial(u_i(t,x)-u_i^*)}{\partial x_1},\cdots,D_{ik}\frac{\partial(u_i(t,x)-u_i^*)}{\partial x_m}\Big)^T.$$

From assumptions (A1)and (A2), we have

$$\begin{aligned}
\int_{\Omega}(u_i(t,x)-u_i^*)\alpha_i(u_i(t,x))[\beta_i(u_i(t,x))-\beta_i(u_i^*)]dx &\geq \underline{\alpha}_i\beta_i\int_{\Omega}(u_i(t,x)-u_i^*)^2dx\\
&= \underline{\alpha}_i\beta_i\|u_i(t,x)-u_i^*\|_2^2. \qquad (6)
\end{aligned}$$

From assumptions (A1), (A3) and Holder inequality, we have

$$\begin{aligned}
&\int_{\Omega}\alpha_i(u_i(t,x))(u_i(t,x)-u_i^*)\sum_{j=1}^{n}a_{ij}[f_j(u_j(t,x))-f_j(u_j^*)]dx\\
\leq\ &\int_{\Omega}\alpha_i(u_i(t,x))\sum_{j=1}^{n}|a_{ij}|\cdot|u_i(t,x)-u_i^*|\cdot|f_j(u_j(t,x))-f_j(u_j^*)|dx\\
\leq\ &\overline{\alpha}_i\sum_{j=1}^{n}|a_{ij}|\int_{\Omega}|u_i(t,x)-u_i^*|F_j|u_j(t,x)-u_j^*|dx\\
\leq\ &\overline{\alpha}_i\sum_{j=1}^{n}|a_{ij}|F_j\|u_i(t,x)-u_i^*\|_2\|u_j(t,x)-u_j^*\|_2 \qquad (7)
\end{aligned}$$

By the same way, we can obtain

$$\begin{aligned}
&\int_{\Omega}\alpha_i(u_i(t,x))(u_i(t,x)-u_i^*)\sum_{j=1}^{n}b_{ij}[g_j(u_j(t-\tau_{ij}(t),x)-g_j(u_j^*)]dx\\
\leq\ &\overline{\alpha}_i\sum_{j=1}^{n}|b_{ij}|G_j\|u_i(t,x)-u_i^*\|_2\|u_j(t-\tau_{ij}(t),x)-u_j^*\|_2 \qquad (8)
\end{aligned}$$

and

$$\int_\Omega \alpha_i(u_i(t,x))(u_i(t,x)-u_i^*)\sum_{j=1}^n c_{ij}\Big[\int_{-\infty}^t K_{ij}(t-s)[h_j(u_j(s,x))-h_j(u_j^*)]ds\Big]dx$$
$$\leq \overline{\alpha}_i\sum_{j=1}^n |c_{ij}|H_j\int_{-\infty}^t K_{ij}(t-s)\|u_i(t,x)-u_i^*\|_2\|u_j(s,x)-u_j^*\|_2 ds. \qquad (9)$$

Apply (5)-(9) to (4), we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|u_i(t,x)-u_i^*\|_2^2\\ &\leq -\underline{\alpha}_i\beta_i\|u_i(t,x)-u_i^*\|_2^2+\overline{\alpha}_i\sum_{j=1}^n|a_{ij}|F_j\|u_i(t,x)-u_i^*\|_2\|u_j(t,x)-u_j^*\|_2\\ &\quad+\overline{\alpha}_i\sum_{j=1}^n|b_{ij}|G_j\|u_i(t,x)-u_i^*\|_2||u_j(t-\tau_{ij}(t),x)-u_j^*||_2\\ &\quad+\overline{\alpha}_i\sum_{j=1}^n|c_{ij}|H_j\int_{-\infty}^t K_{ij}(t-s)\|u_i(t,x)-u_i^*\|_2\|u_j(s,x)-u_j^*\|_2 ds\end{aligned}$$

i.e.

$$\begin{aligned}D^+\|u_i(t,x)-u_i^*\|_2 \;\leq\; & -\underline{\alpha}_i\beta_i\|u_i(t,x)-u_i^*\|_2+\overline{\alpha}_i\sum_{j=1}^n|a_{ij}|F_j\|u_j(t,x)-u_j^*\|_2\\ &+\overline{\alpha}_i\sum_{j=1}^n|b_{ij}|G_j\|u_j(t-\tau_{ij}(t),x)-u_j^*\|_2\\ &+\overline{\alpha}_i\sum_{j=1}^n|c_{ij}|H_j\int_{-\infty}^t K_{ij}(t-s)\|u_j(s,x)-u_j^*\|_2 ds. \qquad (10)\end{aligned}$$

Since $\underline{\alpha}\beta-\overline{\alpha}(|A|F+|B|G+|C|H)$ is an M-matrix, there exists a vector $\xi=(\xi_1,\xi_2,\cdots,\xi_n)^T>0$ such that

$$-\xi_i\underline{\alpha}_i\beta_i+\overline{\alpha}_i\sum_{j=1}^n\xi_j\Big[|a_{ij}|F_j+|b_{ij}|G_j+|c_{ij}|H_j\Big]<0$$

for $i=1,2,\cdots,n$. Constructing the function

$$\Gamma_i(\theta)=\xi_i(\theta-\underline{\alpha}_i\beta_i)+\overline{\alpha}_i\sum_{j=1}^n\xi_j\Big[|a_{ij}|F_j+|b_{ij}|G_je^{\tau\theta}+|c_{ij}|H_jp_{ij}(\theta)\Big]$$

for $i=1,2,\cdots,n$, where $\tau=\max\limits_{1\leq i,j\leq n}\tau_{ij}(t)$.

Obviously, $\Gamma_i(0)<0$, $\Gamma_i(\theta)\to+\infty$ as $\theta\to+\infty$. From the assumption of the delay kernels, we know that $\Gamma_i(\theta)$ is continuous for $i=1,2,\cdots,n$. So there exist

$\lambda_i > 0$ such that

$$\Gamma_i(\lambda_i) = \xi_i(\lambda_i - \underline{\alpha}_i\beta_i) + \overline{\alpha}_i \sum_{j=1}^{n} \xi_j \Big[|a_{ij}|F_j + |b_{ij}|G_j e^{\tau\lambda_i} + |c_{ij}|H_j p_{ij}(\lambda_i)\Big] < 0$$

for $i = 1, 2, \cdots, n$. Taking $\lambda = \min_{1\leq i\leq n}\{\lambda_i\}$, then $\lambda > 0$, and

$$\Gamma_i(\lambda) = \xi_i(\lambda - \underline{\alpha}_i\beta_i) + \overline{\alpha}_i \sum_{j=1}^{n} \xi_j \Big[|a_{ij}|F_j + |b_{ij}|G_j e^{\tau\lambda} + |c_{ij}|H_j p_{ij}(\lambda)\Big] < 0 \tag{11}$$

for $i = 1, 2, \cdots, n$. Let

$$w_i(t) = e^{\lambda t}\|u_i(t, x) - u_i^*\|_2$$

for $i = 1, 2, \cdots, n$. Calculating the upper right derivative $D^+w_i(t)$ of $w_i(t)$ along the solutions of (1), by use of (10), we get

$$\begin{aligned} D^+w_i(t) \ &\leq \ e^{\lambda t}\Big((\lambda - \underline{\alpha}_i\beta_i)\|u_i(t, x) - u_i^*\|_2 + \overline{\alpha}_i \sum_{j=1}^{n} |a_{ij}|F_j\|u_j(t, x) - u_j^*\|_2 \\ &\quad + \overline{\alpha}_i \sum_{j=1}^{n} |b_{ij}|G_j\|u_j(t - \tau_{ij}(t), x) - u_j^*\|_2 \\ &\quad + \overline{\alpha}_i \sum_{j=1}^{n} |c_{ij}|H_j \int_{-\infty}^{t} K_{ij}(t - s)\|u_j(s, x) - u_j^*\|_2 ds\Big) \\ &= \ (\lambda - \underline{\alpha}_i\beta_i)w_i(t) + \overline{\alpha}_i \sum_{j=1}^{n} \Big(|a_{ij}|F_j w_j(t) + |b_{ij}|G_j w_j(t - \tau_{ij}(t))e^{\lambda\tau_{ij}(t)} \\ &\quad + |c_{ij}|H_j \int_{-\infty}^{t} e^{\lambda(t-s)}K_{ij}(t - s)w_j(s)ds\Big) \\ &\leq \ (\lambda - \underline{\alpha}_i\beta_i)w_i(t) + \overline{\alpha}_i \sum_{j=1}^{n} \Big(|a_{ij}|F_j w_j(t) + e^{\lambda\tau}G_j|b_{ij}|w_j(t - \tau_{ij}(t)) \\ &\quad + |c_{ij}|H_j \int_{-\infty}^{t} e^{\lambda(t-s)}K_{ij}(t - s)w_j(s)ds\Big) \end{aligned} \tag{12}$$

for $i = 1, 2, \cdots, n$.

Defining the curve $\gamma = \Big\{z(l) = (\xi_1 l, \xi_2 l, \cdots, \xi_n l)\Big| l > 0\Big\}$ and the set $\Omega(z) = \Big\{u\Big| 0 \leq u \leq z, z \in \gamma\Big\}$. It is obvious that $\Omega(z(l)) \supset \Omega(z(l'))$, when $l > l'$.

Let $l_0 = \frac{(1+\varepsilon)\|\phi - u^*\|_r}{\min_{1\leq i\leq n}\{\xi_i\}}$ (ε is a positive constant), then

$$w_i(s) = e^{\lambda s}||u_i(s, x) - u_i^*||_2 \leq ||u_i(s, x) - u_i^*||_2 = ||\varphi_i(s, x) - u_i^*||_2 \leq \|\phi - u^*\|_r < \xi_i l_0$$

for $i = 1, 2, \cdots, n$, $-\infty < s \leq 0$. In the following, we will prove that

$$w_i(t) < \xi_i l_0 \tag{13}$$

for $t \geq 0$, $i = 1, 2, \cdots, n$. If (13) is not true, then there exist some i and t_1 such that

$$w_i(t_1) = \xi_i l_0, \quad D^+ w_i(t_1) \geq 0 \quad \text{and} \quad w_j(t) \leq \xi_j l_0$$

for $-\infty < t \leq t_1$, $j = 1, 2, \cdots, n$. However, from (12) and (11) we get

$$\begin{aligned} D^+ w_i(t_1) &\leq (\lambda - \underline{\alpha}_i \beta_i) w_i(t_1) + \overline{\alpha}_i \sum_{j=1}^{n} \Big(|a_{ij}| F_j w_j(t_1) + e^{\lambda\tau} |b_{ij} G_j| w_j(t_1 - \tau_{ij}(t_1)) \\ &\quad + |c_{ij}| H_j \int_{-\infty}^{t_1} e^{\lambda(t_1 - s)} K_{ij}(t_1 - s) w_j(s) ds \Big) \\ &\leq \Big((\lambda - \underline{\alpha}_i \beta_i) \xi_i + \overline{\alpha}_i \sum_{j=1}^{n} \xi_j \Big(|a_{ij}| F_j + e^{\lambda\tau} |b_{ij}| G_j + |c_{ij}| H_j p_{ij}(\lambda) \Big) l_0 \\ &< 0 \end{aligned}$$

for $i = 1, 2, \cdots, n$, this is a contradiction, so

$$w_i(t) < \xi_i l_0$$

for $t \geq 0$, $i = 1, 2, \cdots, n$. That is

$$||u_i(t, x) - u_i^*||_2 \leq \xi_i l_0 e^{-\lambda t}$$

for $t \geq 0$, $i = 1, 2, \cdots, n$. Hence

$$\Big[\sum_{i=1}^{n} \|u_i(t, x) - u_i^*\|_2 \Big] \leq M \|\phi - x^*\| e^{-\lambda t}$$

for all $t \geq 0$, where $M = \frac{(1+\varepsilon) \sum_{i=1}^{n} \xi_i}{\min_{1 \leq i \leq n} \{\xi_i\}} > 1$. It means that the equilibrium point of model (1) is globally exponentially stable. The proof is completed.

Corollary 1 *Under assumptions* (A1), (A2) *and* (A3), *the system* (1) *has has a unique equilibrium point, which is globally exponentially stable if any one of the following conditions is true:*

(i) $\underline{\alpha}_i \beta_i > F_i \sum_{j=1}^{n} \overline{\alpha}_j |a_{ji}| + G_i \sum_{j=1}^{n} \overline{\alpha}_j |b_{ji}| + H_i \sum_{j=1}^{n} \overline{\alpha}_j |c_{ji}|, \quad i = 1, 2, \cdots, n.$

(ii) $\underline{\alpha}_i \beta_i > \overline{\alpha}_i \sum_{j=1}^{n} \Big(|a_{ij}| F_j + |b_{ij}| G_j + |c_{ij}| H_j \Big), \quad i = 1, 2, \cdots, n.$

(iii) *There exists a positive vector* $l = (l_1, l_2, \cdots, l_n)^T > 0$ *such that*

$$l_i \underline{\alpha}_i \beta_i > \overline{\alpha}_i \sum_{j=1}^{n} l_j \Big(|a_{ij}| F_j + |b_{ij}| G_j + |c_{ij}| H_j \Big), \quad i = 1, 2, \cdots, n.$$

(iv) *There exists a positive vector* $l = (l_1, l_2, \cdots, l_n)^T > 0$ *such that*

$$l_i \underline{\alpha}_i \beta_i > F_i \sum_{j=1}^{n} \overline{\alpha}_j l_j |a_{ji}| + G_i \sum_{j=1}^{n} \overline{\alpha}_j l_j |b_{ij}| + H_i \sum_{j=1}^{n} \overline{\alpha}_j l_j |c_{ji}|, \quad i = 1, 2, \cdots, n.$$

Proof. In fact, any one of the conditions (i)-(iv) in Corollary 1 can assure $\underline{\alpha}\beta - \overline{\alpha}(|A|F + |B|G + |C|H)$ is a nonsingular M-matrix. The proof is completed.

As $\alpha_i(u_i(t,x)) = \alpha_i$, model (1) may reduce to the following model:

$$\begin{cases} \frac{\partial u_i(t,x)}{\partial t} = \sum\limits_{k=1}^{m} \left(\frac{\partial}{\partial x_k} D_{ik} \frac{\partial u_i(t,x)}{\partial x_k} \right) - \alpha_i \Big[\beta_i(u_i(t,x)) \\ \qquad - \sum\limits_{j=1}^{n} a_{ij} f_j(u_j(t,x)) - \sum\limits_{j=1}^{n} b_{ij} g_j(u_j(t - \tau_{ij}(t), x)) \\ \qquad - \sum\limits_{j=1}^{n} c_{ij} \int_{-\infty}^{t} K_{ij}(t-s) h_j(u_j(s,x)) ds + J_i \Big], \quad i = 1, 2 \cdots, n, \quad x \in \Omega, \\ \frac{\partial u_i}{\partial \tilde{n}} = \left(\frac{\partial u_i}{\partial x_1}, \frac{\partial u_i}{\partial x_2}, \cdots, \frac{\partial u_i}{\partial x_m} \right)^T = 0, \quad t \ge 0, \quad x \in \partial\Omega, \\ u_i(s,x) = \phi_i(s,x), \quad -\infty < s \le 0, \end{cases} \tag{14}$$

For model (14), by applying Theorem 1, we can easily obtain the following results.

Corollary 2 *Under assumptions* (A2) *and* (A3), *the system* (14) *has has a unique equilibrium point, which is globally exponentially stable if* $\beta - (|A|F + |B|G + |C|H)$ *is a nonsingular* M*-matrix.*

Corollary 3 *Under assumptions* (A2) *and* (A3), *the system* (14) *has has a unique equilibrium point, which is globally exponentially stable if any one of the following conditions is true:*

(i) $\beta_i > F_i \sum\limits_{j=1}^{n} |a_{ji}| + G_i \sum\limits_{j=1}^{n} |b_{ji}| + H_i \sum\limits_{j=1}^{n} |c_{ji}|, \quad i = 1, 2, \cdots, n.$

(ii) $\beta_i > \sum\limits_{j=1}^{n} \Big(|a_{ij}| F_j + |b_{ij}| G_j + |c_{ij}| H_j \Big), \quad i = 1, 2, \cdots, n.$

(iii) *There exists a positive vector* $l = (l_1, l_2, \cdots, l_n)^T > 0$ *such that*

$$l_i \beta_i > \sum_{j=1}^{n} l_j \Big(|a_{ij}| F_j + |b_{ij}| G_j + |c_{ij}| H_j \Big), \quad i = 1, 2, \cdots, n.$$

(iv) *There exists a positive vector* $l = (l_1, l_2, \cdots, l_n)^T > 0$ *such that*

$$l_i \beta_i > F_i \sum_{j=1}^{n} l_j |a_{ji}| + G_i \sum_{j=1}^{n} l_j |b_{ij}| + H_i \sum_{j=1}^{n} l_j |c_{ji}|, \quad i = 1, 2, \cdots, n.$$

When the smooth operator $D_{ik} = 0,\ i = 1, 2, \cdots, n, k = 1, 2, \cdots, m$, Model (1) reduce to the generalized Cohen-Grossberg neural networks (2). For model (2), we have

Corollary 4 *Under assumptions* (A1)*,* (A2) *and* (A3)*, the system* (2) *has has a unique equilibrium point, which is globally exponentially stable if* $\underline{\alpha}\beta - \overline{\alpha}(|A|F + |B|G + |C|H)$ *is a nonsingular M-matrix.*

4 Remarks and examples

Remark 1. Some famous neural network models become a special case of model (1). For example, Refs.[4, 5, 16, 23, 27, 34, 36], and as model (1) becomes neural networks model (2), it contains those models studied by many authors, see, for example, Refs.[1, 3, 10, 11, 13, 18, 19, 21, 22, 31, 37, 38],[24]-[28]. Thus the results of this paper can be applied to the recurrent neural networks with and/or without delays. Moreover, our results need only the activation functions f_i, g_i and h_i satisfy the assumption (**A3**), not requiring the activation functions to be bounded and monotone nondecreasing. In addition, we do not also demand that the behavior function $\beta_i(u)$ and variable delay function $\tau_{ij}(t)$ are differentiable. Therefore, we improve some previous results.

Remark 2. In [36], the authors considered a special case of model (1) as $(c_{ij}(t))_{n\times n} = 0$, the sufficient conditions given in Theorem 3.1 not only require f_j and g_j to be bounded, but also demand that the behavior $\beta_i(u)$ is differentiable. It is easy to check that Theorem 3.1 in [36] is included in Corollary 1 in this paper.

Remark 3. In [29], the authors considered a special case of model (14) as $\alpha_i = 1, \beta_i(u_i(t,x)) = \beta_i u_i(t,x)$ (β_i is a positive constant, $i = 1, 2, \cdots, n$). In [27], the authors considered a special case of model (1) as $D_{ik} = 0$, $i = 1, 2, \cdots, n, k = 1, 2, \cdots, m$, that is model (2). It is not difficult to discover that Theorem 1 in [29] is involved in our Corollary 2. However, since there exist reaction-diffusion terms Corollary 4 in this paper is not as good as Theorem 1 in [27].

In order to illustrate the feasibility of our above-established criteria in the preceding sections, we provide concrete examples. Although the selection of the coefficients and functions in the examples is somewhat artificial, the possible application of our theoretical theory is clearly expressed.

Example 1. Consider the following model

$$\begin{cases} \frac{\partial u_i(t,x)}{\partial t} = \sum\limits_{k=1}^{m} \left(\frac{\partial}{\partial x_k} D_{ik} \frac{\partial u_i(t,x)}{\partial x_k} \right) - \alpha_i(u_i(t,x)) \Big[\beta_i(u_i(t,x)) \\ \qquad - \sum\limits_{j=1}^{2} a_{ij} f_j(u_j(t,x)) - \sum\limits_{j=1}^{2} b_{ij} g_j(u_j(t-\tau_{ij}(t),x)) \Big], \quad i = 1, 2, \quad x \in \Omega, \\ \frac{\partial u_i}{\partial \tilde{n}} = \left(\frac{\partial u_i}{\partial x_1}, \frac{\partial u_i}{\partial x_2}, \cdots, \frac{\partial u_i}{\partial x_m} \right)^T = 0, \quad t \geq 0, \quad x \in \partial\Omega, \\ u_i(s,x) = \phi_i(s,x), \quad -4 \leq s \leq 0, \quad i = 1, 2, \end{cases} \tag{15}$$

where the coefficients and functions are taken as

$$\alpha_1(x) = 2 + \sin x,\ \alpha_2(x) = 2 + \cos x, \quad \underline{\alpha}_1 = \underline{\alpha}_2 = 1, \quad \overline{\alpha}_1 = \overline{\alpha}_2 = 3,$$

$$\beta_1(x) = \beta_2(x) = 3x, \quad \beta_1 = \beta_2 = 3,$$

$f_1(x) = f_2(x) = \frac{1}{2}(|x+1| - |x-1|)$, $F_1 = F_2 = 1$, $g_1(x) = g_2(x) = \tanh x$, $G_1 = G_2 = 1$,

$$A = (a_{ij})_{2\times 2} = \begin{pmatrix} 0.2 & -0.4 \\ 0.3 & 0.1 \end{pmatrix}, \quad B = (b_{ij})_{2\times 2} = \begin{pmatrix} -0.2 & 0.3 \\ 0.2 & 0.3 \end{pmatrix},$$

$$(\tau_{ij}(t)) = \begin{pmatrix} \cos^2 t & 2\cos^2 t \\ 3\sin^2 t & 4\sin^2 t \end{pmatrix}, \quad \begin{pmatrix} J_1 \\ J_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}.$$

It is not hard to verify the validity of (**A1**)-(**A3**), and it is easy to calculate that

$$\underline{\alpha}\beta - \overline{\alpha}(|A|F + |B|G|) = \begin{pmatrix} 1.8 & -2.1 \\ -1.5 & 1.8 \end{pmatrix}$$

Therefor $\underline{\alpha}\beta - \overline{\alpha}(|A|F + |B|G|)$ is a nonsingular M-matrix, from Theorem 1, we know that system (15) has one unique equilibrium point, which is globally exponentially stable. but we note that

$$\underline{\alpha}_2\beta_2 - F_2\sum_{j=1}^{2}\overline{\alpha}_j|a_{j1}| - G_2\sum_{j=1}^{2}\overline{\alpha}_j|b_{j1}| = -0.3 < 0.$$

Hence, Theorem 1 in [36] does not hold, this implies that Theorem 1 [36] is not applicable to ascertain the stability of model (15).

Example 2. Consider the following model:

$$\begin{cases} \frac{dx_1(t)}{dt} = -\alpha_1(x_1(t))\Big[\beta_1(x_1(t)) - c_{11}\int\limits_{-\infty}^{t} e^{-(t-s)}h_1(x_1(s))ds \\ \qquad -c_{12}\int\limits_{-\infty}^{t} e^{-2(t-s)}h_2(x_2(s))ds - 2\Big], \\ \frac{dx_2(t)}{dt} = -\alpha_2(x_2(t))\Big[\beta_2(x_2(t)) - c_{21}\int\limits_{-\infty}^{t} e^{-2(t-s)}h_1(x_1(s))ds \\ \qquad -c_{22}\int\limits_{-\infty}^{t} e^{-(t-s)}h_2(x_2(s))ds + 3\Big], \end{cases} \tag{16}$$

where $\alpha_1(x) = 2 + \sin x$, $\alpha_2(x) = 2 - \cos x$, $\beta_1(x) = 3x$, $\beta_2(x) = 3x$, $h_1(x) = h_2(x) = \tanh x$, $C = (c_{ij})_{2\times 2} = \begin{pmatrix} 0.4 & 0.7 \\ -0.5 & 0.4 \end{pmatrix}$.

It is easy to check that assumptions (**A1**)-(**A3**) hold, and $H_1 = H_2 = 1$, $\underline{\alpha}_1 = \underline{\alpha}_2 = 1$, $\overline{\alpha}_1 = \overline{\alpha}_2 = 3$, $\beta_1 = \beta_2 = 3$. Thus

$$\underline{\alpha}\beta - \overline{\alpha}|C|H = \begin{pmatrix} 1.8 & -2.1 \\ -1.5 & 1.8 \end{pmatrix}.$$

Obviously, $\underline{\alpha}\beta - \overline{\alpha}|C|H$ is a nonsingular M-matrix, from Corollary 4, model (16) has one equilibrium point, which is globally exponentially stable.

Remark 4. In [26], authors gave out three sufficient conditions for global exponential stability of Cohen-Grossberg neural networks with continuously distributed delays, that is, Theorem 3.2, Theorem 3.3 and Theorem 3.4. For model (16), we calculate these criteria as follows by using the symbol in this paper, respectively,

$$\lambda_1 := \min_{1\le i\le 2}\left\{\frac{\underline{\alpha}_i\beta_i}{\overline{\alpha}_i} - \sum_{j=1}^{2}|c_{ji}|H_i\right\} = -1 < 0.$$

$$\lambda_2 := \min_{1\le i\le 2}\left\{\frac{2\underline{\alpha}_i\beta_i}{\overline{\alpha}_i} - \sum_{j=1}^{2}(|c_{ij}| + |c_{ji}|H_i^2)\right\} = 0.$$

$$\lambda_1 := \min_{1\le i\le 2}\left\{\frac{2\underline{\alpha}_i\beta_i}{\overline{\alpha}_i} - \sum_{j=1}^{2}(|c_{ij}|H_j + |c_{ji}|H_i)\right\} = 0.$$

Hence, for model (16), Theorem 3.2, Theorem 3.3 and Theorem 3.4 in [26] do not hold, this means that Theorem 3.2, Theorem 3.3 and Theorem 3.4 in [26] are not applicable to ascertain the stability of model (16).

5 Conclusions

In this paper, reaction-diffusion generalized Cohen-Grossberg neural networks with both variable and distributed delays have been studied. Some sufficient conditions for assuring the existence and exponential stability of the equilibrium point have been established. These obtained results are new and they improve previously known results. Moreover, Two examples are given to illustrate the effectiveness of the new results.

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Received: October 23, 2006