Interpretability into Łukasiewicz Algebras

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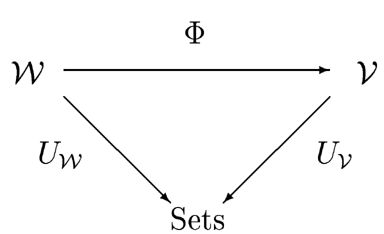
Abstract

In this paper we give a characterization of all the interpretations of the varieties of bounded distributive lattices, De Morgan algebras and Łukasiewicz algebras of order m in the variety of Łukasiewicz algebras of order n.
In the case of distributive lattices we give a structure theorem that is generalized to De Morgan algebras and to Łukasiewicz algebras of order m. In the last two cases we also give the number of such interpretations.

1 Introduction

We say that a variety $\mathcal{V}$ is *interpretable* in a variety $\mathcal{W}$, in symbols, $\mathcal{V} \leq \mathcal{W}$, if for each $\mathcal{V}$-operation $F_t(x_1, \ldots, x_n)$ there is a $\mathcal{W}$-term $f_t(x_1, \ldots, x_n)$ such that if $\langle A, G_t \rangle$ is in $\mathcal{W}$, then $\langle A, f_t^{\boldsymbol{A}} \rangle$ is in $\mathcal{V}$. Intuitively, $\mathcal{V} \leq \mathcal{W}$ means that all algebras in $\mathcal{W}$ can be turned into an algebra in $\mathcal{V}$ by defining the $\mathcal{V}$-operations applying a uniform procedure. This notion of interpretation differs from that used by logicians in that the universe of the algebra remains the same. It was first proposed in [7] and later developed in [5]; for more details and information the reader is referred to the latter monograph.
Another way of thinking about this notion is the following. The above relation defines a functor $\Phi : \mathcal{W} \longrightarrow \mathcal{V}$ which commutes with the underlying set functors, i.e.:



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is commutative; here $U_{\mathcal{V}} : \mathcal{V} \longrightarrow Sets$ and $U_{\mathcal{W}} : \mathcal{W} \longrightarrow Sets$ are the so called forgetful functors which assign to each algebra its universe. Each functor Φ is called an *interpretation of* $\mathcal{V}$ *in* $\mathcal{W}$.

If $\boldsymbol{A} = \langle A; G_t \rangle$ is any algebra and for each $\mathcal{V}$–operation $F_t(x_1, \ldots, x_n)$ there is a term $f_t(x_1, \ldots, x_n)$ in the language of $\boldsymbol{A}$ such that $\langle A; f_t^{\boldsymbol{A}} \rangle$ is in $\mathcal{V}$, the terms $f_t(x_1, \ldots, x_n)$ define an interpretation of $\mathcal{V}$ in $\mathcal{V}(\boldsymbol{A})$, the variety generated by the algebra $\boldsymbol{A}$. One only has to observe that the evaluation of any term in an algebra $\boldsymbol{B}$ in $\mathcal{V}(\boldsymbol{A})$, is determined by its evaluation in $\boldsymbol{A}$ and that both $\langle A; G_t \rangle$ and $\langle B; G_t \rangle$ satisfy the same equations. We sometimes say that $\mathcal{V}$ is interpretable in $\boldsymbol{A}$ and if Φ is the functor, we say that $\Phi(\boldsymbol{A})$ is an interpretation of $\mathcal{V}$ in $\mathcal{V}(\boldsymbol{A})$. This fact is particularly useful if we want to interpret a variety $\mathcal{V}$ in a variety $\mathcal{W}$ that is generated by a single algebra.

In this paper we will study what are all the possible interpretations of the varieties of bounded distributive lattices, De Morgan algebras and Łukasiewicz algebras of order m in the variety $\mathcal{L}_n$ of Łukasiewicz algebras of order n. As we know, this variety is generated by a single algebra, the n element chain, which is a semi–primal algebra. These are the main facts used in the proofs.

The results in sections 3 and 4 are included in [6], the author's doctoral dissertation *Interpretations between Varieties of Algebraic Logic.* The general presentation and most of the proofs are different from the ones that appear there.

2 Definitions and Preliminaries

Throughout this paper $\mathcal{D}_{01}$ will stand for the variety of bounded distributive lattices, $\mathcal{D}M$ the variety of De Morgan algebras, i.e., the class of all algebras $\langle A\,;\, +,\, \cdot\,,\, '\,,\, \mathbf{0}, \mathbf{1} \rangle$ whose similarity type is (2,2,1,0,0) and such that $\langle A,\ +,\ \cdot\ ,\ \mathbf{0},\ \mathbf{1} \rangle$is in $\mathcal{D}_{01}$ and satisfies

1. $(x + y)' = x'\, y'$,
2. $(x\, \cdot\, y)' = x' + y'$,
3. $x'' = x$,

The term x' is called the *quasi-complement* of x. Also, x and x' are said to be *conjugates.* The variety $\mathcal{L}_n$ of Łukasiewicz algebras of order n is the class of all algebras $\langle A\,;\, +, \cdot\,,\, '\,,\, \sigma_1, \ldots,\, \sigma_{n-1},\, \mathbf{0},\, \mathbf{1} \rangle$ of type (2,2,1, ... ,1,0,0) such that $\langle A\,;\, +, \cdot\,,\, '\,,\, \mathbf{0},\, \mathbf{1} \rangle$ is a De Morgan algebra and for $1 \leq i \leq n-1$,

1. $\sigma_i(x + y) = \sigma_i(x) + \sigma_i(y)$ and $\sigma_i(x\, \cdot\, y) = \sigma_i(x)\, \cdot\, \sigma_i(y)$,
2. $\sigma_i(x) + (\sigma_i(x))' = \mathbf{1}$ and $\sigma_i(x)\, \cdot\, (\sigma_i(x))' = \mathbf{0}$,
3. $\sigma_i(\sigma_j(x)) = \sigma_j(x)$, for $1 \leq j \leq n-1$,
4. $\sigma_i(x') = (\sigma_{n-i}(x))'$,
5. $\sigma_i(x)\, \cdot\, \sigma_j(x) = \sigma_i(x)$, for $i \leq j \leq n-1$,
6. $x + \sigma_{n-1}(x) = \sigma_{n-1}(x)$ and $x\, \cdot\, \sigma_1(x) = \sigma_1(x)$,
7. $y\, \cdot\, (x + (\sigma_i(x))' + \sigma_{i+1}(y)) = y$, for $i \neq n-1$.

These axioms are not independent. The reader is referred to [2], [1], [3] and [4] for more information about these classes of algebras.
The following four properties of Łukasiewicz algebras will be used extensively in section 5. The first two are immediate from axioms (1), (5) and (1), respectively. The fourth one was introduced in the original definition of Łukasiewicz algebras instead of axioms (6) and (7); its proof appears in [3].

Lemma 2.1.

($Ł_1$) $\sigma_i(\mathbf{0}) = \mathbf{0}$ *and* $\sigma_i(\mathbf{1}) = \mathbf{1}$, *for* $1 \leq i \leq n-1$.

($Ł_2$) $\sigma_1(x) \leq \cdots \leq \sigma_{n-1}(x)$.

($Ł_3$) *If* $x \leq y$, *then for* $1 < i \leq n-1$, $\sigma_i(x) \leq \sigma_i(y)$.

($Ł_4$) *If* $\sigma_i(x) = \sigma_i(y)$, *for* $1 \leq i < n$, *then* $x = y$.

We will now define a very important Łukasiewicz algebra.

Definition. Let $n = \{0, 1, \ldots, n-1\}$. We define the algebra

$$\mathcal{N} = \langle\, n\,; +,\ \cdot\,,\ ', \sigma_1, \ldots, \sigma_{n-1},\ \mathbf{0},\ \mathbf{1} \rangle,$$

where

$$\begin{aligned} x + y &= \max\ \{x, y\}, \\ x \cdot y &= \min\ \{x, y\}, \\ m' &= n - 1 - m, \qquad \text{for each } m \in n, \\ \mathbf{0} &= 0, \\ \mathbf{1} &= n - 1, \end{aligned}$$

and for $1 \leq i \leq n-1$

$$\sigma_i(m) = \begin{cases} \mathbf{1} & \text{if } i \leq m, \\ \mathbf{0} & \text{if } i > m. \end{cases}$$

It is easy to check that $\mathcal{N}$ is in $\mathcal{L}_n$. The next theorems give some of the most important features of Łukasiewicz algebras that we will use in the sequel. Their proofs and much more can be found in [1], [2] and [3].

Theorem 2.2. (Cignoli) [3]
Let $\boldsymbol{L} \in \mathcal{L}_n$, $n \geq 2$ *and* L *of cardinality greater than 1. Then the following are equivalent.*

1. $\boldsymbol{L}$ *is a chain.*
2. $\boldsymbol{L}$ *is an* $\mathcal{L}_n$*–subalgebra of* $\mathcal{N}$.
3. $\boldsymbol{L}$ *is subdirectly irreducible.*

Corollary 2.3. *The variety* $\mathcal{L}_n$ *is generated by the algebra* $\mathcal{N}$.

This corollary has a very important consequence. As we said in the introduction, any interpretation of a variety $\mathcal{V}$ in $\mathcal{L}_n$ is determined by an interpretation of $\mathcal{V}$ in $\mathcal{N}$, that is to say, by defining new term–defined operations f_t, for each $\mathcal{V}$–operation F_t, such that $\hat{\mathcal{N}} = \langle n; f_t^{\mathcal{N}} \rangle \in \mathcal{V}$.

Theorem 2.4. *$\mathcal{N}$ is a semi–primal algebra.*

As we know, in a semi–primal algebra all functions that preserve subuniverses can be represented by term functions. In the following theorem we will state this precisely in the special cases which we will use, that of unary and binary functions.

Theorem 2.5. *If $f : n \longrightarrow n$ is such that for all $a \in n$, $f(a) \in \{\mathbf{0}, a, a', \mathbf{1}\}$, then there exists a term $\varphi(x)$ such that*

$$\varphi^{\mathcal{N}}(x) = f(x).$$

If $g : n \times n \longrightarrow n$ is such that for all $a, b \in n$, $g(a, b) \in \{\mathbf{0}, a, a', b, b', \mathbf{1}\}$, then there exists a term $\gamma(x, y)$ such that

$$\gamma^{\mathcal{N}}(x, y) = g(x, y).$$

Lemma 2.6. *For any $a, b \in n$ and any $\mathcal{L}_n$–term $\alpha(x)$ or $\beta(x, y)$,*

$$\alpha^{\mathcal{N}}(a) \in \{\mathbf{0}, a, a', \mathbf{1}\} \qquad \text{and} \qquad \beta^{\mathcal{N}}(a, b) \in \{\mathbf{0}, a, a', b, b', \mathbf{1}\}.$$

Proof. Simply observe that $\{\mathbf{0}, a, a', \mathbf{1}\}$ and $\{\mathbf{0}, a, a', b, b', \mathbf{1}\}$ are subuniverses of $\mathcal{N}$. □

Corollary 2.7. *If $a \notin \{\mathbf{0}, \mathbf{1}\}$ and $a = \beta^{\mathcal{N}}(b, c)$ for some term $\beta^{\mathcal{N}}(x, y)$, then either $b \in \{a, a'\}$ or $c \in \{a, a'\}$.*

3 Interpreting $\mathcal{D}_{01}$ in $\mathcal{L}_n$

We will let $\hat{\mathcal{N}} = \langle n\,; \oplus, \odot, \hat{\mathbf{0}}, \hat{\mathbf{1}}\rangle$ be an interpretation of $\mathcal{D}_{01}$ in $\mathcal{L}_n$, that is, $x \oplus y$ and $x \odot y$ are binary $\mathcal{L}_n$-terms, $\hat{\mathbf{0}}$ and $\hat{\mathbf{1}}$ are $\mathcal{L}_n$–constant terms such that $\langle n\,; \oplus, \odot, \hat{\mathbf{0}}, \hat{\mathbf{1}}\rangle$ is a bounded distributive lattice.

Notice that while theorem 2.5 gives us a lot of flexibility, lemma 2.6 restricts the possible values of $x \oplus y$ and $x \odot y$. As for the constants, $\{\hat{\mathbf{0}}, \hat{\mathbf{1}}\} = \{\mathbf{0}, \mathbf{1}\}$.

We will prove several lemmas that will enable us to determine some special cases and a general structure theorem. The strategy is to use lemma 2.6 and the fact that $\hat{\mathcal{N}}$ is a distributive lattice to determine the possible values of the term functions defined by the terms $x \oplus y$ and $x \odot y$.

Throughout this paper, the following well known property of distributive lattices will be used without explicitly mentioning it. If $a \vee b = a \vee c$ and $a \wedge b = a \wedge c$, then $b = c$.

All the lemmas in this section refer to the lattice $\hat{\mathcal{N}}$. The first ten deal with the cases when $\hat{\mathbf{1}}$ is join–reducible and $\hat{\mathbf{0}}$ is meet–reducible. The next three are the cases when $\hat{\mathbf{0}}$ is meet–reducible, when $\hat{\mathbf{1}}$ is join–reducible and when there are some other meet–reducible and join–reducible elements. The main theorem 3.12 summarizes all these.

Lemma 3.1. *There is at most one pair of conjugates a, $a' \in n$, different from $\mathbf{0}$ and $\mathbf{1}$, such that $a \oplus a' = \hat{\mathbf{1}}$.*

Proof. Assume there exists $a,\, b \in n$, $a,\, b$ different from $\mathbf{0}$ and $\mathbf{1}$, a and b not conjugates, such that $a \oplus a' = \hat{\mathbf{1}}$ and $b \oplus b' = \hat{\mathbf{1}}$. By lemma 2.6 and since $\hat{\mathcal{N}}$ is a lattice, this implies that $a \odot a' = \hat{\mathbf{0}}$ and $b \odot b' = \hat{\mathbf{0}}$.
Assume $a \oplus b = \hat{\mathbf{1}}$. Then multiplying by a', we get $(a \odot a') \oplus (b \odot a') = b \odot a' = a'$ and then $a \odot b = a'$, so $b = (a \odot b) \oplus (a' \odot b) = (a \odot b) \oplus a'$ and then Corollary 2.7 forces $a \odot b = b'$. But then $b = b \oplus (a \odot b) = \hat{\mathbf{1}}$, a contradiction, so $a \oplus b \neq \hat{\mathbf{1}}$.
Assume either $a \oplus b = a$ or $a \oplus b = b$. In this case either $a' \oplus b = a' \oplus (a \oplus b) = \hat{\mathbf{1}}$ or $a \oplus b' = (a \oplus b) \oplus b' = \hat{\mathbf{1}}$, and this is the same as case 1. interchanging the roles of a and a' or those of b and b'.
Assume either $a \oplus b = a'$ or $a \oplus b = b'$. In this case either $a \oplus a' = a \oplus b \neq \hat{\mathbf{1}}$ or $b \oplus b' = a \oplus b \neq \hat{\mathbf{1}}$.
Since obviously $a \oplus b \neq \hat{\mathbf{0}}$, under the hypotheses $a \oplus b$ cannot be defined, so we may conclude that there is at most one pair of conjugate elements a and a' such that $a \oplus a' = \hat{\mathbf{1}}$. □

Lemma 3.2. *There is no element different from* $\mathbf{0}$ *and* $\mathbf{1}$ *that covers or is covered by more than two elements.*

Proof. Suppose $a \notin \{\mathbf{0}, \mathbf{1}\}$ covers three different elements b, c, d. That is $a = b \oplus c = b \oplus d = d \oplus c$.
From Corollary 2.7, we may assume w.l.o.g. that $b = a'$ and $c \neq a'$, $d \neq a'$, but then $d \oplus c = a$, contradicting lemma 2.6.
A dual argument shows that a is not covered by more than two elements. □

Lemma 3.3. *If there exist three elements* a, b *and* c *different from* $\hat{\mathbf{1}}$ *such that* $\hat{\mathbf{1}} = a \oplus b = b \oplus c = c \oplus a$, *then* $n = 8$.
Dually, if there exist three elements a, b *and* c *different from* $\hat{\mathbf{0}}$ *such that* $\hat{\mathbf{0}} = a \odot b = b \odot c = c \odot a$, *then* $n = 8$.

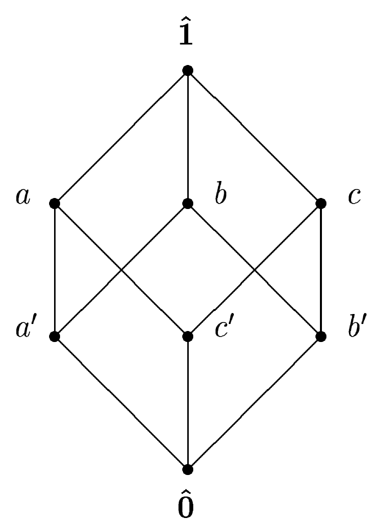

Diagram 1

Proof. Let us assume that there exist three such elements a, b and c as in Diagram 1. ($\hat{\mathbf{1}}$ is not necessarily a cover of a, b and c.)

Suppose $a \odot b = \hat{\mathbf{0}}$. Then $a = a \odot \hat{\mathbf{1}} = a \odot (b \oplus c) = a \odot c$, so $a \oplus c \neq \hat{\mathbf{1}}$, a contradiction, so $a \odot b \in \{a', b'\}$. Similarly, $a \odot c \in \{a', c'\}$ and $b \odot c \in \{b', c'\}$. Moreover the three products are all different or else $a = b$, $a = c$ or $b = c$. In particular this also implies that no two of them are conjugates and that $a \neq a'$, $b \neq b'$ and $c \neq c'$. So we have at least eight elements.
Let $a \odot b = a'$, then by the last remarks, $a \odot c = c'$ and this implies $b \odot c = b'$. Similarly, if $a \odot b = b'$, then $b \odot c = c'$ and $a \odot c = a'$, that is, the choice of $a \odot b$ (or of one of the others) determines the values of $a \odot c$ and of $b \odot c$ and we get the lattice in Diagram 1, (or one with b', a' and c' instead of a', c' and b', respectively.)
Let us now assume that $n > 8$ and let d be different from all of the above.
Suppose $a \oplus d = \hat{\mathbf{1}}$. Then of course $a \odot d \notin \{a, d, \hat{\mathbf{1}}\}$ and also $a \odot d \notin \{a', \hat{\mathbf{0}}\}$, or else $d \in \{b, b'\}$, (or $d \in \{c, c'\}$.) Thus $a \odot d = d'$.
Now $b \oplus d \neq \hat{\mathbf{1}}$, or else the same argument would show that $b \odot d = d'$ and this leads to $a = b$. Similarly, $c \oplus d \neq \hat{\mathbf{1}}$.
Also, $b \oplus d \neq d$, or else $c \oplus d = \hat{\mathbf{1}}$ and $b \oplus d \neq d'$, or else $a = a \oplus d' = a \oplus (b \oplus d) = \hat{\mathbf{1}}$. So $b \oplus d = b$ and similarly $c \oplus d = c$. But then $\hat{\mathbf{0}} = a \odot (b \odot c) = a \odot ((b \oplus d) \odot (c \oplus d)) = a \odot ((b \odot c) \oplus d) = a \odot d = d'$, a contradiction, thus $a \oplus d \neq \hat{\mathbf{1}}$. Similarly we prove that $b \oplus d \neq \hat{\mathbf{1}}$ and $c \oplus d \neq \hat{\mathbf{1}}$.

Suppose now that $a \oplus d \in \{d, d'\}$. Then $d \oplus b = \hat{\mathbf{1}}$ or $d' \oplus b = \hat{\mathbf{1}}$, a contradiction. Finally, the only choice is $a \oplus d = a$, so multiplying this by b', we get $d \odot b' = \hat{\mathbf{0}}$. But then $d \oplus b' \notin \{\hat{\mathbf{1}}, \hat{\mathbf{0}}, d, b'\}$. Also, $d \oplus b' \neq b$, or else $d = a'$ and $d \oplus b' \neq d$, or else $a \oplus d = \hat{\mathbf{1}}$. Since there is no possible value for $a \oplus d$, such an element cannot exist and $n = 8$.
The proof of the dual is similar. ☐

Lemma 3.4. *Assume there exists an $a \notin \{\mathbf{0}, \mathbf{1}\}$ such that $a \oplus a' = \hat{\mathbf{1}}$. If $a \oplus b = \hat{\mathbf{1}}$ for some $b \notin \{a, a', \mathbf{1}, \mathbf{0}\}$, then the subalgebra of $\hat{\mathcal{N}}$ generated by a and b is the lattice in Diagram 2 (a).*

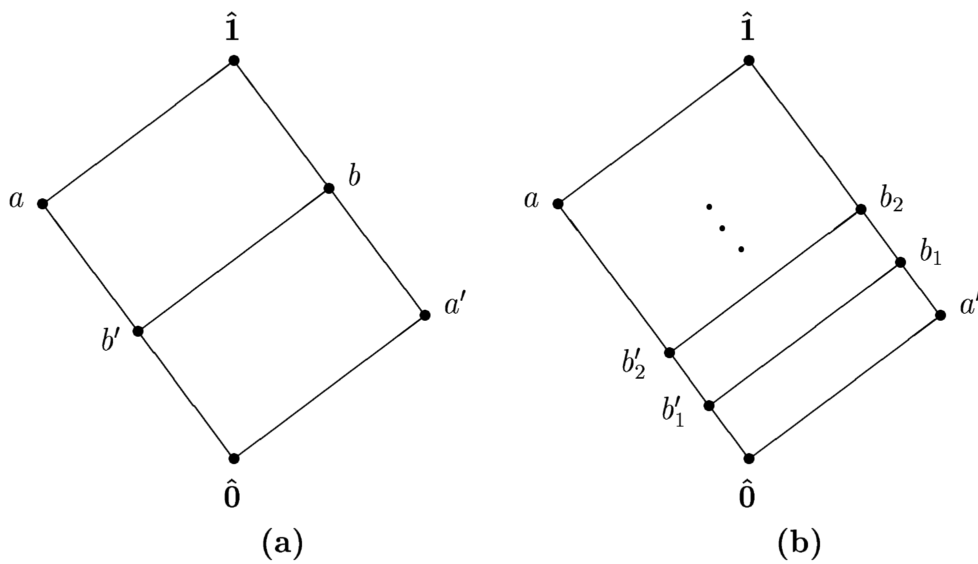

Diagram 2

Proof. Let $b \notin \{\mathbf{0}, \mathbf{1}, a, a'\}$. Since $a \oplus b = \hat{\mathbf{1}}$, $a \odot b \notin \{a, b, \hat{\mathbf{1}}\}$. Also $a \odot b \neq \hat{\mathbf{0}}$, or else $b = a'$ $a \odot b \neq a'$, or else $a \oplus a' = a \neq \hat{\mathbf{1}}$, so $a \odot b = b'$ and thus $a \odot b' = b'$. But then $a' \odot b' = a' \odot (a \odot b') = \hat{\mathbf{0}}$.
A similar dual argument shows that $a' \oplus b' = b$, which completes the proof of our lemma. □

Corollary 3.5. *If $\hat{\mathbf{1}}$ is a cover of a and a', then $n = 4$.*

Lemma 3.6. *Let $n \neq 4, 8$. Assume there exists an $a \notin \{\mathbf{0}, \mathbf{1}\}$, such that $a \oplus a' = \hat{\mathbf{1}}$. If $a \oplus b = \hat{\mathbf{1}}$ for some $b \notin \{\mathbf{0}, \mathbf{1}, a, a'\}$, then for all $c \notin \{\mathbf{0}, \mathbf{1}, a, a', b, b'\}$, either*

$$a \oplus c = \hat{\mathbf{1}} \quad \text{and} \quad a \odot c = c' \quad \text{or}$$
$$a \oplus c' = \hat{\mathbf{1}} \quad \text{and} \quad a \odot c' = c,$$

and thus $\hat{\mathcal{N}}$ is the lattice in Diagram 2 (b). The intermediate elements need not exist.

Proof. By lemma 3.4, the lattice generated by a and b is the lattice in Diagram 2 (a). Let $c \notin \{\mathbf{0}, \mathbf{1}, a, a', b, b'\}$
If $a \oplus c = \hat{\mathbf{1}}$, then by lemma 3.4, the subalgebra of $\hat{\mathcal{N}}$ generated by a and c is the lattice in Diagram 2 (a), with b replaced by c, that is, $a \odot c = c'$.
Since $n \neq 8$, $b \oplus c \neq \hat{\mathbf{1}}$, so either $b \oplus c = b$ or $b \oplus c = c$, in which case either $b \oplus c' = c$ and $b \odot c' = b'$ or $b' \oplus c = b$ and $b' \odot c = c'$, respectively. Since this is the case with any other element d such that $a \oplus d = \hat{\mathbf{1}}$, the theorem follows.
If $a \oplus c = a$, then $a' \odot c = \hat{\mathbf{0}}$ and thus $a' \oplus c \neq \hat{\mathbf{1}}$, since the latter would entail $a = c$. So $a' \oplus c = c'$ and thus $a \oplus c' = \hat{\mathbf{1}}$ and we are back in the previous case.
If either $a \oplus c = c$ or $a \oplus c = c'$, then there is an element between a and $\hat{\mathbf{1}}$. We may assume it is c. But then $b \oplus c = \hat{\mathbf{1}}$ and since $c > a > c'$, $b \odot c = c'$ is the only possibility for $b \odot c$, but this is clearly impossible since in that case $a = c$. □

Lemma 3.7. *If a and b are not conjugates, $a \oplus b = \hat{\mathbf{1}}$ and $a \odot b = \hat{\mathbf{0}}$, then neither $a = a'$ nor $b = b'$.*

Proof. Suppose $a \oplus b = \hat{\mathbf{1}}$, $a \odot b = \hat{\mathbf{0}}$ and $a = a'$. Then $b \neq b'$, since there is only one element x such that $x = x'$.
If $b \oplus b' = \hat{\mathbf{1}}$, then else $a = b'$, so either $b < b'$ or $b' < b$.
If $b < b'$, $a \oplus b' = \hat{\mathbf{1}}$ and in that case $a \odot b' \neq \hat{\mathbf{0}}$, or else $b' = b$. So $a \odot b' = b$, but then $a = a \oplus (a \odot b') = a \oplus b = \hat{\mathbf{1}}$.
On the other hand, if $b' < b$, $a \odot b' = \hat{\mathbf{0}}$ and the dual of the above argument provides a contradiction. □

Theorem 3.8. *If a and b are not conjugates, they are both different from $\mathbf{0}$ and $\mathbf{1}$, $a \oplus b = \hat{\mathbf{1}}$ and $a \odot b = \hat{\mathbf{0}}$, then $n = 6$ or $n = 8$.*

Proof. Notice that by lemma 3.7 we need at least six elements. Also, $a \oplus a' \neq \hat{\mathbf{1}}$ and $b \oplus b' \neq \hat{\mathbf{1}}$ or else a and b are conjugates.
By renaming if necessary, we may assume that $a \oplus a' = a$ and $b \oplus b' = b'$.
This implies that $a' \odot b = \hat{\mathbf{0}}$, so $a' \oplus b \neq \hat{\mathbf{1}}$ or else $a = a'$, contradicting lemma 3.7.
We can easily check that the subalgebra generated by a and b, is the one depicted in Diagram 3.

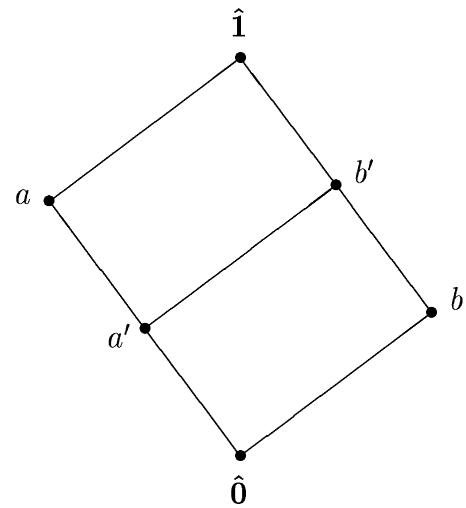

Diagram 3

This proves that if $n = 6$, there is a possible interpretation with the features of the hypothesis.
Let us now assume $n \geq 7$, so let c be different from all of the above. Suppose $a \oplus c = \hat{\mathbf{1}}$. Using a now familiar argument, $a \odot c \notin \{\hat{\mathbf{1}}, a, a', c\}$, the latter would imply $b' = c$. Also, $a \odot c \neq \hat{\mathbf{0}}$, or else $c = b$, so the only possibility is $a \odot c = c'$. Multiplying by b', we get $a' \odot c = c' \odot b'$.
If $b' \oplus c = c$, then $c' \odot b' = a' \odot c = a' \odot (b' \oplus c) = a'$ and if $b' \oplus c = c'$, then $a' \odot c = c' \odot b' = (b' \oplus c) \odot b' = b'$. Both cases contradict lemma 2.6. The only possibility left is $b' \oplus c = \hat{\mathbf{1}}$, so by lemma 3.3, $n = 8$.
Suppose $a \oplus c = c$. Then $b \oplus c = \hat{\mathbf{1}}$, so $b \odot c = c'$ and similarly $b' \oplus c = \hat{\mathbf{1}}$, so $b' \odot c = c'$ and this implies $b = b'$, a contradiction. We get a similar contradiction if we assume $a \oplus c = c'$ and since there are no other possibilities, the theorem is proved. □

Lemma 3.9. *Let $n \neq 6$, 8. If there exist elements a and b such that $a \odot b = \hat{\mathbf{0}}$, $a \oplus b = b'$ and $a \oplus a' \neq \hat{\mathbf{1}}$, then there exists an element $c \in n$ such that the interval $[\hat{\mathbf{0}}, c']$ of $\hat{\mathcal{N}}$ is the lattice depicted in Diagram 4 (a) and c is the $\hat{\mathcal{N}}$–largest such an element, (that is, for any element d such that $d \odot a = \hat{\mathbf{0}}$, $d \odot c = c$.) The intermediate elements need not exist. c' is meet–irreducible.*

Proof. If there is no $x \in n$ other than b such that $x \odot a = \hat{\mathbf{0}}$, we let $c = b$.
Since $n \neq 6, 8$ and $a \oplus a' \neq \hat{\mathbf{1}}$, there is no $x \in n$ such that $x \odot a = \hat{\mathbf{0}}$ and $x \oplus a = \hat{\mathbf{1}}$.
We will now prove that there is no $x \in n$ such that $x \odot a = \hat{\mathbf{0}}$ and $x \oplus a = a'$. If on the contrary there is one, since $n \neq 8$, $b \odot x \neq \hat{\mathbf{0}}$ and obviously $b \odot x \neq b'$.
Suppose $b \odot x \in \{x, x'\}$, then $b \odot a' = b \odot (x \oplus a) = b \odot x \in \{x, x'\}$ and this contradicts lemma 2.6.
Suppose $b \odot x = b$, then $b' \oplus x = (a \oplus b) \oplus x = a \oplus (b \oplus x) = a \oplus x = a'$, which also contradicts lemma 2.6.
So if $x \odot a = \hat{\mathbf{0}}$, then $x \oplus a \neq a'$ and thus $x \oplus a = x'$, as in the Diagram.

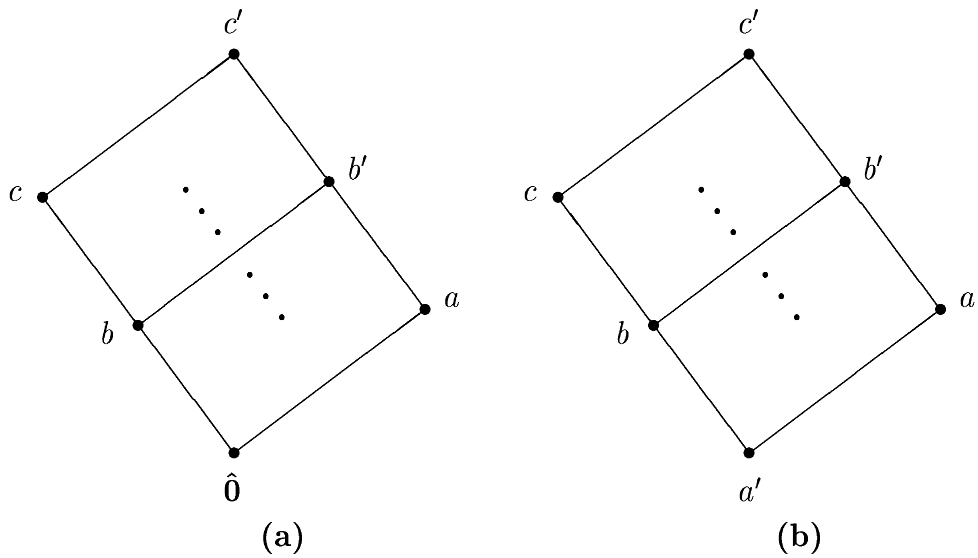

Diagram 4

Now the set of all elements $x \in n$ such that $x \odot a = \hat{\mathbf{0}}$ and $x \oplus a = x'$ has to be linearly ordered since if for two such elements x and y, $x \odot y \neq x, y$, then $x \odot y = \hat{\mathbf{0}}$, contradicting the fact that $n \neq 8$. Take c to be the largest one. By Corollary 2.7, c' is meet–reducible. □

By duality, we can prove the following.

Corollary 3.10. *Let $n \neq 6, 8$. If there exist elements a and c such that $a \oplus c = \hat{\mathbf{1}}$, $a \odot c = c'$, and $a \odot a' \neq \hat{\mathbf{0}}$, then there exists an element $c \in n$ such that the interval $[c', \hat{\mathbf{1}}]$ of $\hat{\mathcal{N}}$ is dual to the lattice depicted in Diagram 4 (a) and c is the $\hat{\mathcal{N}}$–least such an element. The intermediate elements need not exist. Also, c' is join–irreducible.*

Lemma 3.11. *If there exist a, c both different from $\mathbf{0}$ and $\mathbf{1}$ such that $a \oplus c = c'$ and $a \odot c = a'$, then the interval $[a', c']$ is the lattice in Diagram 4 (b).
Moreover, if there is no element b such that $a \oplus b = \hat{\mathbf{1}}$, then there exists the $\hat{\mathcal{N}}$–largest such an element c. The intermediate elements need not exist.*

Proof. Let a and c be two such elements and let b be any other element in $[c', a']$.
Suppose $c \oplus b = b$. Then $c' \geq a \oplus b = a \oplus c \oplus b = c' \oplus b \geq c'$, so $c \oplus b = a'$, contradicting lemma 2.6. A similar contradiction is obtained if $a \oplus b = b'$.
So either $c \oplus b = c'$, in which case one obtains the lattice in Diagram 4 (b), or $c \oplus b = c$ and we obtain that lattice with b and b' interchanged.
If there is no element b such that $a \oplus b = \hat{\mathbf{1}}$, the largest such an element c exists by an argument similar to the one used in lemma 3.9. □

For the main theorem of this section we will use the following notation. If $\mathcal{A}$ and $\mathcal{B}$ are two lattices, $\top_{\mathcal{A}}$ is the largest element of $\mathcal{A}$ and $\perp_{\mathcal{B}}$ is the least element of $\mathcal{B}$.

We define $\mathcal{A} \dagger \mathcal{B}$ as the lattice obtained by identifying $\top_{\mathcal{A}}$ and $\bot_{\mathcal{B}}$ and extending the order in the natural way, i.e. if $x,\ y \in A \cup B$ then

$$x \leq y \text{ iff } \begin{cases} x,\ y \in A, \quad \text{and} \quad x \leq_{\mathcal{A}} y \\ x \in A,\ y \in B \\ x,\ y \in B, \quad \text{and} \quad x \leq_{\mathcal{B}} y. \end{cases}$$

Theorem 3.12. *Let $n \neq 6, 8$. Then any interpretation of $\mathcal{D}_{01}$ in $\mathcal{N}$ is of the form*

$$\mathcal{A}_1 \dagger \mathcal{A}_2 \dagger \cdots \dagger \mathcal{A}_m,$$

where for each $i \leq m$, $\mathcal{A}_i$ is either a chain or one of the lattices in Diagram 5. Conversely, each such lattice gives rise to an interpretation of $\mathcal{D}_{01}$ in $\mathcal{N}$. In each case the intermediate elements need not exist.

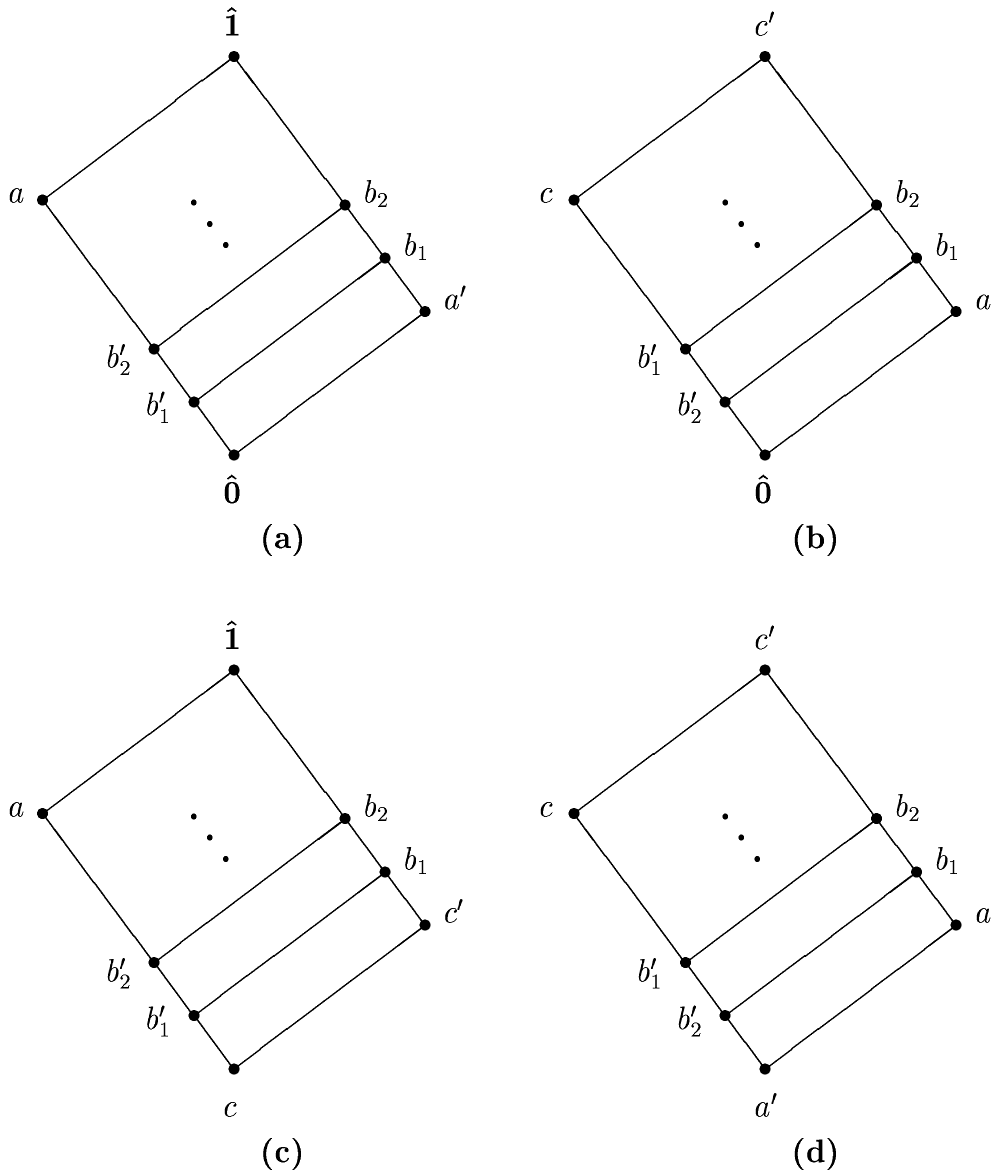

Diagram 5

Proof.
If there is no meet–reducible element in $\hat{\mathcal{N}}$, the interpretation is a chain.

If there exist meet–reducible elements, then there are several cases.
Case 1: The $\mathcal{N}$–least meet–reducible element is $\hat{\mathbf{0}}$ and there is an element a such that $a \odot a' = \hat{\mathbf{0}}$. Then by lemma 3.4, the interpretation is a lattice as in Diagram 5 (a).
Case 2: $\hat{\mathbf{0}}$ is the $\mathcal{N}$–least meet–reducible element and there are elements a, b, not conjugates, such that $a \odot b = \hat{\mathbf{0}}$.
Since $n \neq 6, 8$, $a \oplus b$ must be either a' or b'. We may assume w.l.o.g. that $a \oplus b = a'$. Then by lemma 3.9, there exists a $\mathcal{N}$–greatest element c such that the interval $[\hat{\mathbf{0}}, c']$ is a lattice as in Diagram 5 (b). We let $c_1 = c'$ and $\mathcal{A}_1 = [\hat{\mathbf{0}}, c_1]$. Observe that since c_1 must be meet irreducible, it has a unique cover.

1. If there is no meet–reducible element x such that $c_1 < x < \hat{\mathbf{1}}$, we let $\mathcal{A}_2 = [c_1, \hat{\mathbf{1}}]$ and thus $\hat{\mathcal{N}} = \mathcal{A}_1 \dagger \mathcal{A}_2$. Observe that $\mathcal{A}_2$ is a chain.

2. If there is one, let c_2 be the $\mathcal{N}$–least meet–reducible element greater than c_1. We let $\mathcal{A}_2 = [c_1, c_2]$. Again $\mathcal{A}_2$ is a chain of length at least 2.

   Since c_2 is meet reducible, there exists an element a such that $a \odot c_2' = c_2$. Again we have two possibilities, either $a \oplus c_2' = \hat{\mathbf{1}}$ or $a \oplus c_2' = a'$.

   (a) In the first case, by lemma 3.10 the interval $[c_2, \hat{\mathbf{1}}]$ is a lattice as the one in Diagram 5 (c). Let $\mathcal{A}_3 = [c_2, \hat{\mathbf{1}}]$ and $\hat{\mathcal{N}} = \mathcal{A}_1 \dagger \mathcal{A}_2 \dagger \mathcal{A}_3$.

   (b) In the second case, by lemma 3.11, there exists the largest element c such that the interval $[c_2, c']$ is a lattice as the one in Diagram 5 (d). We let $c_3 = c'$ and $\mathcal{A}_3 = [c_2, c_3]$.

We can now continue as in the previous step, searching for c_4, the next meet–reducible element, if one exists, and proceed as we did with c_2. The process must eventually terminate and we have $\hat{\mathcal{N}} = \mathcal{A}_1 \dagger \mathcal{A}_2 \dagger \cdots \dagger \mathcal{A}_m$.
Case 3: $\hat{\mathbf{0}}$ is not the $\mathcal{N}$–least meet–reducible element. Then since $\hat{\mathbf{0}}$ is meet–irreducible, it has a single immediate successor. Let c_1 be the $\hat{\mathcal{N}}$–least meet–reducible element and define $\mathcal{A}_1 = [\hat{\mathbf{0}}, c_1]$. $\mathcal{A}_1$ is a chain. Now proceed as in step 2 with c_1 in place of $\hat{\mathbf{0}}$. This completes the proof of necessity.

That each such lattice gives rise to an interpretation of $\mathcal{D}_{01}$ in $\mathcal{N}$ is immediate from theorem 2.5. □

4 Interpreting $\mathcal{DM}$ in $\mathcal{L}_n$

We will now study the interpretations of the variety $\mathcal{DM}$ of De Morgan algebras in the variety of Łukasiewicz algebras. In this case we also have to interpret the unary operation $'$, the quasi–complement. Of course, since De Morgan algebras are distributive lattices, all that was proved in the previous section holds for them.
Throughout this section, we will let $\hat{\mathcal{N}} = \langle n\,; \oplus, \odot, ^{\ominus}, \hat{\mathbf{0}}, \hat{\mathbf{1}} \rangle$ be an interpretation of $\mathcal{DM}$ in $\mathcal{L}_n$, where $\langle n\,; \oplus, \odot, \hat{\mathbf{0}}, \hat{\mathbf{1}} \rangle$ is an interpretation of $\mathcal{D}_{01}$ in $\mathcal{L}_n$ as in section 3 and $^{\ominus}$ is a unary operation, the interpretation of the quasi–complement $'$.

The first lemma is a straightforward consequence of the definition of a quasi–complement and lemma 2.6.

Lemma 4.1.

1. *The quasi–complement* $^{\ominus}$ *is one–to–one.*
2. $\hat{\mathbf{0}}^{\ominus} = \hat{\mathbf{1}}$ *and* $\hat{\mathbf{1}}^{\ominus} = \hat{\mathbf{0}}$.
3. *If* $a^{\ominus} = a$, *then* $a'^{\ominus} = a'$.
4. *If* $a \leq b$, *then* $b^{\ominus} \leq a^{\ominus}$.

Theorem 4.2. *If* $n \neq 4,\ 6$, *then the underlying lattice of every interpretation of* $\mathcal{DM}$ *in* $\mathcal{L}_n$ *is a chain and the new quasi–complement coincides with the old one.*

Proof. Let $\hat{\mathcal{N}}$ be an interpretation of $\mathcal{DM}$ in $\mathcal{L}_n$ and suppose there exist a meet–reducible element c. We have several cases.
If $c = \hat{\mathbf{0}}$ and $\hat{\mathbf{0}} = a \odot a'$, then underlying lattice of $\hat{\mathcal{N}}$ is like the lattice in Diagram 5 (a) and since $n \neq 4,\ 6$, there exist at least four elements b_1, b_2, b_1' and b_2' like the ones depicted in the Diagram. But then $b_1 \oplus b_2' = b_2$ and $b_1 \geq b_1^{\ominus}$. So $b_1 \geq b_1^{\ominus} \odot b_2'^{\ominus} = b_2^{\ominus} \in \{b_2, b_2'\}$ and this is not possible.
If $c = \hat{\mathbf{0}}$ and $\hat{\mathbf{0}} = a \odot b$, where a and b are not conjugates, then since $n \neq 6$, the underlying lattice of $\hat{\mathcal{N}}$ is like the lattice in Diagram 5 (b). In this case, since $b' > c'$, $\hat{\mathbf{1}} = b^{\ominus} \oplus c^{\ominus} \in \{c', b'\}$.
If $c = \hat{\mathbf{0}}$, $n = 8$ and the underlying lattice of $\hat{\mathcal{N}}$ is the one in Diagram 1, then $a \odot c = c'$, but taking quasi–complements, $a' \oplus c' = c \neq a$.
If $c > \hat{\mathbf{0}}$. Then in the decomposition of $\hat{\mathcal{N}}$, $\mathcal{A}_1$ is a chain and $\mathcal{A}_2$ is either the lattice in Diagram 5 (c) or the one in 5 (d). These cases are similar to case 2.

So in any case we get a contradiction, thus there is no meet–reducible element and $\hat{\mathcal{N}}$ is a chain.

Assume now that there exists an a such that $a^{\ominus} = a$. By lemma 4.1.3, $a'^{\ominus} = a'$. So if $a < a'$ and by lemma 4.1.4, $a' = a'^{\ominus} < a^{\ominus} = a$, a contradiction. A similar contradiction arises if we assume that $a' < a$. This implies that $a = a'$ and thus $a^{\ominus} = a = a'$. Now since in a chain there is at most one element such that $a = a'$, for any other b, $b^{\ominus} = b'$. So for any x, $x^{\ominus} = x'$. □

In the following theorem we will prove that if $n = 4$, there are two possible definitions for the quasi–complement.

Theorem 4.3. *There are 8 interpretations of* $\mathcal{DM}$ *in* $\mathcal{L}_4$.

Proof. Let $n = \{\mathbf{0}, a, a', \mathbf{1}\}$. Recall that the underlying lattice of the interpretation must be isomorphic to one of the lattices of Diagram 6.

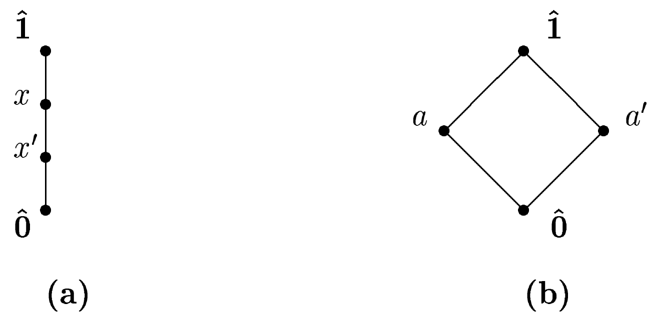

Diagram 6

If the lattice interpretation is the lattice in Diagram 6 (a), since $\hat{\mathbf{1}}$ can be either $\mathbf{0}$ or $\mathbf{1}$, we have two choices. For each of these, x can be either a or a' and that gives us 4 possibilities.
An argument similar to the one in the previous theorem shows that for all $x^{\ominus} = x'$. This gives us four interpretations.
If the lattice interpretation is the lattice in Diagram 6 (b), again $\hat{\mathbf{1}}$ can be either $\mathbf{0}$ or $\mathbf{1}$, so we have two choices, but in this case, by symmetry we have only one choice for a. For each of these there are two possible quasi-complements, namely,

$$\begin{aligned} x^{\ominus} &= x' \\ x^{\ominus} &= (\sigma_1(x))' + x(\sigma_3(x))'. \end{aligned}$$

The first function defines the four element Boolean algebra. The second function assigns $\mathbf{0}$, a, a' and $\mathbf{1}$ to $\mathbf{1}$, a, a' and $\mathbf{0}$, respectively. It is well known fact that these two are $\mathcal{D}M$ algebras. This provides the other four interpretations. □

Theorem 4.4. *There are 32 interpretations of $\mathcal{D}M$ in $\mathcal{L}_6$.*

Proof. Let $6 = \{\hat{\mathbf{0}}, a, b, a', b', \mathbf{1}\}$. If the underlying lattice of the interpretation is a chain, its first element $\hat{\mathbf{0}}$ has to be either $\mathbf{0}$ or $\mathbf{1}$. For each of those, the second element can be filled by any of the four elements a, a', b or b', the third has only two possibilities since the others are determined by the previous selections and lemma 4.1.4. That gives us 16 possible interpretations.
If the underlying lattice of the interpretation is the lattice in Diagram 2 (a) and $b^{\ominus} = b$, then by lemma 4.1.4, $b'^{\ominus} > b^{\ominus}$, which would force $b'^{\ominus} = \hat{\mathbf{1}}$, contradicting lemma 4.1.1, so $b^{\ominus} = b'$.But then $a^{\ominus} \odot b' = (a \oplus b)^{\ominus} = \hat{\mathbf{0}}$, so $a^{\ominus} = a$. So for all x, $x^{\ominus} = x'$. The reader can easily check that the old quasi–complement works well. A similar analysis to that of the previous paragraph shows there are another 16 interpretations of this sort.
Finally, using the same arguments of Theorem 4.2, one can check that for the other possible underlying lattices for an interpretation, there is no acceptable definition for the quasi–complement. For instance, in the lattice in Diagram 3, $a \oplus b = a \oplus b' = \hat{\mathbf{1}}$, so $a^{\ominus} \odot b^{\ominus} = a^{\ominus} \odot b'^{\ominus} = \hat{\mathbf{0}}$. But in this lattice this is possible only if $b^{\ominus} = b'^{\ominus} = b$, a contradiction. There are essentially three other underlying lattices. □

Theorem 4.5. *The number of interpretations of $\mathcal{DM}$ in $\mathcal{L}_n$ is*

$$\begin{array}{ll} 2^{\frac{n}{2}}(\frac{n}{2}-1)! & \text{if } n \text{ is even, } n \neq 4,\ 6, \\ 2^{\frac{n-1}{2}}(\frac{n-1}{2}-1)! & \text{if } n \text{ is odd.} \end{array}$$

Proof. The proof is a straight forward generalization of the $n = 6$ case. One must observe that in the odd case, there is one single element c for which $c' = c$ and by lemma lemma 4.1.4 it must be assigned to the "midpoint" of the underlying lattice. □

5 Interpreting $\mathcal{L}_m$ in $\mathcal{L}_n$

In the previous section we proved that De Morgan interpretations are pretty tight. We will now extend those results to Łukasiewicz algebras, that is, we have to define the new unary operations $\hat{\sigma}_1, \hat{\sigma}_2, \ldots, \hat{\sigma}_{m-1}$.
Throughout this section $\hat{\mathcal{N}} = \langle n\,; \oplus, \odot, ^{\ominus}, \hat{\sigma}_1 \ldots, \hat{\sigma}_{m-1}, \hat{\mathbf{0}}, \hat{\mathbf{1}}\rangle$ will be an interpretation of $\mathcal{L}_m$ in $\mathcal{L}_n$, where $\langle n\,; \oplus, \odot, ^{\ominus}, \hat{\mathbf{0}}, \hat{\mathbf{1}}\rangle$ is an interpretation of $\mathcal{DM}$ in $\mathcal{L}_n$ as in section 4 and the $\hat{\sigma}_i$'s are unary operations, the interpretation of the σ_i's. Of course this means that except for $n = 4$ and 6, $\langle n\,; \oplus, \odot, ^{\ominus}, \hat{\mathbf{0}}, \hat{\mathbf{1}}\rangle$ is a chain and the quasi–complements $^{\ominus}$ and $'$ coincide, so we will analize these two cases separately.

Lemma 5.1. *If $\hat{\mathbf{4}}$ is an interpretation of $\mathcal{L}_m$ in $\mathcal{L}_4$ and its underlying lattice is not a chain, then $\hat{\mathbf{4}}$ is the four element Boolean algebra and for all x, $\hat{\sigma}_1(x) = \hat{\sigma}_2(x) = \cdots = \hat{\sigma}_{m-1}(x) = x$.*

Proof. Assume that the underlying lattice of the interpretation is the lattice in Diagram 6 (b) and that $a^{\ominus} = a$ and $a'^{\ominus} = a'$. Then $\hat{\sigma}_{m-1}(a) \neq a,\ a'$ or else $\hat{\sigma}_{m-1}(a) \oplus (\hat{\sigma}_{m-1}(a)\,)^{\ominus} \in \{a, a'\}$, contradicting axiom (2).
By axiom (1), since $a \odot a' = \hat{\mathbf{0}}$, $\hat{\sigma}_{m-1}(a) \odot \hat{\sigma}_{m-1}(a') = \hat{\mathbf{0}}$, so either $\hat{\sigma}_{m-1}(a) = \hat{\mathbf{0}}$ or $\hat{\sigma}_{m-1}(a') = \hat{\mathbf{0}}$. But $\hat{\sigma}_{m-1}(a) = \hat{\mathbf{0}}$ (and similarly $\hat{\sigma}_{m-1}(a') = \hat{\mathbf{0}}$) is clearly impossible because by (Ł$_2$) we would have $\hat{\sigma}_1(a) = \hat{\sigma}_2(a) = \cdots = \hat{\sigma}_{m-1}(a) = \hat{\mathbf{0}}$, that is to say, for all $i \leq m-1$, $\hat{\sigma}_i(a) = \hat{\sigma}_i(\hat{\mathbf{0}})$, which in turn by (Ł$_4$) implies $a = \hat{\mathbf{0}}$.
If the De Morgan interpretation is the four element Boolean algebra, then it is a well known fact that the only possibility for the $\hat{\sigma}_i$'s is the identity. See [3]. □

Lemma 5.2. *If $\hat{\mathbf{6}}$ is an interpretation of $\mathcal{L}_m$ in $\mathcal{L}_6$, then its underlying lattice is a chain.*

Proof. Suppose the lattice reduct of $\hat{\mathbf{6}}$ is not a chain, then by Theorem 4.4, it is the one that appears in Diagram 2 (a) and $x^{\ominus} = x'$ for all x.
If $\hat{\sigma}_i(b) \in \{b, b'\}$, then $\hat{\mathbf{1}} = \hat{\sigma}_i(b) \oplus (\hat{\sigma}_i(b))^{\ominus} = b \oplus b' = b$, so for any i, $\hat{\sigma}_i(b) \in \{\hat{\mathbf{0}}, \hat{\mathbf{1}}\}$. Similarly, $\hat{\sigma}_i(b') \in \{\hat{\mathbf{0}}, \hat{\mathbf{1}}\}$.
As in the previous lemma, $\hat{\sigma}_{m-1}(b) \neq \hat{\mathbf{0}}$. So $\hat{\sigma}_{m-1}(b) = \hat{\sigma}_{m-1}(b') = \hat{\mathbf{1}}$, which, by (Ł$_3$) and since $a \geq b'$, implies $\hat{\sigma}_{m-1}(a) = \hat{\mathbf{1}}$. But then, $\hat{\mathbf{0}} = \hat{\sigma}_{m-1}(a \odot a') = \hat{\sigma}_{m-1}(a) \odot \hat{\sigma}_{m-1}(a') = \hat{\sigma}_{m-1}(a')$, which as we know implies $a' = \hat{\mathbf{0}}$, a contradiction. So there is no possible definition for $\hat{\sigma}_{m-1}(b)$ and the lattice reduct must be a chain. □

Lemma 5.3. *Let $\hat{\mathcal{N}}$ be an interpretation of $\mathcal{L}_m$ in $\mathcal{L}_n$ for which the underlying lattice is a chain. Then for $0 \le i \le m-1$, $\hat{\sigma}_i(x) \in \{\hat{\mathbf{0}},\ \hat{\mathbf{1}}\}$.*
If we let $\mu(a)$ be the least k such that $\hat{\sigma}_k(a) = \hat{\mathbf{1}}$, then μ defines a one–to–one correspondence between the non-zero elements of $\hat{\mathcal{N}}$ and the $\hat{\sigma}_i$'s. Moreover, if the De Morgan reduct of $\hat{\mathcal{N}}$ is $\hat{\mathbf{0}} = a_0 < a_1 < \cdots < a_{n-1} = \hat{\mathbf{1}}$, setting $\mu(\hat{\mathbf{0}}) = n$, $\mu(a_j) \ge n-j$.

Proof. We first observe that since the lattice reduct of $\hat{\mathcal{N}}$ is a chain, every element, in particular $\hat{\mathbf{1}}$ is join irreducible, so by axiom (2), for all $i \le m-1$ and any x, $\hat{\sigma}_i(x) \in \{\hat{\mathbf{0}},\ \hat{\mathbf{1}}\}$.
Next, recall that by (Ł$_1$), for all $i \le n-1$, $\hat{\sigma}_i(\hat{\mathbf{0}}) = \hat{\mathbf{0}}$, so for $a \ne \hat{\mathbf{0}}$, there exists some k such that $\hat{\sigma}_k(a) = \hat{\mathbf{1}}$. If not, for all $i \le m-1$, $\hat{\sigma}_i(a) = \hat{\mathbf{0}} = \hat{\sigma}_i(\hat{\mathbf{0}})$ and by (Ł$_4$), $a = \hat{\mathbf{0}}$, a contradiction.
Let $a \ne \hat{\mathbf{0}}$ and $b \ne \hat{\mathbf{0}}$. We now observe that if $a \ne b$, then $\mu(a) \ne \mu(b)$. If not, by (Ł$_2$), for $r \ge \mu(a) = \mu(b)$, $\hat{\sigma}_r(a) = \hat{\mathbf{1}} = \hat{\sigma}_r(b)$ and for $r < \mu(a)$, $\hat{\sigma}_r(a) = \hat{\mathbf{0}} = \hat{\sigma}_r(b)$, so again using (Ł$_4$), we get $a = b$, a contradiction.
Suppose that $k = \mu(a_j) < n-j$, for some $0 < j < n$. Then $\hat{\sigma}_{k-1}(a_j) = \hat{\mathbf{0}}$. This implies that $\hat{\sigma}_{k-1}(a_{j+1}) = \hat{\mathbf{1}}$ or else by (Ł$_2$) and (Ł$_3$), $\hat{\sigma}_r(a_j) = \hat{\sigma}_r(a_{j+1})$, for all $1 \le r < n$, and by (Ł$_4$), $a_j = a_{j+1}$. So $\mu(a_{j+1}) < n-j-1$.
In a similar way we prove that for $s \le k-1$, $\hat{\sigma}_{k-s}(a_{j+s}) = \hat{\mathbf{1}}$, in particular, $\hat{\sigma}_1(a_{j+k-1}) = \hat{\mathbf{1}}$, so by (Ł$_2$), $\hat{\sigma}_r(a_{j+k-1}) = \hat{\mathbf{1}}$, for $1 \le r < n$. But by (Ł$_1$) and (Ł$_4$), this implies that $a_{j+k-1} = a_{n-1}$, that is $j+k-1 = n-1$, contradicting our assumption.
This completes the proof that $\mu(a_j) \ge n-j$. □

Notice that the function μ determines the $\hat{\sigma}_i$'s as follows:

$$\hat{\sigma}_i(a_j) = \begin{cases} \hat{\mathbf{0}} & \text{if } j < \mu(a_j), \\ \hat{\mathbf{1}} & \text{if } j \ge \mu(a_j). \end{cases}$$

for all $1 \le i \le m-1$ and $0 \le j \le n-1$. Also, since $\mu(x)$ is one-to-one and the number of non–zero elements of n is $n-1$, there has to be at least as many $\hat{\sigma}_i$'s. This provides another proof of our next Theorem 5.4.1.

Theorem 5.4.

1. *If $m < n$, there is no interpretation of $\mathcal{L}_m$ in $\mathcal{L}_n$.*
2. *If m is even and n is odd, then there is no interpretation of $\mathcal{L}_m$ in $\mathcal{L}_n$.*

Proof. One should observe that $\hat{\mathcal{N}}$ is an $\mathcal{L}_m$–algebra and it is a chain, so by Theorem 2.2, $\hat{\mathcal{N}}$ is a an $\mathcal{L}_m$–subalgebra of $\mathcal{M}$. This immediately implies that $n \le m$. The second assertion follows from the fact that $\mathcal{M}$ does not have subalgebras of odd cardinality. □

Theorem 5.5. *Let $m \ge n$. Then the number of interpretations of $\mathcal{L}_m$ in $\mathcal{L}_n$ is determined as follows.*

1. *If m is even and n is odd there is no interpretation of $\mathcal{L}_m$ in $\mathcal{L}_n$.*

2. *In any other case, for each De Morgan interpretation in $\mathcal{L}_n$, there are $\binom{h(m)}{h(n)}$ interpretations of $\mathcal{L}_m$ in $\mathcal{L}_n$, where for any positive integer p,*

$$h(p) = \begin{cases} \frac{p}{2} - 1 & \text{if } p \text{ is even,} \\ \frac{p-1}{2} - 1 & \text{if } p \text{ is odd.} \end{cases}$$

3. *If $n = 4$ and $m \geq 4$, there are two more interpretations of $\mathcal{L}_m$ in $\mathcal{L}_4$.*

Proof. Let $\hat{\mathcal{N}}$ be an interpretation of $\mathcal{L}_m$ in $\mathcal{L}_n$ such that the De Morgan reduct of the interpretation is the chain $\hat{\mathbf{0}} = a_0 < a_1 < \cdots < a_{n-1} = \hat{\mathbf{1}}$ and $a_j^{\ominus} = a_j' = a_{n-j}$. Our problem here is to count the number of possible functions μ defined in lemma 5.3. We know that they are one–to–one and $\mu(a_j) \geq n - j$, but that is not all we know. By axiom (4) μ has to have a symmetry with respect to the midpoint of $\mathcal{N}$ if n is odd or its midpoints if n is even. Recall that axiom (4) states that $\hat{\sigma}_i(a) = \hat{\sigma}_{m-i}(a^{\ominus})^{\ominus}$. In this case this means that $\hat{\sigma}_i(a_j) = (\hat{\sigma}_{m-i}(a_{n-j}))'$.

Case (1): This is Theorem 5.4, 2.

Case (2): Both m and n are even.
In this case $\hat{\sigma}_{\frac{m}{2}}(a_{\frac{n}{2}}) = (\hat{\sigma}_{m-\frac{m}{2}}(a'_{\frac{n}{2}}))' = (\hat{\sigma}_{\frac{m}{2}}(a_{\frac{n}{2}-1}))'$, and since $a_{\frac{n}{2}-1} < a_{\frac{n}{2}}$, by (Ł$_2$), $\hat{\sigma}_{\frac{m}{2}}(a_{\frac{n}{2}-1}) \leq \hat{\sigma}_{\frac{m}{2}}(a_{\frac{n}{2}})$. These two imply that

$$\hat{\sigma}_{\frac{m}{2}}(a_{\frac{n}{2}-1}) = \hat{\mathbf{1}} \quad \text{and} \quad \hat{\sigma}_{\frac{m}{2}}(a_{\frac{n}{2}}) = \hat{\mathbf{0}}.$$

The information gathered so far is summarized in the following chart.

	$\hat{\sigma}_1$	$\hat{\sigma}_2$	$\cdots$	$\hat{\sigma}_{\frac{m-n}{2}-1}$		$\cdots$		$\hat{\sigma}_{\frac{m}{2}}$	$\cdots$	$\hat{\sigma}_{m-1}$
a_0	$\hat{\mathbf{0}}$			$\vdots$				$\hat{\mathbf{0}}$		$\hat{\mathbf{0}}$
a_1								$\hat{\mathbf{0}}$		$\hat{\mathbf{1}}$
$\vdots$								$\vdots$		$\vdots$
$a_{\frac{n}{2}-1}$								$\hat{\mathbf{0}}$		$\hat{\mathbf{1}}$
$a_{\frac{n}{2}}$				$\vdots$			?	$\hat{\mathbf{1}}$		$\hat{\mathbf{1}}$
							$\hat{\mathbf{1}}$	$\hat{\mathbf{1}}$		$\hat{\mathbf{1}}$
$\vdots$					?	$\iddots$		$\vdots$		$\vdots$
a_{n-2}	?	?	$\cdots$	?	$\hat{\mathbf{1}}$	$\cdots$	$\hat{\mathbf{1}}$	$\hat{\mathbf{1}}$		$\hat{\mathbf{1}}$
a_{n-1}	$\hat{\mathbf{1}}$	$\hat{\mathbf{1}}$	$\cdots$	$\hat{\mathbf{1}}$	$\hat{\mathbf{1}}$	$\cdots$	$\hat{\mathbf{1}}$	$\hat{\mathbf{1}}$	$\cdots$	$\hat{\mathbf{1}}$

Case (3): Both m and n are odd.
Then $\hat{\sigma}_{\frac{m-1}{2}}(a_{\frac{n-1}{2}}) = (\hat{\sigma}_{\frac{m+1}{2}}(a'_{\frac{n-1}{2}}))' = (\hat{\sigma}_{\frac{m+1}{2}}(a_{\frac{n-1}{2}}))'$, but by (Ł$_2$), $\hat{\sigma}_{\frac{m+1}{2}}(a_{\frac{n-1}{2}}) \geq \hat{\sigma}_{\frac{m-1}{2}}(a_{\frac{n-1}{2}})$, so these two together with (Ł$_3$) imply

$$\hat{\sigma}_{\frac{m+1}{2}}(a_{\frac{n-1}{2}}) = \hat{\mathbf{1}} \quad \text{and} \quad \hat{\sigma}_{\frac{m-1}{2}}(a_{\frac{n-1}{2}}) = \hat{\mathbf{0}}.$$

Also, if $\hat{\sigma}_{\frac{m-1}{2}}(a_{\frac{n+1}{2}}) = \hat{\mathbf{0}}$, by (Ł$_4$), $a_{\frac{n+1}{2}} = a_{\frac{n-1}{2}}$, a contradiction, so $\hat{\sigma}_{\frac{m-1}{2}}(a_{\frac{n+1}{2}}) = \hat{\mathbf{1}}$. We summarize this in the following chart.

	$\hat{\sigma}_1$	$\hat{\sigma}_2$	$\cdots$	$\hat{\sigma}_{\frac{m-n}{2}}$		$\cdots$		$\hat{\sigma}_{\frac{m-1}{2}}$	$\hat{\sigma}_{\frac{m+1}{2}}$	$\cdots$	$\hat{\sigma}_{m-1}$
a_0	$\hat{\mathbf{0}}$			$\vdots$				$\hat{\mathbf{0}}$	$\hat{\mathbf{0}}$		$\hat{\mathbf{0}}$
a_1	$\hat{\mathbf{0}}$										$\hat{\mathbf{1}}$
$\vdots$								$\vdots$			$\vdots$
$a_{\frac{n-1}{2}}$	$\hat{\mathbf{0}}$							$\hat{\mathbf{0}}$	$\hat{\mathbf{1}}$	$\cdots$	$\hat{\mathbf{1}}$
				$\vdots$			?	$\hat{\mathbf{1}}$	$\hat{\mathbf{1}}$		$\hat{\mathbf{1}}$
							$\hat{\mathbf{1}}$	$\hat{\mathbf{1}}$	$\hat{\mathbf{1}}$		$\hat{\mathbf{1}}$
$\vdots$					?	$\iddots$		$\vdots$	$\vdots$		$\vdots$
a_{n-2}	?	?	$\cdots$	?	$\hat{\mathbf{1}}$	$\cdots$	$\hat{\mathbf{1}}$	$\hat{\mathbf{1}}$	$\hat{\mathbf{1}}$		$\hat{\mathbf{1}}$
a_{n-1}	$\hat{\mathbf{1}}$	$\hat{\mathbf{1}}$	$\cdots$	$\hat{\mathbf{1}}$	$\hat{\mathbf{1}}$	$\cdots$	$\hat{\mathbf{1}}$	$\hat{\mathbf{1}}$	$\hat{\mathbf{1}}$	$\cdots$	$\hat{\mathbf{1}}$

Case (4): m is odd and n is even.
Then if $\hat{\mathbf{0}} = \hat{\sigma}_{\frac{m-1}{2}}(a_{\frac{n}{2}}) = (\hat{\sigma}_{\frac{m+1}{2}}(a_{\frac{n}{2}-1}))'$, so $\hat{\sigma}_{\frac{m+1}{2}}(a_{\frac{n}{2}-1}) = \hat{\mathbf{1}}$ and thus by (Ł$_3$), $\hat{\sigma}_{\frac{m+1}{2}}(a_{\frac{n}{2}}) \geq (\hat{\sigma}_{\frac{m+1}{2}}(a_{\frac{n}{2}-1})) = \hat{\mathbf{1}}$, and also $\hat{\sigma}_{\frac{m-1}{2}}(a_{\frac{n}{2}-1}) \leq \hat{\sigma}_{\frac{m-1}{2}}(a_{\frac{n}{2}}) = \hat{\mathbf{0}}$. Putting these together, by (Ł$_4$), we get $a_{\frac{n}{2}} = a_{\frac{n}{2}-1}$, a contradiction. So

$$\begin{aligned} \hat{\sigma}_{\frac{m-1}{2}}(a_{\frac{n}{2}-1}) &= \hat{\mathbf{0}} = \hat{\sigma}_{\frac{m+1}{2}}(a_{\frac{n}{2}-1}) \\ \hat{\sigma}_{\frac{m-1}{2}}(a_{\frac{n}{2}}) &= \hat{\mathbf{1}} = \hat{\sigma}_{\frac{m+1}{2}}(a_{\frac{n}{2}}). \end{aligned}$$

We summarize this in the following chart.

	$\hat{\sigma}_1$	$\hat{\sigma}_2$	$\cdots$	$\hat{\sigma}_{\frac{m-n}{2}}$		$\cdots$		$\hat{\sigma}_{\frac{m-1}{2}}$	$\hat{\sigma}_{\frac{m+1}{2}}$	$\cdots$	$\hat{\sigma}_{m-1}$
a_0	$\hat{\mathbf{0}}$			$\vdots$				$\hat{\mathbf{0}}$	$\hat{\mathbf{0}}$		$\hat{\mathbf{0}}$
a_1	$\hat{\mathbf{0}}$							$\hat{\mathbf{0}}$	$\hat{\mathbf{0}}$		$\hat{\mathbf{1}}$
$\vdots$								$\vdots$			$\vdots$
$a_{\frac{n}{2}-1}$								$\hat{\mathbf{0}}$	$\hat{\mathbf{0}}$		$\hat{\mathbf{1}}$
$a_{\frac{n}{2}}$							?	$\hat{\mathbf{1}}$	$\hat{\mathbf{1}}$		$\hat{\mathbf{1}}$
							$\hat{\mathbf{1}}$	$\hat{\mathbf{1}}$	$\hat{\mathbf{1}}$		$\hat{\mathbf{1}}$
$\vdots$					?	$\iddots$		$\vdots$	$\vdots$		$\vdots$
a_{n-2}	?	?	$\cdots$	?	$\hat{\mathbf{1}}$	$\cdots$	$\hat{\mathbf{1}}$	$\hat{\mathbf{1}}$	$\hat{\mathbf{1}}$		$\hat{\mathbf{1}}$
a_{n-1}	$\hat{\mathbf{1}}$	$\hat{\mathbf{1}}$	$\cdots$	$\hat{\mathbf{1}}$	$\hat{\mathbf{1}}$	$\cdots$	$\hat{\mathbf{1}}$	$\hat{\mathbf{1}}$	$\hat{\mathbf{1}}$	$\cdots$	$\hat{\mathbf{1}}$

In the charts above we see that

$$\hat{\sigma}_i(a_j) = \begin{cases} \hat{\mathbf{0}} & \text{if } 1 \leq j < n/2 \text{ and } 1 \leq i \leq m/2, \\ \hat{\mathbf{1}} & \text{if } j \geq n/2 \text{ and } i \geq m/2. \end{cases}$$

Observe that by axiom (4), the values of $\hat{\sigma}_i(a_j)$ for $j < n/2$ and $i \geq (m+1)/2$, are determined by those of $\hat{\sigma}_i(a_j)$ for $j \geq n/2$ and $i < m/2$. Also, we must take into

account (Ł$_2$), (Ł$_3$) and (Ł$_4$), which imply that $\hat{\sigma}_i(a_j)$ must increase both with i and with j.
So in order to find all possible functions μ, one only has to determine how many "?" one has to replace by $\hat{\mathbf{0}}$'s in the lower left hand side of the charts.
Assuming l is the number of rows and k is the number of columns, this is the same as the number of integers less than l which can be expressed as a sum of k positive integers, this number is $\binom{l}{k}$.

Conversely, by Theorem 2.5 any such partition defines an interpretation of $\mathcal{L}_m$ in $\mathcal{L}_n$. So for appropriate l and k, the number of interpretations equals the number of these partitions. Now it is a matter of determining the particular l's and k's in each of the three cases and the theorem follows. Notice that by (Ł$_1$) the last line in each chart is fixed.

If $n = 4$ and its De Morgan reduct is the four element Boolean algebra, then as we mentioned before, we have another interpretation if we let for all x, $\hat{\sigma}_1(x) = \hat{\sigma}_2(x) = \cdots = \hat{\sigma}_{m-1}(x) = x$. □

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