Finite Fields and Their Applications 25 (2014) 37–48

Contents lists available at ScienceDirect

Finite Fields and Their Applications

www.elsevier.com/locate/ffa

Poles of the Igusa local zeta function of some hybrid polynomials

Edwin León-Cardenal [a], Denis Ibadula [b,c,*,1], Dirk Segers [d,2]

[a] Centro de Investigacion y de Estudios Avanzados del Instituto Politecnico Nacional, Departamento de Matematicas–Unidad Queretaro, Libramiento Norponiente, #2000, Fracc. Real de Juriquilla, Santiago de Queretaro, Qro. 76230, Mexico
[b] Faculty of Mathematics and Informatics, Ovidius University of Constanta, Mamaia Bd. 124, RO 900527, Constanta, Romania
[c] Institute of Mathematics "Simion Stoilow" of the Romanian Academy, P.O. Box 1-764, 014700, Bucharest, Romania
[d] University of Leuven, Department of Mathematics, Celestijnenlaan 200B, B-3001, Leuven (Heverlee), Belgium

ARTICLE INFO

Article history:
Received 30 November 2011
Received in revised form 13 June 2013
Accepted 31 July 2013
Available online 14 September 2013
Communicated by Daqing Wan

MSC:
primary 14G10, 11S40
secondary 11T23

Keywords:
Hybrid polynomials
Igusa zeta function
Stationary phase formula

ABSTRACT

Hybrid polynomials were introduced by Hauser [6] in connection with the problem of extending Hinoraka's resolution of singularities theorem to fields of positive characteristic. In this paper we study the local zeta function associated to some hybrid polynomials defined over a non-archimedean local field of positive characteristic, by using essentially the π-adic stationary phase formula. We show the rationality of these local zeta functions and we describe completely its poles. For this class of polynomials we also met the classical results about exponential sums mod π^m.

1. Introduction

Fix a prime number p. Let K be a non-archimedean local field with ring of integers $\mathcal{O}_K$. Let $\mathcal{P}_K$ denote the maximal ideal of $\mathcal{O}_K$ and take π as a fixed uniformizing parameter of K. Let the residue field of K be $\mathbb{F}_q$, the field with $q = p^r$ elements. For $x \in K$ we denote by $ord(x)$ the valuation

* Corresponding author at: Ovidius University, Faculty of Mathematics and Informatics, Bd. Mamaia 124, 900527 Constanta, Romania.

E-mail addresses: eleonc81@gmail.com (E. León-Cardenal), denis@univ-ovidius.ro (D. Ibadula), dirk.segers1976@gmail.com (D. Segers).

1 This paper is supported by the Sectorial Operational Programme Human Resources Development (SOP HRD), financed from the European Social Fund and by the Romanian Government under the contract number SOP HRD/89/1.5/S/62988.

2 Postdoctoral Fellow of the Fund for Scientific Research – Flanders (Belgium).

1071-5797/$ – see front matter
http://dx.doi.org/10.1016/j.ffa.2013.07.009

of K, such that $ord(\pi)=1$ and $|x|_K := |x| = q^{-ord(x)}$. The *angular component* of $x \in K$ is defined as $ac\, x = x\pi^{-ord(x)}$.

Let $f(x) \in \mathcal{O}_K[x]$, $x=(x_1,\ldots,x_n)$ be a non-constant polynomial, and $\chi : \mathcal{O}_K^\times \to \mathbb{C}^\times$ a multiplicative character of $\mathcal{O}_K^\times$, the group of units of $\mathcal{O}_K$. We set $\chi(0)=0$. Then the *Igusa local zeta function* is defined as

$$Z(f,s,\chi,K) := Z_f(s,\chi) = \int\limits_{\mathcal{O}_K^n} \chi\left(ac\, f(x)\right)\left|f(x)\right|^s |dx|, \quad s \in \mathbb{C},\ Re(s) > 0,$$

where $|dx|$ is the Haar measure on K^n normalized such that the measure of $\mathcal{O}_K^n$ is one.

One may see easily that $Z_f(s,\chi)$ is holomorphic on $Re(s)>0$. Furthermore, by using resolution of singularities Igusa showed that $Z_f(s,\chi)$ admits a meromorphic continuation to the whole $\mathbb{C}$ as a rational function of q^{-s} when $char(K)=0$, see [7]. He also proved that the poles of $Z_f(s,\chi)$ can be described using the data provided by a resolution of singularities of $f^{-1}(0)$. In this way, however, the set of candidate poles is too big, and it is a current open problem the determination of the actual poles for arbitrary n, see the corresponding comments in [3] and [15].

By using p-adic cell decomposition, Denef proved in [4] the rationality of $Z_f(s,\chi_{triv})$ in characteristic zero.

$Z_f(s,\chi_{triv})$ is directly related to the number of solutions of the congruences $f(x) \equiv 0 \bmod \pi^e\mathcal{O}_K$, for $e \in \mathbb{N}$, that are codified by a Poincaré series. For $f(x) \in \mathcal{O}_K[x]$ *the Poincaré series of* f is the formal power series

$$P_f(t) = \sum_{e=0}^{\infty} N_e q^{-ne} t^e,$$

where $N_e = Card\{x \in (\mathcal{O}_K/\pi\mathcal{O}_K)^n \mid f(x) \equiv 0 \bmod \pi^e\mathcal{O}_K\}$, for $e \geqslant 1$. By convention, $N_0 = 1$, see [1]. The relation between the Poincaré series associated to f and the Igusa zeta function of f is given by

$$P_f(t) = \frac{1 - tZ_f(s,\chi_{triv})}{1-t},$$

with $t := q^{-s}$ (see e.g. [11, Theorem 8.2.2]). Hence, one can obtain explicit formulas for all N_e, for $e \geqslant 0$, from the explicit form of $Z_f(s,\chi_{triv})$. And thus the rationality of $Z_f(s,\chi_{triv})$ is a relevant problem. For a general character χ it is known that $Z_f(s,\chi)$ is related with exponential sums mod π^e, and more generally with oscillatory integrals depending on a p-adic parameter, see e.g. [3,13,15].

Much less is known in positive characteristic, due to the absence of a general theorem of resolution of singularities, or an equivalent method of p-adic cell decomposition. However there is another tool that has been very useful in this setting: *the stationary phase method*. The stationary phase formula, SPF for short, was introduced in [10] by Igusa; there he suggested that a systematic study of the SPF might lead to a proof of the rationality of $Z_f(s,\chi)$ in arbitrary characteristic. This tool has been used extensively by Zúñiga-Galindo, who has shown in [14] that $Z_f(s,\chi_{triv})$ is a rational function of q^{-s} when f is a semiquasihomogeneous polynomial with coefficients in an arbitrary non-archimedean local field. He also showed in [15], that for almost all χ, the local zeta functions $Z_f(s,\chi)$ are rational functions of q^{-s}, provided that f is a non-degenerate polynomial with respect to its Newton polyhedra. In both papers the author gives a short list of candidate poles for the local zeta function.

Hybrid polynomials were introduced by Hauser in [5] in connection with the obstructions to resolution of singularities in positive characteristic. Hauser asserts that the main obstructions can only occur in a concrete series of polynomials, the hybrid ones. It seems that they have a strange behavior under linear coordinates changes when considered modulo p, see e.g. [6]. In this paper we use the SPF to describe the poles of the Igusa zeta function of some class of hybrid polynomials. Furthermore, Theorem 1 asserts that $Z_f(s,\chi)$ is a rational function of q^{-s} with three possible factors on the denominator.

We deal with the case of hybrid polynomials in three variables. Recently Cossart and Piltant showed in [2] the existence of an embedded resolution of singularities in the case of surfaces in $\mathbb{A}^3_K$ in positive characteristic. This fact and a 'suitable tame condition' would imply that the Igusa zeta function of hybrid polynomials is a rational function, see e.g. [8, p. 309], and then we will be able to give a list of candidate poles in this case. However, in practice, for a given polynomial, it is difficult to find an explicit embedded resolution and even if one can find it, the list of candidate poles given by the numerical invariants associated to such an embedded resolution gives a very long list of candidate poles. Our work provides a list with just three candidate poles.

The class of hybrid polynomials that we consider are weighted polynomials in the sense of [14]. We obtain the same two poles as in the case considered in [14], but also a new extra pole, which is related with the characteristic of the ground field.

Our second result, Theorem 2 is the estimation of certain exponential sums attached to the polynomials f. These results are consistent with the classical results about exponential sums, see e.g. [3,15].

The paper is organized as follows: in Section 2 we introduce the stationary phase formula, in Section 3 we introduce the hybrid polynomials and the local zeta functions attached to them, we also state Theorem 1. Section 4 is devoted to the proof of Theorem 1. Finally in Section 5 we study exponential sums mod π^m, there we state and prove Theorem 2.

2. Stationary phase formula

In [10] Igusa introduced the so-called stationary phase formula which is very useful to compute explicitly some p-adic integrals. He suggested that a systematic study of the SPF might lead to a proof of the rationality of $Z_f(s,\chi)$ in arbitrary characteristic. This suggestion was motivated by his results about local zeta functions of some prehomogeneous vector spaces, see [10] and [9]. Other authors have used the SPF to compute local zeta functions of several type of polynomials, among them [12,14,15].

Let L be a ring and $f(x) \in L[x]$, we denote by $V_f(L)$ the corresponding L-hypersurface and by $Sing_f(L)$ the L-singular locus. We denote by $\bar{x}$ the image of an element of O_K^3 under the canonical homomorphism $O_K^3 \to (O_K/\pi O_K)^3 \cong \mathbb{F}_q^3$. We denote by $\overline{f(x)}$ the polynomial obtained by reducing modulo π the coefficients of $f(x) \in O_K[x] \setminus O_K$.

We fix a lifting R of $\mathbb{F}_q$ in O_K. By definition, the set R^3 is mapped bijectively onto $\mathbb{F}_q^3$ by the canonical homomorphism $O_K^3 \to (O_K/\pi O_K)^3$. Let $\bar{D}$ be a subset of $\mathbb{F}_q^3$ and D its preimage under the canonical homomorphism.

Let $S(f,D)$ denote the subset of R^3 mapped bijectively to the set $Sing_{\bar{f}}(\mathbb{F}_q) \cap \bar{D}$. We use the simplified notation $S(f)$ in the case of $D = O_K^3$. We define also

$$\nu(\bar{f},D,\chi) := \begin{cases} q^{-3}\, Card\{\bar{P} \in \bar{D} \mid \bar{P} \notin V_{\bar{f}}(\mathbb{F}_q)\} & \text{if } \chi = \chi_{triv}, \\ q^{-3c_\chi} \sum_{\{P \in D \mid \bar{P} \notin V_{\bar{f}}(\mathbb{F}_q)\} \bmod \mathcal{P}_K^{c_\chi}} \chi(ac\, f(P)) & \text{if } \chi \neq \chi_{triv}, \end{cases}$$

where c_χ is the conductor of χ, and

$$\sigma(\bar{f},D,\chi) := \begin{cases} q^{-3}\, Card\{\bar{P} \in \bar{D} \mid \bar{P} \text{ is a smooth point of } V_{\bar{f}}(\mathbb{F}_q)\} & \text{if } \chi = \chi_{triv}, \\ 0 & \text{if } \chi \neq \chi_{triv}. \end{cases}$$

If $D = O_K^3$ we use the simplified notation $\nu(\bar{f},\chi)$, $\sigma(\bar{f},\chi)$. We also set $Z_f(s,\chi,D)$ for the integral

$$\int_D \chi\big(ac\, f(x,y,z)\big) \big|f(x,y,z)\big|^s\, |dx\, dy\, dz|.$$

Lemma 1 *(Stationary phase formula). With the above notations, we have*

$$Z_f(s,\chi,D)=\nu(\bar{f},D,\chi)+\sigma(\bar{f},D,\chi)\frac{(1-q^{-1})q^{-s}}{1-q^{-1-s}}$$
$$+\int_{S(f,D)}\chi\left(ac\, f(x,y,z)\right)\left|f(x,y,z)\right|^{s}|dx\,dy\,dz|,$$

where $Re(s)>0$.

Proof. The proof given by Igusa in [11], for the case $\chi=\chi_{triv}$, can be generalized literally to arbitrary characters. □

3. Local zeta functions of hybrid polynomials

From now on, let K be a non-archimedean local field of characteristic p. Let k, r, and s be positive integers such that $p\nmid rs$, $p\mid r+s+k$ and $\bar{r}+\bar{s}\leqslant p$, where $\bar{r}$ and $\bar{s}$ denote the residues of r and s modulo p. If t is an arbitrary constant from the ground field K, the *hybrid polynomials* are polynomials of the form

$$g(x,y,z)=x^p+P(y,z),\quad \text{with } P(y,z)=y^r z^s\sum_{i=0}^{k}\binom{k+r}{i+r}y^i(tz-y)^{k-i},\tag{3.1}$$

see [6].

Take $r=s=1$ and suppose that $p\mid k+2$ and that t^{k+1} is a p-th power of an arbitrary unit from $\mathcal{O}_K^{\times}$. Hence, one gets

$$g(x,y,z)=x^p+yz\sum_{i=0}^{k}\binom{k+1}{i+1}y^i(tz-y)^{k-i}.\tag{3.2}$$

In what follows we compute the explicit form and the poles of $Z_f(s,\chi)$ for f a hybrid polynomial as in (3.2). Note that this type of polynomials are weighted polynomials (in the sense of [14]) of degree $n+1$ and weight $w=(w_1,w_2,w_3)=((n+1)/p,1,1)$. Our purpose is to extend the methods and results of [14].

Remark 1. Note that the class of hybrid polynomials that we consider is an infinite class due to the conditions over t and k.

First we change variables in (3.2) as $(x,y,z)\to(x,y+tz,z)$ in order to obtain

$$g^{+}(x,y,z)=x^p+z\sum_{i=0}^{k}\binom{k+1}{i+1}(y+tz)^{i+1}(-y)^{k-i}.$$

By using the identity $\binom{k+1}{i}=\binom{k+1}{k+1-i}$, we get

$$g^{+}(x,y,z)=x^p+z\sum_{i=1}^{k+1}\binom{k+1}{i}(y+tz)^{i}(-y)^{k-i+1}$$

$$=x^p+z\sum_{i=0}^{k+1}\binom{k+1}{k+1-i}(y+tz)^i(-y)^{k-i+1}-z(-y)^{k+1}$$
$$=x^p+z(y+tz-y)^{k+1}-z(-y)^{k+1}$$
$$=x^p+z\big[t^{k+1}z^{k+1}+(-1)^k y^{k+1}\big]. \tag{3.3}$$

Eq. (3.3) shows that the computation of the local zeta function of hybrid polynomials is reduced to the computation of $Z_f(s,\chi)$ for polynomials of type

$$f(x,y,z)=x^p+\alpha y^n z+\beta z^{n+1}, \tag{3.4}$$

where $\alpha\in\mathcal{O}_K^\times$, $\beta^{1/p}\in\mathcal{O}_K^\times$ (i.e. β is the p-th power of an element from $\mathcal{O}_K^\times$), n is a positive natural number which is at least 2 such that $p\mid n+1$.

We start the required computation by setting

$$A_w=\big\{(x,y,z)\in\mathcal{O}_K^3 \bigm| ord\,x\geqslant(n+1)/p,\ ord\,y\geqslant 1,\ ord\,z\geqslant 1\big\}.$$

If T is the transformation given by $T(x,y,z)=(\pi^{w_1}x,\pi^{w_2}y,\pi^{w_3}z)$, then $T(\mathcal{O}_K^3)=A_w$ and

$$Z_f(s,\chi,A_w)=\int_{\mathcal{O}_K^3}\chi\big(ac\,f\circ T(x,y,z)\big)\big|f\circ T(x,y,z)\big|^s\big|J(T)\big|\,|dx\,dy\,dz|,$$

where $J(T)=\pi^{w_1+w_2+w_3}$ is the determinant of the Jacobian matrix associated to T. It follows that

$$Z_f(s,\chi,A_w)=\int_{A_w}\chi\big(ac\,f(x,y,z)\big)\big|f(x,y,z)\big|^s\,|dx\,dy\,dz|=q^{-|w|-(n+1)s}Z_f(s,\chi),$$

where $|w|=(n+1)/p+2$. Hence

$$Z_f(s,\chi)=\frac{1}{1-q^{-|w|-(n+1)s}}Z_f\big(s,\chi,A_w^c\big), \tag{3.5}$$

where A_w^c is the complement of the set A_w. By decomposing A_w^c as a disjoint union of the following seven sets:

$$B_1=\big\{(x,y,z)\in\mathcal{O}_K^3 \bigm| ord\,x<(n+1)/p,\ ord\,y\geqslant 1,\ ord\,z\geqslant 1\big\},$$
$$B_2=\big\{(x,y,z)\in\mathcal{O}_K^3 \bigm| ord\,x\geqslant(n+1)/p,\ ord\,y=0,\ ord\,z\geqslant 1\big\},$$
$$B_3=\big\{(x,y,z)\in\mathcal{O}_K^3 \bigm| ord\,x\geqslant(n+1)/p,\ ord\,y\geqslant 1,\ ord\,z=0\big\},$$
$$B_4=\big\{(x,y,z)\in\mathcal{O}_K^3 \bigm| ord\,x\geqslant(n+1)/p,\ ord\,y=0,\ ord\,z=0\big\},$$
$$B_5=\big\{(x,y,z)\in\mathcal{O}_K^3 \bigm| ord\,x<(n+1)/p,\ ord\,y\geqslant 1,\ ord\,z=0\big\},$$
$$B_6=\big\{(x,y,z)\in\mathcal{O}_K^3 \bigm| ord\,x<(n+1)/p,\ ord\,y=0,\ ord\,z\geqslant 1\big\},$$
$$B_7=\big\{(x,y,z)\in\mathcal{O}_K^3 \bigm| ord\,x<(n+1)/p,\ ord\,y=0,\ ord\,z=0\big\},$$

we obtain

$$Z_f(s,\chi,A_w^c)=\sum_{i=1}^{7}Z_f(s,\chi,B_i). \tag{3.6}$$

Integrals over B_i will be computed by using several changes of variables of the form:

$$x=\pi^i x_1, \qquad y=\pi^j y_1, \qquad z=\pi^k z_1, \tag{3.7}$$

where $i, j, k\in\mathbb{N}$ and x_1, y_1, z_1 will depend on the integration domain.

Theorem 1. *Let $f(x,y,z)=x^p+\alpha y^n z+\beta z^{n+1}\in K[x,y,z]$, where $\alpha\in\mathcal{O}_K^\times$, $\beta^{1/p}\in\mathcal{O}_K^\times$, $n\in\mathbb{N}\setminus\{0\}$ such that $p\mid(n+1)$. Then the Igusa local zeta function of $(f(x,y,z),\chi)$ is a rational function of q^{-s} of the form:*

$$Z_f(s,\chi)=\frac{L(q^{-s},\chi)}{(1-q^{-1-s})(1-q^{-|w|-(n+1)s})(1-q^{-p-n-pns})},$$

where $|w|=((n+1)/p)+2$ and $L(q^{-s},\chi)$ is a polynomial with complex coefficients and degree independent of χ.

The proof of this theorem is reserved to Section 5.

Corollary 1. *Take f as above. If s is a pole of $Z_f(s,\chi)$ then*

$$s=-1+\frac{2\pi i\mathbb{Z}}{\log q}, \quad \text{or} \quad s=-\frac{1}{p}-\frac{2}{n+1}+\frac{2\pi i\mathbb{Z}}{(n+1)\log q}, \quad \text{or} \quad s=-\frac{1}{p}-\frac{1}{n}+\frac{2\pi i\mathbb{Z}}{pn\log q}.$$

Corollary 2. *Let $f(x,y,z)=x^p+yz\sum_{i=0}^{k}\binom{k+1}{i+1}y^i(tz-y)^{k-i}\in K[x,y,z]$, where k is a positive natural number such that $p\mid(k+2)$ and t is a constant in the ground field K such that t^{k+1} is a p-th power of an element of $\mathcal{O}_K^\times$. Then the Igusa local zeta function of $(f(x,y,z),\chi)$ is a rational function of q^{-s} of the form*

$$Z_f(s,\chi)=\frac{L(q^{-s},\chi)}{(1-q^{-1-s})(1-q^{-((k+2)/p)-2-(k+2)s})(1-q^{-p-k-1-p(k+1)s})},$$

where $L(q^{-s},\chi)$ is a polynomial with complex coefficients and degree independent of χ. Furthermore, the real parts of the candidate poles for $Z_f(s,\chi)$ are -1, $-\frac{1}{p}-\frac{2}{k+2}$ and $-\frac{1}{p}-\frac{1}{k+1}$.

Proof. It follows from Theorem 1, the relation (3.3) and the preservation of the measure under transformation $(x,y,z)\mapsto(x,y+tz,z)$. □

Remark 2. It is very interesting that for the considered class of hybrid polynomials a new candidate pole appeared: $-\frac{1}{p}-\frac{1}{k+1}$. Moreover, we also get the two candidate poles that we have expected, -1, $-\frac{1}{p}-\frac{2}{k+2}$ which are connected with the degree of the polynomial and with its weight, see e.g. [14].

Example 1. Let K be a non-archimedean local field with residue field $\mathbb{F}_q$, where $q=3^a$ and $q\not\equiv 1 \bmod 5$. Take $\chi=\chi_{triv}$. Consider the hybrid polynomial

$$g(x,y,z)=x^3+yz\sum_{i=0}^{4}\binom{5}{i+1}y^i(z-y)^{4-i}.$$

Then $g^+(x,y,z)=x^3+y^5z+z^6\in K[x,y,z]$, and it has degree $n+1=6$ and weight $w=(2,1,1)$.

According to Corollary 1, the real parts of the candidate poles for $Z_g(s,\chi)$ are -1, $-\frac{2}{3}$ and $\frac{8}{15}$. An explicit calculation of the seven integrals of (3.6) gives

$$\begin{aligned}Z_g(t,\chi)={}&\frac{1}{(q-t)(q^8-t^{15})(q^4-t^6)}\big\{(q-1)t^{19}+\big(-q^2+q\big)t^{18}+\big(-q^3-3q+3q^2+1\big)t^{17}\\&+\big(6q^3+5q-q^4-1-9q^2\big)t^{16}+\big(-q+2q^2+q^4-q^3-q^5\big)t^{15}\\&+\big(-q^3+2q^4-q^5\big)t^{13}+\big(q^4+q^6-2q^5\big)t^{12}+\big(2q^5-q^6-q+3q^2-3q^3\big)t^{11}\\&+\big(q^2+3q^4-q^5-3q^3\big)t^{10}+\big(q^8-2q^7+q^6\big)t^9+\big(-q^8+2q^7-q^6\big)t^7\\&+\big(-q^2+3q^3-3q^4+q^5\big)t^6+\big(q^{10}-3q^4+q^8-q^6+q^3-2q^9+3q^5\big)t^5\\&+\big(q^9-q^{10}\big)t^4+\big(-q^{10}+q^{11}\big)t^3+\big(3q^9+q^{11}-3q^{10}-q^8\big)t^2\\&+\big(7q^{10}-4q^{11}-3q^9\big)t+2q^9+3q^{11}-3q^{12}-4q^{10}+2q^{13}\big\}.\end{aligned}$$

Hence all the candidate poles are true poles.

4. Proof of Theorem 1

First we compute each one of the integrals in (3.6).

4.1. Integral over B_1

For a a natural number, $0\leqslant a<w_1$ we define the set

$$B_1^a=\big\{(x,y,z)\in\mathcal{O}_K^3\;\big|\;ord\,x=a,\ ord\,y\geqslant 1,\ ord\,z\geqslant 1\big\},$$

and thus $B_1=\coprod_{0\leqslant a<w_1}B_1^a$. Moreover $Z_f(s,\chi,B_1)=\sum_{0\leqslant a<w_1}Z_f(s,\chi,B_1^a)$.

In B_1^a, we use the change of variables (3.7) with $i=a$, $j=1$ and $k=1$; also $x_1\in\mathcal{O}_K^\times$, $y_1,z_1\in\mathcal{O}_K$. The new integration domain will be $D_1=\mathcal{O}_K^\times\times\mathcal{O}_K^2$ and the measure $|dx\,dy\,dz|$ will change in $q^{-(a+2)}|dx_1\,dy_1\,dz_1|$. Thus,

$$\begin{aligned}Z_f(s,\chi,B_1)&=\sum_{0\leqslant a<w_1}q^{-(a+2)}Z\big(\pi^{ap}x^p+\pi^{n+1}z\big(\alpha y^n+\beta z^n\big),s,\chi,B_1\big)\\&=\sum_{0\leqslant a<w_1}q^{-(a+2)-aps}Z\big(x^p+\pi^{(n+1)-ap}z\big(\alpha y^n+\beta z^n\big),s,\chi,B_1\big)\\&=\sum_{0\leqslant a<w_1}q^{-(a+2)-aps}\int_{D_1}\chi\big(ac\big(x^p+\pi^{(n+1)-ap}z\big(\alpha y^n+\beta z^n\big)\big)\big)\,|dx\,dy\,dz|\\&=F_1\big(q^{-s},\chi\big),\end{aligned}\tag{4.1}$$

where $F_1(q^{-s},\chi)$ is a polynomial function. Hence this integral will not contribute to the poles of $Z_f(s,\chi)$.

4.2. Integral over B_2

For computing the integral over B_2, we apply the change of variables (3.7) with $i=w_1$, $j=0$ and $k=1$. The new domain of integration is now $D_2=\mathcal{O}_K\times\mathcal{O}_K^\times\times\mathcal{O}_K$. Then

$$\begin{aligned} Z_f(s,\chi,B_1) &= q^{-1-w_1-s} \\ &\quad \times \int_{D_2} \chi\left(ac\left(\pi^n x^p + z\left(\alpha y^n + \beta\pi^n z^n\right)\right)\right)\left|\pi^n x^p + z\left(\alpha y^n + \beta\pi^n z^n\right)\right|^s |dx\,dy\,dz| \\ &= \frac{F_2(q^{-s},\chi)}{1-q^{-1-s}}, \end{aligned} \tag{4.2}$$

by using Lemma 1.

4.3. Integral over B_3

Here we use the change of variables (3.7) with $i = w_1,\ j = 1,\ k = 0$ giving $D_3 = (\mathcal{O}_K)^2 \times \mathcal{O}_K^\times$. Hence,

$$\begin{aligned} Z_f(s,\chi,B_3) &= q^{-(w_1+1)} \\ &\quad \times \int_{D_3} \chi\left(ac\left(\pi^{n+1} x^p + z\left(\alpha\pi^n y^n + \beta z^n\right)\right)\right)\left|\pi^{n+1} x^p + z\left(\alpha\pi^n y^n + \beta z^n\right)\right|^s |dx\,dy\,dz| \\ &= F_3\left(q^{-s},\chi\right). \end{aligned} \tag{4.3}$$

In consequence this integral will not contribute to the poles of the Igusa local zeta function of f.

4.4. Integral over B_4

For computing the integral over B_4, after a change of variables of type (3.7) with $i = w_1,\ j = k = 0$, the integration domain changes to $D_4 = \mathcal{O}_K \times (\mathcal{O}_K^\times)^2$. Then

$$\begin{aligned} Z_f(s,\chi,B_4) &= q^{-w_1} \\ &\quad \times \int_{D_4} \chi\left(ac\left(\pi^{n+1} x^p + z\left(\alpha y^n + \beta z^n\right)\right)\right)\left|\pi^{n+1} x^p + z\left(\alpha y^n + \beta z^n\right)\right|^s |dx\,dy\,dz| \\ &= \frac{F_4(q^{-s},\chi)}{1-q^{-1-s}}, \end{aligned} \tag{4.4}$$

as one may check by using Lemma 1.

4.5. Integral over B_5

Here we proceed similarly to the case of integrating over B_1. We fix a natural number a such that $0 \leqslant a < w_1$ and we define the set B_5^a to be

$$\left\{(x,y,z) \in \mathcal{O}_K^3 \;\middle|\; ord\,x = a,\ ord\,y \geqslant 1,\ ord\,z = 0\right\}.$$

Next we change variables as in (3.7), with $i = a,\ j = 1$ and $k = 0$. This will transform the integration domain to $D_5 = \mathcal{O}_K^\times \times \mathcal{O}_K \times \mathcal{O}_K^\times$. Hence

$$Z_f(s,\chi,B_5) = q^{-1} \int_{D_5} \chi\left(ac\left(x^p + z\left(\alpha\pi^n y^n + \beta z^n\right)\right)\right)\left|x^p + z\left(\alpha\pi^n y^n + \beta z^n\right)\right|^s |dx\,dy\,dz|$$

$$+\sum_{1\leqslant a<w_1} q^{-(a+1)}$$
$$\times\int_{D_5}\chi\bigl(ac\bigl(\pi^{ap}x^p+z\bigl(\alpha\pi^n y^n+\beta z^n\bigr)\bigr)\bigr)\bigl|\pi^{ap}x^p+z\bigl(\alpha\pi^n y^n+\beta z^n\bigr)\bigr|^s\,|dx\,dy\,dz|$$
$$=q^{-1}\int_{D_5}\chi\bigl(ac\bigl(x^p+z\bigl(\alpha\pi^n y^n+\beta z^n\bigr)\bigr)\bigr)\bigl|x^p+z\bigl(\alpha\pi^n y^n+\beta z^n\bigr)\bigr|^s\,|dx\,dy\,dz|$$
$$+\sum_{1\leqslant a<w_1} q^{-(a+1)}F'_5\bigl(a,q^{-s},\chi\bigr),$$

since $|\pi^{ap}x^p+z(\alpha\pi^n y^n+\beta z^n)|=1$. Now by using Lemma 2, that we will prove later on, we have that

$$Z_f(s,\chi,B_5)=\frac{F_5(q^{-s},\chi)}{(1-q^{-1-s})(1-q^{-n-p-nps})},\tag{4.5}$$

where $F_5(q^{-s},\chi)$ is a polynomial with complex coefficients and degree independent of χ.

4.6. Integral over B_6

Again we fix a natural number a such that $0\leqslant a<w_1$, we set

$$B_6^a=\bigl\{(x,y,z)\in\mathcal{O}_K^3 \bigm| ord\,x=a,\ ord\,y=0,\ ord\,z\geqslant 1\bigr\}.$$

Then we use (3.7) to change variables with $i=a$, $j=0$ and $k=1$, getting the new integration domain $D_6=(\mathcal{O}_K^\times)^2\times\mathcal{O}_K$. Therefore

$$Z_f(s,\chi,B_6)=q^{-1}\int_{D_6}\chi\bigl(ac\bigl(x^p+\pi z\bigl(\alpha y^n+\pi^n\beta z^n\bigr)\bigr)\bigr)\bigl|x^p+\pi z\bigl(\alpha y^n+\pi^n\beta z^n\bigr)\bigr|^s\,|dx\,dy\,dz|$$
$$+\sum_{1\leqslant a<w_1} q^{-(a+1)}$$
$$\times\int_{D_6}\chi\bigl(ac\bigl(\pi^{ap}x^p+\pi z\bigl(\alpha y^n+\pi^n\beta z^n\bigr)\bigr)\bigr)\bigl|\pi^{ap}x^p+\pi z\bigl(\alpha y^n+\pi^n\beta z^n\bigr)\bigr|^s\,|dx\,dy\,dz|$$
$$=F'_6\bigl(q^{-s},\chi\bigr)+\frac{F''_6(q^{-s},\chi)}{1-q^{-1-s}}=\frac{F_6(q^{-s},\chi)}{1-q^{-1-s}},\tag{4.6}$$

where we have used the fact that $|x^p+\pi z(\alpha y^n+\pi^n\beta z^n)|=1$ for the first integral and Lemma 1 for the integrals inside the sum.

4.7. Integral over B_7

We fix a natural number a between 0 and w_1, and after the suitable change of variables of type (3.7) with $i=a$, $j=k=0$, the new integration domain is $D_7=(\mathcal{O}_K^\times)^3$. Then

$$Z_f(s,\chi,B_7)=\sum_{0\leqslant a<w_1} q^{-(a+1)} \times \int_{D_7} \chi\left(ac\left(\pi^{pa}x^p+z\left(\alpha y^n+\beta z^n\right)\right)\right)\left|\pi^{pa}x^p+z\left(\alpha y^n+\beta z^n\right)\right|^s |dx\,dy\,dz| = \frac{F_7(q^{-s},\chi)}{1-q^{-1-s}}, \tag{4.7}$$

as we may see by using Lemma 1.

Finally the theorem follows from Eqs. (3.5) and (4.1)–(4.7). □

We end this section by proving the auxiliary lemma that we used when computing $Z_f(s,\chi,B_5)$.

Lemma 2. *Let $h(x,y,z)=x^p+\alpha\pi^n y^n z+\beta z^{n+1}$, and let D be the set $\mathcal{O}_K^\times\times\mathcal{O}_K\times\mathcal{O}_K^\times$. Then*

$$Z_h(s,\chi,D)=\begin{cases}\frac{H(q^{-s},\chi)}{(1-q^{-1-s})(1-q^{-n-p-nps})} & \text{if } \chi^{n+1}|_{\mathcal{O}_K^\times}=1,\\ 0 & \text{if } \chi^{n+1}|_{\mathcal{O}_K^\times}\neq 1,\end{cases}$$

where H is a polynomial with complex coefficients and with degree independent from χ.

Proof. We change variables as $(x,y,z)\mapsto(uw^{\frac{n+1}{p}},vw,w)$ and note that this transformation preserves the measure. Then

$$\begin{aligned}Z_h(s,\chi,D)&=\int_D \chi\left(ac\left(w^{n+1}\left[u^p+\alpha\pi^n v^n+\beta\right]\right)\right)\left|u^p+\alpha\pi^n v^n+\beta\right|^s |dx\,dy\,dz|\\&=\int_{\mathcal{O}_K^\times\times\mathcal{O}_K}\left|u^p+\alpha\pi^n v^n+\beta\right|^s\left\{\chi\left(ac\left(u^p+\alpha\pi^n v^n+\beta\right)\right)\int_{\mathcal{O}_K^\times}\chi\left(ac\,w^{n+1}\right)|dw|\right\}|du\,dv|\\&=\begin{cases}\int_{\mathcal{O}_K^\times\times\mathcal{O}_K}\chi(ac(u^p+\alpha\pi^n v^n+\beta))|u^p+\alpha\pi^n v^n+\beta|^s\,|du\,dv| & \text{if } \chi^{n+1}|_{\mathcal{O}_K^\times}=1,\\ 0 & \text{if } \chi^{n+1}|_{\mathcal{O}_K^\times}\neq 1.\end{cases}\end{aligned}$$

In order to compute the integral for the case $\chi^{n+1}|_{\mathcal{O}_K^\times}=1$, we take $g(u,v)=u^p+\alpha\pi^n v^n+\beta$ and $D'=\mathcal{O}_K^\times\times\mathcal{O}_K$, then $\bar{g}(\bar{u},\bar{v})=\bar{\beta}+\bar{u}^p$. In consequence $S(\bar{g},D')=\{(-\bar{\beta}^{1/p},0)\}$ and

$$Z_g(s,\chi,D')=\nu(\bar{g},D',\chi)+\sigma(\bar{g},D',\chi)\frac{(1-q^{-1})q^{-s}}{1-q^{-1-s}} + \int_{\mathcal{O}_K^2}\chi\left(ac\left(\pi^p u^p+\alpha\pi^n v^n\right)\right)\left|\pi^p u^p+\alpha\pi^n v^n\right|^s |du\,dv|. \tag{4.8}$$

Note that $\pi^p u^p+\alpha\pi^n v^n=\pi^p[u^p+\alpha\pi^{n-p}v^n]$ and put $g_1(u,v)=u^p+\alpha\pi^{n-p}v^n$. Then

$$\int_{\mathcal{O}_K^2}\chi\left(ac\left(\pi^p u^p+\alpha\pi^n v^n\right)\right)\left|\pi^p u^p+\alpha\pi^n v^n\right|^s |du\,dv| = q^{-ps}Z_{g_1}(s,\chi). \tag{4.9}$$

Finally by using [15, Theorem A] for g_1, we have that there exists a polynomial $H(q^{-s}, \chi)$ which degree is independent from χ, such that

$$Z_{g_1}(s, \chi) = \frac{H(q^{-s}, \chi)}{(1 - q^{-1-s})(1 - q^{-n-p-nps})}. \tag{4.10}$$

The result now follows from Eqs. (4.8), (4.9) and (4.10). □

5. Exponential sums

Let Ψ be an additive character trivial on $\mathcal{O}_K$ but not on $\mathcal{P}_K^{-1}$. A such character is named *standard*. We put $z = u\pi^{-m}$, $m \in \mathbb{N} \setminus \{0\}$, $u \in \mathcal{O}_K^\times$. To a given polynomial $f(x, y, z) \in K[x, y, z]$ one associates the following exponential sum:

$$E(z, K, f) = q^{-3m} \sum_{x \in (\mathcal{O}_K/\mathcal{P}_K^m)^3} \Psi\big(uf(x)/\pi^m\big). \tag{5.1}$$

Proposition 1. *Let f be a hybrid polynomial of the form $f(x, y, z) = x^p + \alpha y^n z + \beta z^{n+1}$, where $\alpha \in \mathcal{O}_K^\times$, $\beta^{1/p} \in \mathcal{O}_K^\times$, n is a positive natural number such that $p \mid n + 1$. Further we assume that $n + 1 = p^l k$ with $p \nmid k$.*

If c_χ denotes the conductor of χ, then $Z_f(s, \chi) = 0$ unless $c(\chi) \leqslant p^l$.

Proof. We note first that since f is quasihomogeneous of degree $n + 1$ we have

$$Z_f(s, \chi) \equiv 0 \quad \text{if } \chi^{n+1} \neq 1.$$

If $Z_f(s, \chi)$ is not identically zero then

$$1 = \chi^{n+1}(1 + \pi\mathcal{O}_K) = \chi\big(1 + \pi^{p^l k}\mathcal{O}_K\big)$$

and thus $c_\chi \leqslant p^l$. And then we have the desired conclusion. □

Theorem 2.

(i) *Let f be a hybrid polynomial as in (3.4). Then for $|z|$ big enough*

$$E(z, K, f) = A\chi(acz)|z|^{-1} + \sum_{k=0}^{n} B_k \chi(acz)|z|^{-\frac{|w|}{n+1} + \frac{2\pi ik}{(n+1)\log q}} + \sum_{k=0}^{pn-1} C_k \chi(acz)|z|^{-\frac{1}{p} - \frac{1}{n} + \frac{2\pi ik}{pn\log q}},$$

where the A, B_k and C_k are complex constants and at least one of them is different from zero.

(ii) *For $|z|$ big enough,*

$$\big|E(z, K, f)\big| \leqslant D(K)|z|^{-\frac{1}{p} - \frac{1}{n}},$$

where $D(K)$ is a positive constant.

Proof. The results follow from Theorem 1, Proposition 1 and [3, Proposition 1.4.5], by writing $Z_f(s, \chi)$ in partial fractions. □

Acknowledgments

The authors want to thank to the referees for their careful reading of the article and for several useful suggestions. The authors are grateful to Wilson Zúñiga-Galindo for pointing their attention at this problem and for many useful discussions they had with him concerning this paper.

References

[1] Z.I. Borevich, I.R. Shafarevich, Number Theory, Academic Press, New York, 1966.
[2] V. Cossart, O. Piltant, Resolution of singularities of threefolds in positive characteristic. II, J. Algebra 321 (7) (2009) 1836–1976.
[3] J. Denef, Report on Igusa's local zeta functions, in: Seminaire Bourbaki 741, Astérisque 201/202/203 (1990–1991).
[4] J. Denef, Poles of p-adic complex powers and Newton polyhedra, Nieuw Arch. Wiskd. 13 (1995) 289–295.
[5] H. Hauser, Why the characteristic zero proof of resolution of singularities fails in positive characteristic, manuscript, 2003, 51 pp., available at http://homepage.univie.ac.at/herwig.hauser/index.html#Publikationen.
[6] H. Hauser, On the problem of resolution of singularities in positive characteristic (or: A proof we are still waiting for), Bull. Am. Math. Soc. 47 (1) (2010) 1–30.
[7] J.I. Igusa, Complex powers and asymptotic expansions. I, J. Reine Angew. Math. 268/269 (1974) 110–130.
[8] J.I. Igusa, Complex powers and asymptotic expansions. II, J. Reine Angew. Math. 278/279 (1975) 307–321.
[9] J.I. Igusa, Local zeta functions of certain prehomogeneous vector spaces, Am. J. Math. 114 (2) (1992) 251–296.
[10] J.I. Igusa, A stationary phase formula for p-adic integrals and its applications, in: Algebraic Geometry and Its Applications, Springer-Verlag, 1994, pp. 175–194.
[11] J.I. Igusa, An Introduction to the Theory of Local Zeta Functions, Stud. Adv. Math., vol. 14, Amer. Math. Soc., International Press, 2000.
[12] D. Meuser, M. Robinson, The Igusa local zeta functions of elliptic curves, Math. Comput. 71 (238) (2002) 815–823.
[13] D. Segers, The asymptotic behaviour of the number of solutions of polynomial congruences, An. St. Univ. Ovidius Constanta 19 (2011) 255–262.
[14] W.A. Zúñiga-Galindo, Igusa's local zeta function of semiquasihomogeneous polynomials, Trans. Am. Math. Soc. 353 (2001) 3193–3207.
[15] W.A. Zúñiga-Galindo, Local zeta functions and Newton polyhedra, Nagoya Math. J. 172 (2003) 31–58.