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Compact composition operators on the Dirichlet space and capacity of sets of contact points ☆

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Abstract

We prove several results about composition operators on the Dirichlet space $\mathcal{D}_*$. For every compact set $K \subseteq \partial\mathbb{D}$ of logarithmic capacity $\operatorname{Cap} K = 0$, there exists a Schur function φ both in the disk algebra $A(\mathbb{D})$ and in $\mathcal{D}_*$ such that the composition operator C_φ is in all Schatten classes $S_p(\mathcal{D}_*)$, $p > 0$, and for which $K = \{e^{it};\ |\varphi(e^{it})| = 1\} = \{e^{it};\ \varphi(e^{it}) = 1\}$. For every bounded composition operator C_φ on $\mathcal{D}_*$ and every $\xi \in \partial\mathbb{D}$, the logarithmic capacity of $\{e^{it};\ \varphi^*(e^{it}) = \xi\}$ is 0. Every compact composition operator C_φ on $\mathcal{D}_*$ is compact on $\mathfrak{B}^{\Psi_2}$ and on H^{Ψ_2}; in particular, C_φ is in every Schatten class S_p, $p > 0$, both on H^2 and on $\mathfrak{B}^2$. There exists a Schur function φ such that C_φ is compact on H^{Ψ_2}, but which is not even bounded on $\mathcal{D}_*$. There exists a Schur function φ such that C_φ is compact on $\mathcal{D}_*$, but in no Schatten class $S_p(\mathcal{D}_*)$.
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1. Introduction, notation and background

1.1. Introduction

Recall that a Schur function is an analytic self-map of the open unit disk $\mathbb{D}$. Every Schur function φ generates a bounded composition operator C_φ on the Hardy space H^2, given by $C_\varphi(f) = f \circ \varphi$. Let us also introduce the set E_φ of contact points of the symbol with the unit circle (equipped with its normalized Haar measure m), namely:

$$E_\varphi = \{ e^{it};\ |\varphi^*(e^{it})| = 1 \}. \tag{1.1}$$

In terms of E_φ, a well-known necessary condition for compactness of C_φ on H^2 is that $m(E_\varphi) = 0$. This set E_φ is otherwise more or less arbitrary. Indeed, it was proved in [12] that there exist compact composition operators C_φ on H^2 such that the Hausdorff dimension of E_φ is 1. This was generalized in [10]: for every Lebesgue-negligible compact set K of the unit circle $\mathbb{T}$, there is a Hilbert–Schmidt composition operator C_φ on H^2 such that $E_\varphi = K$, and in [24]:

Theorem 1.1. *(See [24].) For every Lebesgue-negligible compact set K of the unit-circle $\mathbb{T}$ and every vanishing sequence (ε_n) of positive numbers, there is a composition operator C_φ on H^2 such that $E_\varphi = K$ and such that its approximation numbers satisfy $a_n(C_\varphi) \leqslant C e^{-n\varepsilon_n}$.*

We are interested here in a different Hilbert space of analytic functions, on which not every Schur function defines a bounded composition operator, namely the Dirichlet space $\mathcal{D}$. Recall its definition: the Dirichlet space $\mathcal{D}$ is the space of analytic functions $f : \mathbb{D} \to \mathbb{C}$ such that:

$$\|f\|_{\mathcal{D}}^2 := |f(0)|^2 + \int_{\mathbb{D}} |f'(z)|^2 \, dA(z) < +\infty. \tag{1.2}$$

If $f(z) = \sum_{n=0}^{\infty} c_n z^n$, one has:

$$\|f\|_{\mathcal{D}}^2 = |c_0|^2 + \sum_{n=1}^{\infty} n |c_n|^2. \tag{1.3}$$

Then $\| \ \|_{\mathcal{D}}$ is a norm on $\mathcal{D}$, making $\mathcal{D}$ a Hilbert space. Whereas every Schur function φ generates a bounded composition operator C_φ on the Hardy space H^2, it is no longer the case for the Dirichlet space (see [27], Proposition 3.12, for instance). In this paper, we shall actually work, for convenience, with the subspace $\mathcal{D}_*$ of functions $f \in \mathcal{D}$ such that $f(0) = 0$.

In [11], the study of compact composition operators on the Dirichlet space $\mathcal{D}$ associated with a Schur function φ in connection with the set E_φ was initiated. In particular, it was proved there that if the composition operator C_φ is Hilbert–Schmidt on $\mathcal{D}$, then the logarithmic capacity $\operatorname{Cap} E_\varphi$ of E_φ is 0, but, on the other hand, there are compact composition operators on $\mathcal{D}$ for which this capacity is positive. The optimality of this theorem was later proved in [10] under the following form:

Theorem 1.2 *(O. El-Fallah, K. Kellay, M. Shabankhah, H. Youssfi). For every compact set K of the unit circle $\mathbb{T}$ with logarithmic capacity* $\operatorname{Cap} K$ *equal to* 0*, there exists a Hilbert–Schmidt composition operator C_φ on $\mathcal{D}$ such that $E_\varphi = K$.*

In this paper, we shall improve on this last result. We prove in Section 4 (Theorem 4.1) that for every compact set $K \subseteq \partial\mathbb{D}$ of logarithmic capacity $\operatorname{Cap} K = 0$, there exists a Schur function $\varphi \in A(\mathbb{D}) \cap \mathcal{D}_*$ such that the composition operator C_φ is in all Schatten classes $S_p(\mathcal{D}_*)$, $p > 0$, and for which $E_\varphi = K$ (and moreover $E_\varphi = \{\mathrm{e}^{it};\ \varphi(\mathrm{e}^{it}) = 1\}$). On the other hand, in Section 2, we show (Theorem 2.1) that for every bounded composition operator C_φ on $\mathcal{D}_*$ and every $\xi \in \partial\mathbb{D}$, the logarithmic capacity of $E_\varphi(\xi) = \{\mathrm{e}^{it};\ \varphi(\mathrm{e}^{it}) = \xi\}$ is 0.

In link with Hardy and Bergman spaces, we prove, in Section 2 yet, that every compact composition operator C_φ on $\mathcal{D}_*$ is compact on the Bergman–Orlicz space $\mathfrak{B}^{\Psi_2}$ and on the Hardy–Orlicz space H^{Ψ_2}. In particular, C_φ is in every Schatten class S_p, $p > 0$, both on the Hardy space H^2 and on the Bergman space $\mathfrak{B}^2$ (Theorem 2.9). However, there exists a Schur function φ such that C_φ is compact on H^{Ψ_2}, but which is not even bounded on $\mathcal{D}_*$ (Theorem 2.10).

In Section 3, we give a characterization of the membership of composition operators in the Schatten classes $S_p(\mathcal{D}_*)$, $p > 0$ (actually in $S_p(\mathcal{D}_{\alpha,*})$, where $\mathcal{D}_{\alpha,*}$ is the weighted Dirichlet space). We deduce that for every $p > 0$, there exists a symbol φ such that $C_\varphi \in S_p(\mathcal{D}_*)$, but $C_\varphi \notin S_q(\mathcal{D}_*)$ for any $q < p$, and that there exists another symbol φ such that $C_\varphi \in S_q(\mathcal{D}_*)$ for every $q < p$, but $C_\varphi \notin S_p(\mathcal{D}_*)$ (Theorem 3.3). We also show that there exists a Schur function φ such that C_φ is compact on $\mathcal{D}_*$, but in no Schatten class $S_p(\mathcal{D}_*)$ (Theorem 3.4).

1.2. Notation and background

We denote by $\mathbb{D}$ the unit open disk of the complex plane and by $\mathbb{T} = \partial\mathbb{D}$ the unit circle. A is the normalized area measure $dx\,dy/\pi$ of $\mathbb{D}$ and m the normalized Lebesgue measure $dt/2\pi$ on $\mathbb{T}$.

As said before, a Schur function is an analytic self-map of $\mathbb{D}$ and the associated composition operator is defined, formally, by $C_\varphi(f) = f \circ \varphi$. The function φ is called the symbol of C_φ. We denote by φ^* the non-tangential boundary values of φ on $\partial\mathbb{D}$. Let us recall that it follows from Lindelöf's Theorem that if φ and σ are two Schur functions, then $(\sigma \circ \varphi)^* = \sigma^* \circ \varphi^*$ almost everywhere (see [30], Theorem 2, or [8], Proposition 2.25).

The Dirichlet space $\mathcal{D}$ and its subspace $\mathcal{D}_*$ are defined above. In this paper, we call $\mathcal{D}_*$ the *Dirichlet space*.

An orthonormal basis of $\mathcal{D}_*$ is formed by $e_n(z) = z^n/\sqrt{n}$, $n \geqslant 1$. The reproducing kernel on $\mathcal{D}_*$, defined by $f(a) = \langle f, K_a\rangle$ for every $f \in \mathcal{D}_*$, is given by $K_a(z) = \sum_{n=1}^{\infty} \overline{e_n(a)} e_n(z)$, so that:

$$K_a(z) = \log \frac{1}{1 - \bar{a}z}. \tag{1.4}$$

Compactness of composition operators on $\mathcal{D}$ was characterized in terms of Carleson measure by D. Stegenga [32] and by B. MacCluer and J. Shapiro in terms of angular derivative, as well as in terms of Carleson measure [27]. Another characterization, though not very different, will be more useful for us here. It was given by N. Zorboska [38], p. 2020, but she attributed it to J. Shapiro: for $\varphi \in \mathcal{D}$, C_φ is bounded on $\mathcal{D}$ if and only if:

$$\sup_{h\in(0,2)}\sup_{|\xi|=1}\frac{1}{A[W(\xi,h)]}\int\limits_{W(\xi,h)} n_\varphi(w)\,dA(w)<\infty, \tag{1.5}$$

where $W(\xi,h)=\{w\in\mathbb{D};\ 1-|w|\leqslant h \text{ and } |\arg(w\bar{\xi})|\leqslant \pi h\}$ is the Carleson window of size $h\in(0,2)$ center at $\xi\in\mathbb{T}$ and n_φ is the counting function of φ:

$$n_\varphi(w)=\sum_{\varphi(z)=w}1,\quad w\in\varphi(\mathbb{D}) \tag{1.6}$$

(we set $n_\varphi(w)=0$ for $w\in\mathbb{D}\setminus\varphi(\mathbb{D})$). In particular, every Schur function with bounded valence defines a bounded composition operator on $\mathcal{D}$.

Moreover, C_φ is compact if and only if:

$$\sup_{|\xi|=1}\frac{1}{A[W(\xi,h)]}\int\limits_{W(\xi,h)} n_\varphi(w)\,dA(w)\underset{h\to0}{\longrightarrow}0. \tag{1.7}$$

For further informations on the Dirichlet space, one may consult the two surveys [1] and [29], for example.

1.2.1. Logarithmic capacity

The notion of logarithmic capacity is tied to the study of the Dirichlet space by the following seminal and sharp result of Beurling [3]; see also [14].

Theorem 1.3 *(Beurling). For every function $f(z)=\sum_{n=0}^\infty c_n z^n\in\mathcal{D}$, there exists a set $E\subseteq\partial\mathbb{D}$, with logarithmic capacity 0, such that, if $t\in\mathbb{T}\setminus E$, then the radial limit $f^*(\mathrm{e}^{it}):=\lim_{r\to1^-}f(r\mathrm{e}^{it})$ exists (in $\mathbb{C}$). Moreover, the result is optimal: if a compact set $E\subseteq\mathbb{T}$ has zero logarithmic capacity, there exists $f(z)=\sum_{n=0}^\infty c_n z^n\in\mathcal{D}$ such that $f^*(\mathrm{e}^{it})$ does not exist on E.*

Let us recall some definitions (see [14], Chapitre III, [7], Chapter 21, §7, or [29], Section 4, for example).

Let μ be a probability measure supported by a compact subset K of $\mathbb{T}$. The *potential* U_μ of μ is defined, for every $z\in\mathbb{C}$, by:

$$U_\mu(z)=\int\limits_K\log\frac{\mathrm{e}}{|z-w|}\,d\mu(w).$$

The *energy* I_μ of μ is defined by:

$$I_\mu=\int\limits_K U_\mu(z)\,d\mu(z)=\iint\limits_{K\times K}\log\frac{\mathrm{e}}{|z-w|}\,d\mu(w)\,d\mu(z).$$

The *logarithmic capacity* of a Borel set $E\subseteq\mathbb{T}$ is:

$$\mathrm{Cap}\,E=\sup\nolimits_\mu \mathrm{e}^{-I_\mu},$$

where the supremum is over all Borel probability measures μ with compact support contained in E. Hence E is of logarithmic capacity 0 (which is the case we are interested in) if and only if $I_\mu = \infty$ for all probability measures compactly carried by E. If $\operatorname{Cap} E = 0$, then E has null Lebesgue measure (see [14], Chapitre III, Théorème I) (hence $\operatorname{Cap} E > 0$ if E is a non-void open subset of $\mathbb{T}$), but the converse is wrong, as shown by Cantor's middle-third set $\mathfrak{C}$. A compact set K such that $\operatorname{Cap} K = 0$ is totally disconnected [7], Corollary 21.7.7.

If E is a compact set with $\operatorname{Cap} E > 0$, there is a unique probability measure carried by E that minimizes the energy I_μ (see [7], Theorem 21.10.2, or [14], Chapitre III, Proposition 4). Such a measure is called the *equilibrium measure* of E.

If μ is the equilibrium measure of the compact set K, we have Frostman's Theorem (see [7], Theorem 21.7.12, or [14], Chapitre III, Proposition 5 and Proposition 6): $U_\mu(z) \leqslant I_\mu$ for every $z \in \mathbb{C}$ and

$$U_\mu(z) = I_\mu \quad \text{for almost all } z \in K \tag{1.8}$$

(this actually holds outside a set of capacity zero, but we only use that this holds almost everywhere).

Suppose that the compact set K has zero logarithmic capacity. For $\varepsilon > 0$, let $K_\varepsilon = \{z \in \mathbb{T};\ \operatorname{dist}(z, K) \leqslant \varepsilon\}$, μ_ε its equilibrium measure, and I_{μ_ε} its energy. Then (see [7], Proposition 21.7.15):

$$\lim_{\varepsilon \to 0} I_{\mu_\varepsilon} = \infty. \tag{1.9}$$

2. Bounded and compact composition operators

2.1. Boundedness

In [11], E.A. Gallardo-Gutiérrez and M.J. González showed that for every Hilbert–Schmidt composition operator C_φ on $\mathcal{D}_*$, the logarithmic capacity of the set $E_\varphi = \{e^{i\theta} \in \partial\mathbb{D};\ |\varphi^*(e^{i\theta})| = 1\}$ is zero. On the other hand, they showed that there are compact composition operators on $\mathcal{D}_*$ for which E_φ has positive logarithmic capacity. We shall see that if we replace $|\varphi|$ by φ in the definition of E_φ, the result is very different.

Theorem 2.1. *For every bounded composition operator C_φ on $\mathcal{D}_*$ and every $\xi \in \partial\mathbb{D}$, the logarithmic capacity of $E_\varphi(\xi) = \{e^{it};\ \varphi^*(e^{it}) = \xi\}$ is* 0.

It is worth pointing out that the conclusion of the theorem does not hold for arbitrary $\varphi \in \mathcal{D}$ (see below, at the end of the section).

In order to prove Theorem 2.1, we first state the following characterization of Hilbert–Schmidt composition operators on $\mathcal{D}_*$. This result is stated in [11], but not entirely proved.

Lemma 2.2. *Let $\varphi \in \mathcal{D}_*$ be an analytic self-map of $\mathbb{D}$. Then C_φ is Hilbert–Schmidt on $\mathcal{D}_*$ if and only if*

$$\int_{\mathbb{D}} \frac{|\varphi'(z)|^2}{(1 - |\varphi(z)|^2)^2}\, dA(z) < \infty. \tag{2.1}$$

Proof. Let $e_n(z) = z^n/\sqrt{n}$; then $(e_n)_{n\geqslant 1}$ is an orthonormal basis of $\mathcal{D}_*$ and

$$\sum_{n=1}^{\infty} \left\| C_\varphi(e_n) \right\|^2 = \sum_{n=1}^{\infty} \frac{\|\varphi^n\|^2}{n} = \int_{\mathbb{D}} \frac{|\varphi'(z)|^2}{(1-|\varphi(z)|^2)^2}\, dA(z).$$

Hence (2.1) is satisfied if C_φ is Hilbert–Schmidt. To get the converse, we need to show that (2.1) implies that C_φ is bounded on $\mathcal{D}_*$. Let $f \in \mathcal{D}_*$ and write $f(z) = \sum_{n=1}^{\infty} c_n z^n$. Then $C_\varphi f = \sum_{n=1}^{\infty} c_n \varphi^n$ and

$$\begin{aligned} \|C_\varphi f\| &\leqslant \sum_{n=1}^{\infty} |c_n| \left\|\varphi^n\right\| \leqslant \left(\sum_{n=1}^{\infty} n|c_n|^2\right)^{1/2} \left(\sum_{n=1}^{\infty} \frac{\|\varphi^n\|^2}{n}\right)^{1/2} \\ &= \left(\int_{\mathbb{D}} \frac{|\varphi'(z)|^2}{(1-|\varphi(z)|^2)^2}\, dA(z)\right)^{1/2} \|f\|. \end{aligned}$$

Then (2.1) implies that C_φ is Hilbert–Schmidt. □

Now Theorem 2.1 will follow from the next proposition.

Proposition 2.3. *There exists an analytic self-map σ of $\mathbb{D}$, belonging to $\mathcal{D}_*$ and to the disk algebra $A(\mathbb{D})$, such that $\sigma(1) = 1$ and $|\sigma(\xi)| < 1$ for $\xi \in \partial\mathbb{D} \setminus \{1\}$ and such that the associated composition operator C_σ is Hilbert–Schmidt on $\mathcal{D}_*$.*

Taking this proposition for granted for a while, we can prove the theorem.

Proof of Theorem 2.1. Making a rotation, we may, and do, assume that $\xi = 1$. Then, if σ is the map of Proposition 2.3, $C_\varphi C_\sigma = C_{\sigma\circ\varphi}$ is Hilbert–Schmidt. By [11], the set $E_{\sigma\circ\varphi}$ has zero logarithmic capacity. But σ has modulus 1 only at 1; hence $e^{i\theta} \in E_{\sigma\circ\varphi}$ if and only if $e^{i\theta} \in E_\varphi(1)$ (recall that σ is continuous on $\overline{\mathbb{D}}$). □

To prove Proposition 2.3, it will be convenient to use the following criteria, where $\varphi_a(z) = \frac{z-a}{1-\bar{a}z}$.

Lemma 2.4. *Let $f \in \mathcal{D}$ such that $\Re\mathrm{e}\, f \geqslant 1$. Then if $\sigma = \varphi_a \circ e^{-1/f}$, where $a = e^{-1/f(0)}$, the composition operator C_σ is Hilbert–Schmidt on $\mathcal{D}_*$.*

Proof. Let $\sigma_0 = e^{-1/f}$. If $u = \Re\mathrm{e}\, f$ and $v = \Im\mathrm{m}\, f$, one has:

$$|\sigma_0|^2 = \exp\left(-\frac{2u}{u^2+v^2}\right) \quad \text{and} \quad \left|\sigma_0'\right|^2 = \frac{u'^2+v'^2}{(u^2+v^2)^2} \exp\left(-\frac{2u}{u^2+v^2}\right).$$

Then $|\sigma_0| < 1$ and so σ_0 is a self-map of $\mathbb{D}$. Since $u \geqslant 1 > 0$, one has $|\sigma_0'|^2 \leqslant (u'^2+v'^2)/(u^2+v^2)^2 \leqslant u'^2 + v'^2 = |f'|^2$; hence $\sigma_0 \in \mathcal{D}$.

For $0 \leqslant x \leqslant 2$, one has $1 - e^{-x} \geqslant x/4$. Therefore, since $u \geqslant 1$ implies $2u/(u^2 + v^2) \leqslant 2/u \leqslant 2$, one has:

$$1 - |\sigma_0|^2 \geqslant \frac{u}{2(u^2 + v^2)}.$$

It follows that:

$$\frac{|\sigma_0'|^2}{(1 - |\sigma_0|^2)^2} \leqslant \frac{u'^2 + v'^2}{(u^2 + v^2)^2} \frac{4(u^2 + v^2)^2}{u^2} \leqslant 4\big(u'^2 + v'^2\big) = 4\big|f'\big|^2.$$

Since $f \in \mathcal{D}$, $|f'|^2$ has a finite integral and therefore (2.1) is satisfied. It follows that C_{σ_0} is Hilbert–Schmidt on $\mathcal{D}$ and hence $C_\sigma = C_{\sigma_0} \circ C_{\varphi_a}$ is Hilbert–Schmidt on $\mathcal{D}_*$, since $\sigma(0) = 0$. □

Proof of Proposition 2.3. Let Ω be the domain defined by:

$$\Omega = \big\{z \in \mathbb{C};\ \Re\mathrm{e}\, z > 1 \text{ and } |\Im\mathrm{m}\, z| < 1/(\Re\mathrm{e}\, z)^2\big\}.$$

Let f be a conformal map from $\mathbb{D}$ onto Ω such that $f(1) = \infty$. Since $A(\Omega) < \infty$, we have $f \in \mathcal{D}$. By Lemma 2.4, the function $\sigma = e^{-1/f}$ has the required properties. □

Remark. The referee suggests us to give an example showing that the conclusion of Theorem 2.1 does not hold for arbitrary $\varphi \in \mathcal{D}$. Here is the example. Recall that, for $0 < \alpha \leqslant 1$, $f : \mathbb{T} \to \mathbb{C}$ belongs to Lip_α if, for some $C > 0$, $|f(x) - f(y)| \leqslant C|x - y|^\alpha$, for every $x, y \in \mathbb{T}$.

Theorem 2.5. *There exists a Schur function $\varphi \in \mathcal{D}_* \cap A(\mathbb{D})$ and $\xi \in \mathbb{T}$ for which the set $E_\varphi(\xi) = \{e^{it};\ \varphi(e^{it}) = \xi\}$ has positive logarithmic capacity.*

The proof is based on the two following results. We say that a set E is a peak for a subset A of the disk algebra $A(\mathbb{D})$ if there exists a function $q \in A$ which peaks on E, as defined in (4.1).

Theorem 2.6 *(J. Bruna). (See [4].) Let E be a compact subset of the unit circle and denote by d_E the distance function to E. Let $0 < \alpha < 1$. If the function $d_E^{-\alpha}$ belongs to $L^{1+\varepsilon}$ for some $\varepsilon > 0$, then E is a peak for $A_\alpha := A(\mathbb{D}) \cap \mathrm{Lip}_\alpha$. In particular, denoting by l_n, $n \geqslant 1$, the lengths of the disjoint open intervals composing $\mathbb{T} \setminus E$, then the set E is a peak for A_α if:*

$$\sum_{n \geqslant 1} l_n^{1-\alpha-\varepsilon} < +\infty \quad \textit{for some } \varepsilon > 0, \tag{2.2}$$

Note that (2.2) is fulfilled by every Cantor set of ratio λ with $2\lambda^{1-\alpha} < 1$.

Lemma 2.7. *If $\varphi \in A_\alpha$ with $\alpha > 1/2$, then $\varphi \in \mathcal{D} \cap A(\mathbb{D})$.*

Proof of Theorem 2.5. Take α such that $1/2 < \alpha < 1$ and a Cantor set E of ratio λ with $2\lambda^{1-\alpha} < 1$. Since the ratio is constant, the logarithmic capacity of E is positive (see [31], p. 41, for example). Let f be a function $f \in A_\alpha$ which peaks on E, given by Theorem 2.6. Since $\alpha > 1/2$, f belongs to $\mathcal{D}$, by Lemma 2.7. Let now $\varphi(z) = \frac{f(z) - f(0)}{1 - \overline{f(0)} f(z)} = \varphi_{f(0)}[f(z)]$. Then $\varphi \in \mathcal{D}$, because $\varphi'(z) = (\varphi_{f(0)})'[f(z)]f'(z)$ and $(\varphi_{f(0)})' \in \mathcal{C}(\overline{\mathbb{D}})$ is bounded in $\mathbb{D}$; hence $f' \in L^2(\mathbb{D})$ implies $\varphi' \in L^2(\mathbb{D})$. Since $\varphi(0) = 0$, we get $\varphi \in \mathcal{D}_*$. Moreover, $\xi = \frac{1-f(0)}{1-\overline{f(0)}} \in \mathbb{T}$ and $E_\varphi(\xi) = E$. □

Proof of Lemma 2.7. This is well known, but since we have no reference, we give a short proof, for convenience.

Let $f(z) = \sum_{n=0}^{\infty} c_n z^n$ a function in A_α and set $\gamma(t) = f(\mathrm{e}^{it})$. Parseval's formula applied to the first difference $\gamma(t+2h) - \gamma(t)$ gives:

$$\sum_{n=1}^{\infty} |c_n|^2 \sin^2 nh \ll h^{2\alpha}.$$

In particular $\sum_{2^N \leqslant n < 2^{N+1}} |c_n|^2 \sin^2 nh \ll h^{2\alpha}$, for every $N \geqslant 0$. Taking $h = 2^{-N}$ and using $\sin^2 nh \gg 1$ for $2^N \leqslant n < 2^{N+1}$, we get $\sum_{2^N \leqslant n < 2^{N+1}} |c_n|^2 \ll 2^{-2\alpha N}$, so that $\sum_{2^N \leqslant n < 2^{N+1}} n|c_n|^2 \ll 2^{(1-2\alpha)N}$. Summing up over N, we get $\sum_{n=1}^{\infty} n|c_n|^2 < +\infty$ since $2^{(1-2\alpha)} < 1$.

An alternative shorter proof is the following. Since $f \in A_\alpha$, one has, for some constant $C > 0$ (see [9], Theorem 5.1), $|f'(z)| \leqslant C(1-|z|)^{\alpha-1}$. Using polar coordinates, we get $\int_{\mathbb{D}} |f'(z)|^2\, dA(z) \leqslant 2\int_0^1 C^2(1-r)^{2\alpha-2} r\, dr < +\infty$, since $2\alpha - 2 > -1$. □

Let us give a simple proof of the following known result, which implies that the set of peak points of φ is finite if $\varphi \in \mathrm{Lip}_1$ (see [28]).

Proposition 2.8. *Let E be a compact subset of the unit circle and $0 < \alpha \leqslant 1$. If E is peak for A_α, then the function $d_E^{-\alpha}$ belongs to $L^{1,\infty}$, i.e.:*

$$m\big(d_E^{-\alpha} > t\big) \leqslant C/t \quad \textit{for all } t > 0.$$

In particular, $d_E^{-\alpha} \in L^{1-\varepsilon}$ for all $0 < \varepsilon < 1$, equivalently:

$$\sum_{n \geqslant 1} l_n^{1-\alpha+\varepsilon} < +\infty \quad \textit{for all } \varepsilon > 0.$$

For $\alpha = 1$, E is necessarily finite.

Proof. Let $E_\varepsilon = \{d_E < \varepsilon\}$. By a change of variable, it is the same to prove:

$$m(E_\varepsilon) \leqslant C\varepsilon^\alpha.$$

Let $\varphi \in A_\alpha$ peaking on E. In particular, $\varphi(\mathbb{D}) \subseteq \mathbb{D}$ and φ is a symbol for H^2, implying that the image measure m_φ is a Carleson measure. We set $\gamma(u) = \varphi(\mathrm{e}^{iu})$ and have $|\gamma(u) - \gamma(v)| \leqslant M|u-v|^\alpha$, for some constant M. Let now $t \in E_\varepsilon$ and $s \in E$ with $|t-s| \leqslant \varepsilon$. We see that:

$$\big|\gamma(t) - 1\big| = \big|\gamma(t) - \gamma(s)\big| \leqslant M|t-s|^\alpha \leqslant M\varepsilon^\alpha,$$

meaning that $E_\varepsilon \subseteq \varphi^{-1}[S(1, M\varepsilon^\alpha)]$. Therefore, we have:

$$m(E_\varepsilon) \leqslant m_\varphi\big[S\big(1, M\varepsilon^\alpha\big)\big] \leqslant C\varepsilon^\alpha$$

for some constant C and we are done.

For $\alpha = 1$, let $t_1, \dots, t_N$ be distinct points of E and $\varepsilon = \min_{j\neq k} |t_j - t_k| > 0$. Then, the intervals $(t_j - \varepsilon, t_j + \varepsilon)$ are disjoint and included in E_ε, so that $2N\varepsilon \leqslant m(E_\varepsilon) \leqslant C\varepsilon$, showing that $\#E \leqslant C/2$. □

2.2. Compactness

For the next result, recall that an Orlicz function Ψ is a nondecreasing convex function such that $\Psi(0) = 0$ and $\Psi(x)/x \to \infty$ as x goes to infinity. We refer to [18] for the definition of Hardy–Orlicz and Bergman–Orlicz spaces. In the following result, one set $\Psi_2(x) = \exp(x^2) - 1$.

Theorem 2.9. *Every compact composition operator C_φ on $\mathcal{D}_*$ is compact on the Bergman–Orlicz space $\mathfrak{B}^{\Psi_2}$ and on the Hardy–Orlicz space H^{Ψ_2}. In particular, C_φ is in every Schatten class S_p, $p > 0$, both on the Hardy space H^2 and on the Bergman space $\mathfrak{B}^2$.*

Proof. Consider the normalized reproducing kernels $\tilde{K}_a = K_a/\|K_a\|$, $a \in \mathbb{D}$. When $|a|$ goes to 1, they tend to 0 uniformly on compact sets of $\mathbb{D}$; hence $\|C_\varphi^*(\tilde{K}_a)\|$ tends to 0, by compactness of the adjoint operator C_φ^*. But $C_\varphi^*(K_a) = K_{\varphi(a)}$ and $\|K_a\|^2 = \langle K_a, K_a\rangle = \log\frac{1}{1-|a|^2}$, so we get:

$$\lim_{|a|\to 1} \frac{\log\frac{1}{1-|\varphi(a)|^2}}{\log\frac{1}{1-|a|^2}} = 0. \tag{2.3}$$

This condition means that C_φ is compact on the Bergman–Orlicz space $\mathfrak{B}^{\Psi_2}$ (see [18], p. 69) and implies that C_φ is in all Schatten classes $S_p(\mathfrak{B}^2)$, $p > 0$ (see [20]).

In the same way, it suffices to show that C_φ is compact on H^{Ψ_2}, because that implies that C_φ is in all Schatten classes $S_p(H^2)$ (see [17], Theorem 5.2).

Compactness of C_φ on H^Ψ is equivalent to say (see [18], Theorem 4.18) that:

$$\begin{aligned}\rho_\varphi(h) &:= \sup_{|\xi|=1} m\big(\big\{e^{it};\ \varphi^*\big(e^{it}\big) \in W(\xi, h)\big\}\big)\\ &= o_{h\to 0}\left[\frac{1}{\Psi(A\Psi^{-1}(1/h))}\right] \quad \text{for every } A > 0.\end{aligned}$$

When $\Psi = \Psi_2$, this means that $\rho_\varphi(h) = o(h^A)$ for every $A > 0$. Now, by [19], Theorem 4.2, this is also equivalent to say that:

$$\sup_{|\xi|=1} \frac{1}{A[W(\xi, h)]} \int_{W(\xi,h)} N_\varphi(w)\, dA(w) = o\big(h^A\big) \quad \text{for every } A > 0, \tag{2.4}$$

where N_φ is the Nevanlinna counting function of φ:

$$N_\varphi(w) = \sum_{\varphi(z)=w} \big(1 - |z|^2\big), \quad w \in \varphi(\mathbb{D}), \tag{2.5}$$

and $N_\varphi(w) = 0$ otherwise.

But (2.3) is equivalent to the fact that for every $\varepsilon > 0$ there exists $\delta_\varepsilon > 0$ such that:

$$1 - \left|\varphi(z)\right| \geqslant \delta_\varepsilon \left(1 - |z|\right)^\varepsilon, \quad \forall z \in \mathbb{D}. \tag{2.6}$$

Hence $N_\varphi(w) \leqslant 2\delta_\varepsilon^{-1}(1 - |w|)^{1/\varepsilon} n_\varphi(w)$. It follows that (since $1 - |w| \leqslant h$ for $w \in W(\xi, h)$):

$$\frac{1}{A[W(\xi,h)]} \int\limits_{W(\xi,h)} N_\varphi(w)\, dA(w) \leqslant 2\delta_\varepsilon^{-1} h^{1/\varepsilon} \frac{1}{A[W(\xi,h)]} \int\limits_{W(\xi,h)} n_\varphi(w)\, dA(w),$$

which is $o(h^{1/\varepsilon})$, uniformly for $|\xi| = 1$, by (1.7). □

Remarks. 1. One may argue that compactness of C_φ on H^{Ψ_2} implies its compactness on $\mathfrak{B}^{\Psi_2}$ (see [20], Proposition 4.1, or [22], Theorem 9). One may also use the forthcoming Corollary 3.2 saying that $C_\varphi \in S_p(H^2)$ implies that $C_\varphi \in S_p(\mathfrak{B}^2)$.

2. To show the compactness of C_φ on H^{Ψ_2}, we used its compactness on $\mathcal{D}_*$ twice. However, due to the fact that $\varepsilon > 0$ is arbitrary, we may replace $o(h^{1/\varepsilon})$ by $O(h^{1/\varepsilon})$; hence to end the proof, we only have to use (1.5), *i.e.* the boundedness of C_φ on $\mathcal{D}_*$, instead of (1.7).

Note that (2.3) does not suffice to have compactness on H^{Ψ_2} (in [18], Proposition 5.5, we construct a Blaschke product satisfying (2.3)).

In the opposite direction, we have the following result.

Theorem 2.10. *There exists a Schur function φ such that C_φ is compact on H^{Ψ_2}, but which is not even bounded on $\mathcal{D}_*$.*

To prove this theorem, we first begin with the following key lemma.

Lemma 2.11. *There exists a constant $\kappa_1 > 0$ such that for any $f \in \mathcal{H}(\mathbb{D})$ having radial limits f^* a.e. and which satisfies, for some $\alpha \in \mathbb{R}$:*

$$\begin{cases} \Im\mathrm{m}\, f(0) < \alpha \quad \text{and} \\ f(\mathbb{D}) \subseteq \{z \in \mathbb{C};\ 0 < \Re\mathrm{e}\, z < \pi\} \cup \{z \in \mathbb{C};\ \Im\mathrm{m}\, z < \alpha\}, \end{cases} \tag{2.7}$$

we have, for all $y \geqslant \alpha$:

$$m\left(\left\{z \in \mathbb{T};\ \Im\mathrm{m}\left[f^*(z)\right] \geqslant y\right\}\right) \leqslant \kappa_1 \mathrm{e}^{\alpha - y}.$$

Proof. Suppose that f satisfies (2.7), and define $f_1(z) = -if(z) + \frac{\pi}{2}i - \alpha$. Then either $\Re\mathrm{e}[f_1(z)] < 0$, or $-\frac{\pi}{2} < \Im\mathrm{m}[f_1(z)] < \frac{\pi}{2}$ for every $z \in \mathbb{D}$. Therefore, defining $h(z) = 1 + \exp[f_1(z)]$, we have $h: \mathbb{D} \to \mathbb{H}$, that is $\Re\mathrm{e}[h(z)] > 0$ for every $z \in \mathbb{D}$.

Finally define $h_1(z) = h(z) - i\,\Im\mathrm{m}[h(0)]$. Then $h_1: \mathbb{D} \to \mathbb{H}$ and $h_1(0) \in \mathbb{R}$ (and so $h_1(0) > 0$). Kolmogorov's inequality on the weak-type (1, 1) of the Hilbert transform (see [23], Chapitre 6, Théorème II.6, or [2], Chapter 3, Proposition 4.6) yields that, for some absolute constant C_1, one has, for every $\lambda > 0$:

$$m\left(\left\{z \in \mathbb{T};\ \left|h_1^*(z)\right| \geqslant \lambda\right\}\right) \leqslant C_1 \frac{h_1(0)}{\lambda}. \tag{2.8}$$

Observe that, since $\Im\mathrm{m}[f(0)] < \alpha$, we have $\Re\mathrm{e}[f_1(0)] < 0$, and then:

$$\left|\Im\mathrm{m}\left[h(0)\right]\right| < 1 \quad \text{and} \quad h_1(0) = \Re\mathrm{e}\left[h(0)\right] < 2. \tag{2.9}$$

Suppose now that, for $y > \alpha$ and $z \in \mathbb{D}$, we have $\Im\mathrm{m}[f(z)] > y$; then $\exp[f_1(z)] \in \mathbb{H}$, and $|h(z)| \geqslant |\exp[f_1(z)]| > \mathrm{e}^{y-\alpha}$. Taking radial limits we get, up to a set of null Lebesgue-measure:

$$\left\{z \in \mathbb{T};\ \Im\mathrm{m}\left[f^*(z)\right] \geqslant y\right\} \subseteq \left\{z \in \mathbb{T};\ \left|h^*(z)\right| \geqslant \mathrm{e}^{y-\alpha}\right\}.$$

We consider two cases: $\mathrm{e}^{y-\alpha} \geqslant 2$ and $\mathrm{e}^{y-\alpha} < 2$. When $\mathrm{e}^{y-\alpha} \geqslant 2$, then $|h^*(z)| \geqslant \mathrm{e}^{y-\alpha}$ yields:

$$\left|h_1^*(z)\right| \geqslant \mathrm{e}^{y-\alpha} - \left|\Im\mathrm{m}\left[h(0)\right]\right| > \mathrm{e}^{y-\alpha} - 1 \geqslant \frac{1}{2}\mathrm{e}^{y-\alpha},$$

by the first part of (2.9). Then, using (2.8) and the second part of (2.9), we have:

$$\begin{aligned} m\left(\left\{z \in \mathbb{T};\ \Im\mathrm{m}\left[f^*(z)\right] \geqslant y\right\}\right) &\leqslant m\left(\left\{z \in \mathbb{T};\ \left|h_1^*(z)\right| > (1/2)\mathrm{e}^{y-\alpha}\right\}\right) \\ &\leqslant \frac{2C_1 h_1(0)}{\mathrm{e}^{y-\alpha}} \leqslant \frac{4C_1}{\mathrm{e}^{y-\alpha}}, \end{aligned}$$

and, in this case, the lemma is proved, if one takes $\kappa_1 \geqslant 4C_1$.

When $\mathrm{e}^{y-\alpha} < 2$, then $\mathrm{e}^{\alpha-y} > 1/2$, and, because:

$$m\left(\left\{z \in \mathbb{T};\ \Im\mathrm{m}\left[f^*(z)\right] \geqslant y\right\}\right) \leqslant 1 < \kappa_1 \mathrm{e}^{\alpha-y},$$

since $\kappa_1 > 2$, the lemma is proved. □

Now, we give a general construction of Schur functions with suitable properties.

Proposition 2.12. *Let $\mathfrak{g}\colon (0,\infty) \to (0,\infty)$ be a continuous and non-increasing function such that*:

$$\lim_{t \to 0^+} \mathfrak{g}(t) = +\infty \quad \textit{and} \quad \lim_{t \to +\infty} \mathfrak{g}(t) = 0.$$

Let $\mathfrak{h}\colon (0,\infty) \to (0,\infty]$ be a lower semicontinuous function such that $M := \sup\{\mathfrak{h}(t);\ t \geqslant \pi\} < +\infty$ and consider the simply connected domain:

$$\Omega = \left\{x + iy;\ x \in (0,\infty) \quad \textit{and} \quad \mathfrak{g}(x) < y < \mathfrak{g}(x) + \mathfrak{h}(x)\right\}.$$

Let $\mathfrak{f}\colon \overline{\mathbb{D}} \to \overline{\Omega} \cup \{\infty\}$ be a conformal mapping from $\mathbb{D}$ onto Ω such that $\mathfrak{f}(0) = \pi + i(\mathfrak{g}(\pi) + \mathfrak{h}(\pi)/2)$.

Then the symbol $\varphi\colon \mathbb{D} \to \mathbb{D}$ defined by $\varphi(z) = \exp[-\mathfrak{f}(z)]$, for every $z \in \mathbb{D}$, satisfies, for some ε_0, $k_0 > 0$:

1) *For all* $h \in (0, \varepsilon_0)$:

$$m\bigl(\bigl\{z \in \mathbb{T};\ \bigl|\varphi^*(z)\bigr| > 1 - h\bigr\}\bigr) \leqslant k_0 \exp\bigl(-\mathfrak{g}(2h)\bigr). \tag{2.10}$$

2) *Assume that, for some* $r \in (0, \infty]$ *and integers* $0 \leqslant n < N \leqslant \infty$, *one has* $\{\mathfrak{h}(t);\ t \leqslant r\} \subseteq (2n\pi, 2N\pi]$. *Then, for all* $z \in \mathbb{D}$, *such that* $|z| > \mathrm{e}^{-r}$, *we have* $n \leqslant n_\varphi(z) \leqslant N$.

In particular, $\{z \in \mathbb{D};\ |z| > \mathrm{e}^{-r}\} \subseteq \varphi(\mathbb{D}) \subseteq \mathbb{D} \setminus \{0\}$, *when* $n \geqslant 1$.

Remarks. 1. When $N = 1$, the map φ is univalent.

2. When $r = \infty$ and $n \geqslant 1$, we have $\varphi(\mathbb{D}) = \mathbb{D} \setminus \{0\}$.

3. With $\mathfrak{g}(t) = 1/t$, the operator C_φ is compact on H^{Ψ_2}, therefore belongs to all Schatten classes $S_p(H^2)$, $p > 0$.

4. When $N < \infty$, the operator C_φ is bounded on the Dirichlet space.

5. When $n \geqslant 1$, the operator C_φ is not compact on the Dirichlet space (since the averages on the windows of the function n_φ cannot uniformly vanish).

Proof of Proposition 2.12. We shall apply Lemma 2.11 with $\alpha = M + \mathfrak{g}(\pi)$.

Suppose that, for $z \in \mathbb{T}$ and $0 < h < 1$, we have $|\varphi^*(z)| > 1 - h$. Then, if h is small enough,

$$\mathrm{e}^{-2h} < 1 - h < \bigl|\varphi^*(z)\bigr| = \exp\bigl(-\Re\mathrm{e}\bigl[f^*(z)\bigr]\bigr),$$

and therefore $2h > \Re\mathrm{e}[\mathfrak{f}^*(z)]$. But observe that $\mathfrak{f}^*(z) \in \overline{\Omega} \cup \{\infty\}$, and so, if $2h > \Re\mathrm{e}[f^*(z)]$, we necessarily have $\Im\mathrm{m}[\mathfrak{f}^*(z)] \geqslant \mathfrak{g}(2h)$. Again, if h is small enough, we have $y = \mathfrak{g}(2h) > \alpha$, and may apply the lemma to obtain:

$$m\bigl(\bigl\{z \in \mathbb{T};\ \bigl|\varphi^*(z)\bigr| > 1 - h\bigr\}\bigr) \leqslant m\bigl(\bigl\{z \in \mathbb{T};\ \Im\mathrm{m}\bigl[\mathfrak{f}^*(z)\bigr] \geqslant \mathfrak{g}(2h)\bigr\}\bigr) \leqslant \kappa_1 \mathrm{e}^{\alpha - \mathfrak{g}(2h)}.$$

We get (2.10).

On the other hand, let $Z \in \mathbb{D}$ such that $|Z| > \mathrm{e}^{-r}$, we can write $Z = \mathrm{e}^{-x}\mathrm{e}^{i\theta}$ with $x < r$. We can find $\theta_i's$ such that $\mathfrak{g}(x) < \theta_1 < \cdots < \theta_s < \mathfrak{g}(x) + \mathfrak{h}(x)$ and $\theta_j \equiv \theta[2\pi]$ with $n \leqslant s \leqslant N$. For each j, there exists a unique $z_j \in \mathbb{D}$, such that $\Re\mathrm{e}\, \mathfrak{f}(z_j) = x$ and $\Im\mathrm{m}\, \mathfrak{f}(z_j) = \theta_j$; hence $\varphi(z_j) = Z$. Moreover no other $z \in \mathbb{D}$ can satisfy $\varphi(z) = Z$. Hence $n_\varphi(Z) = s$. □

Proof of Theorem 2.10. As said before, if one takes $\mathfrak{g}(t) = 1/t$ in Proposition 2.12, then C_φ is compact on H^{Ψ_2} and hence is in all Schatten classes $S_p(H^2)$, $p > 0$. On the other hand, if one chooses also $\mathfrak{h}(t) = 1/t$, then, for every $r > 0$, $\{\mathfrak{h}(t);\ t \leqslant r\} = [1/r, \infty)$ and for $|z| > \mathrm{e}^{-r}$, we get that $n_\varphi(z) \geqslant [1/(2\pi r)]$ (the integer part of $1/(2\pi r)$). It follows that, for some constant $c > 0$, one has, with $\mathrm{e}^{-r} = 1 - h$:

$$\frac{1}{A[W(\xi, h)]} \int\limits_{W(\xi,h)} n_\varphi(z)\, dA(z) \geqslant c \frac{1}{\log[1/(1-h)]} \xrightarrow[h \to 0]{} \infty.$$

Therefore, C_φ is not bounded on $\mathcal{D}_*$, by (1.5). □

Remarks. 1. Actually, as we may take $\mathfrak{g}$ growing as we wish, the proof shows, using [18], Theorem 4.18, that for every Orlicz function Ψ, one can find a Schur function φ such that C_φ is not bounded on $\mathcal{D}_*$, though compact on the Hardy–Orlicz space H^Ψ.

2. This construction also allows to produce a *univalent* map φ, with an arbitrary small Carleson function $\rho_\varphi(h) = \sup_{|\xi|=1} m(\{e^{it};\ \varphi^*(e^{it}) \in W(\xi, h)\})$, and such that C_φ is not compact on the Dirichlet space (note we cannot replace "compact" by "bounded" since any Schur function with a bounded valence is bounded on the Dirichlet space).

Indeed, take $\mathfrak{h}(t) = 2\pi$ and $\mathfrak{g}$ be $\mathcal{C}^1$: $\mathfrak{g}(t) = 1/t$ for instance. We have $N = 1$ and so φ is univalent. Now it suffices to notice that the range of the curve

$$\Gamma = \left\{e^{-x-i\mathfrak{g}(x)};\ x \in (0,\infty)\right\} = \left\{\left(t\cos\left(1/\ln(t)\right), t\sin\left(1/\ln(t)\right)\right);\ t \in (0,1)\right\} \subseteq \mathbb{D}$$

has a null area measure. The range of φ is $\mathbb{D} \setminus (\Gamma \cup \{0\})$ and for each $w \notin \Gamma$, we have $n_\varphi(w) = 1$ Then, for $h \in (0,1)$, we have:

$$\frac{1}{h^2}\int_{W(1,h)} n_\varphi(w)\,dA(w) = \frac{1}{h^2}\int_{W(1,h)\setminus\Gamma} dA(w) = \frac{1}{h^2}A\left[W(1,h)\setminus\Gamma\right]$$
$$= \frac{1}{h^2}A\left[W(1,h)\right] \approx 1,$$

and so C_φ in not compact on $\mathcal{D}_*$, by (1.7). □

3. Composition operators in Schatten classes

3.1. Characterization

In this section, we give a characterization of the membership in the Schatten classes of composition operators on $\mathcal{D}_*$. This characterization will be deduced from Luecking's one for composition operators on the Bergman space. Actually, we shall give it for weighted Dirichlet spaces $\mathcal{D}_{\alpha,*}$. Boundedness and compactness have been characterized by B. MacCluer and J. Shapiro in [27] and, in other terms, by N. Zorboska in [38].

Recall that for $\alpha > -1$, the weighted Dirichlet space $\mathcal{D}_\alpha$ is the space of analytic functions $f:\mathbb{D} \to \mathbb{C}$ such that

$$\int_{\mathbb{D}} \left|f'(z)\right|^2\left(1-|z|^2\right)^\alpha\,dA(z) < \infty. \tag{3.1}$$

This is a Hilbert space for the norm given by:

$$\|f\|_\alpha^2 = \left|f(0)\right|^2 + (\alpha+1)\int_{\mathbb{D}} \left|f'(z)\right|^2\left(1-|z|^2\right)^\alpha\,dA(z) < \infty. \tag{3.2}$$

The standard Dirichlet space $\mathcal{D}$ corresponds to $\alpha = 0$; the Hardy space H^2 to $\alpha = 1$ and the standard Bergman space to $\alpha = 2$. For more general weights, see [15].

We denote by $\mathcal{D}_{\alpha,*}$ the subspace of the $f \in \mathcal{D}_\alpha$ such that $f(0) = 0$.

If φ is a Schur function, one defines its *weighted Nevanlinna counting function* $N_{\varphi,\alpha}$ at $w \in \Omega := \varphi(\mathbb{D})$ as the number of pre-images of w with the weight $(1-|z|^2)^\alpha$:

$$N_{\varphi,\alpha}(w) = \sum_{\varphi(z)=w} \left(1-|z|^2\right)^\alpha. \tag{3.3}$$

For $w \in \mathbb{D} \setminus \varphi(\mathbb{D})$, we set $N_{\varphi,\alpha}(w) = 0$. One has $N_{\varphi,1} = N_\varphi$ and $N_{\varphi,0} = n_\varphi$.

With this notation, recall the change of variable formula:

$$\int_{\mathbb{D}} F\left[\varphi(z)\right]\left|\varphi'(z)\right|^2\left(1-|z|^2\right)^\alpha \, dA(z) = \int_{\Omega} F(w) N_{\varphi,\alpha}(w)\, dA(w). \tag{3.4}$$

Denote by $R_{n,j}$, $n \geqslant 0$, $0 \leqslant j \leqslant 2^n - 1$, the Hastings–Luecking windows:

$$R_{n,j} = \left\{ z \in \mathbb{D};\ 1-2^{-n} \leqslant |z| < 1-2^{-n-1} \text{ and } \frac{2j\pi}{2^n} \leqslant \arg z < \frac{2(j+1)\pi}{2^n} \right\}.$$

We can now state.

Theorem 3.1. *Let $\alpha > -1$. Let φ be a Schur function and $p > 0$. Then $C_\varphi \in S_p(\mathcal{D}_{\alpha,*})$ if and only if*:

$$\sum_{n=0}^{\infty} \sum_{j=0}^{2^n-1} \left[2^{n(\alpha+2)} \int_{R_{n,j}} N_{\varphi,\alpha}(w)\, dA(w) \right]^{p/2} < \infty. \tag{3.5}$$

If φ is univalent, (3.5) *can be replaced by the purely geometric condition*:

$$\sum_{n=0}^{\infty} \sum_{j=0}^{2^n-1} \left[2^{n(\alpha+2)} A_\alpha(R_{n,j} \cap \Omega)\right]^{p/2} < \infty, \tag{3.6}$$

where A_α is the weighted measure $dA_\alpha(w) = (\alpha+1)(1-|w|^2)^\alpha\, dA(w)$.

Remark. Of course, every operator in a Schatten class is compact, but we may note that condition (3.5) implies the compactness of C_φ, by [38], Theorem 1 (and [16], Proposition 3.3).

Proof of Theorem 3.1. First, we compute $C_\varphi^* C_\varphi$. Let us fix f and g in the Dirichlet space $\mathcal{D}_{\alpha,*}$. We have:

$$(\alpha+1) \int_{\mathbb{D}} \left[\left(C_\varphi^* C_\varphi\right)(f)\right]'(z)\overline{g'(z)}\left(1-|z|^2\right)^\alpha \, dA(z)$$

$$= \langle f \circ \varphi, g \circ \varphi \rangle_{\mathcal{D}_{\alpha,*}} = (\alpha+1) \int_{\mathbb{D}} \left(f' \circ \varphi\right)(z)\overline{\left(g' \circ \varphi\right)(z)}\left|\varphi'(z)\right|^2\left(1-|z|^2\right)^\alpha \, dA(z).$$

By the change of variable formula, we get:

$$\int_{\mathbb{D}} \big[\big(C_\varphi^* C_\varphi\big)(f)\big]'(z)\overline{g'(z)}\big(1-|z|^2\big)^\alpha\,dA = \int_{\mathbb{D}} f'(w)\overline{g'(w)}N_{\varphi,\alpha}(w)\,dA(w),$$

which is equivalent to:

$$\int_{\mathbb{D}} \big[\big(C_\varphi^* C_\varphi\big)(f)\big]'(z)\overline{G(z)}\big(1-|z|^2\big)^\alpha\,dA(z) = \int_{\mathbb{D}} f'(w)\overline{G(w)}N_{\varphi,\alpha}(w)\,dA(w)$$

for every function G belonging to the weighted Bergman space $\mathfrak{B}^2_\alpha$.

That means that $[(C_\varphi^* C_\varphi)(f)]' - f'.N_{\varphi,\alpha}/(1-|w|^2)^\alpha$ is orthogonal to the weighted Bergman space $\mathfrak{B}^2_\alpha$. But $[(C_\varphi^* C_\varphi)(f)]' \in \mathfrak{B}^2_\alpha$. Hence $[(C_\varphi^* C_\varphi)(f)]'$ is the orthogonal projection onto $\mathfrak{B}^2_\alpha$ of the function $f'.N_{\varphi,\alpha}/(1-|w|^2)^\alpha$. Thus (see [35], §6.4.1, or [37], §4.4), we obtain that for every $z \in \mathbb{D}$:

$$\begin{aligned}\big[\big(C_\varphi^* C_\varphi\big)(f)\big]'(z) &= (\alpha+1)\int_{\mathbb{D}} \frac{f'(w)}{(1-\bar{w}z)^{\alpha+2}}\,\frac{N_{\varphi,\alpha}(w)}{(1-|w|^2)^\alpha}\big(1-|w|^2\big)^\alpha\,dA(w)\\ &= (\alpha+1)\int_{\mathbb{D}} \frac{f'(w)}{(1-\bar{w}z)^{\alpha+2}}\,d\mu(w)\\ &= (\alpha+1)T_\mu\big(f'\big)(z),\end{aligned}$$

where μ is the positive measure A with weight $N_{\varphi,\alpha}$ and T_μ is the Toeplitz operator on $\mathfrak{B}^2_\alpha$ is introduced in [25] (let us point out that α in [25] corresponds to $-(\alpha+1)$ in our work).

In other words, introducing the map $\Delta(h) = h'$, which is an isometry from $\mathcal{D}_{\alpha,*}$ onto $\mathfrak{B}^2_\alpha$, we have $\Delta \circ (C_\varphi^* C_\varphi) = T_\mu \circ \Delta$. We have the following diagram:

$$\begin{array}{ccc} \mathcal{D}_{\alpha,*} & \xrightarrow{C_\varphi^* C_\varphi} & \mathcal{D}_{\alpha,*} \\ \Delta\big\downarrow & & \big\downarrow\Delta \\ \mathfrak{B}^2_\alpha & \xrightarrow{T_\mu} & \mathfrak{B}^2_\alpha \end{array}$$

Hence the approximation numbers of T_μ (viewed as an operator on $\mathfrak{B}^2_\alpha$) and the ones of $C_\varphi^* C_\varphi$ (viewed as an operator on $\mathcal{D}_{\alpha,*}$) are the same. In particular, the membership in the Schatten classes are the same and the final result follows from the main theorem in [25]: $C_\varphi \in S_p(\mathcal{D}_{\alpha,*})$ if and only if $C_\varphi^* C_\varphi \in S_{p/2}(\mathcal{D}_{\alpha,*})$ and that holds if and only if:

$$\sum_{n=0}^{\infty}\sum_{j=0}^{2^n-1}\big[2^{n(\alpha+2)}\mu(R_{n,j})\big]^{p/2} < \infty.$$

Hence $C_\varphi \in S_p(\mathcal{D}_{\alpha,*})$ if and only if:

$$\sum_{n=0}^{\infty}\sum_{j=0}^{2^n-1}\left[2^{n(\alpha+2)}\int\limits_{R_{n,j}} N_{\varphi,\alpha}(w)\,dA(w)\right]^{p/2}<\infty,$$

and that ends the proof of Theorem 3.1. □

Remark. In the same way, we can obtain other characterizations for $\mathcal{D}_{\alpha,*}$ by using the ones for $\mathfrak{B}^2_\alpha$ given in [26] and [36]: $C_\varphi\in S_p(\mathfrak{B}^2_\alpha)$ if and only if $N_{\varphi,\alpha+2}(z)/(\log(1/|z|))^{\alpha+2}\in L^{p/2}(\lambda)$, where $d\lambda(z)=(1-|z|^2)^{-2}\,dA(z)$ is the Möbius invariant measure on $\mathbb{D}$, and, when φ has bounded valence and $p\geqslant 2$, if and only if $(1-|z|^2)/(1-|\varphi(z)|^2)\in L^{p(\alpha+2)/2}(\lambda)$. Such a result can be found in [34].

3.2. Applications

We give several applications of the previous theorem.

Corollary 3.2. *Let $-1<\alpha\leqslant\beta$, $p>0$, and φ be a Schur function. Then $C_\varphi\in S_p(\mathcal{D}_{\alpha,*})$ implies that $C_\varphi\in S_p(\mathcal{D}_{\beta,*})$.*

In particular, $C_\varphi\in S_p(\mathcal{D}_*)$ implies that $C_\varphi\in S_p(H^2)$, which in turn implies that $C_\varphi\in S_p(\mathfrak{B}^2)$.

Proof. Assume that $C_\varphi\in S_p(\mathcal{D}_{\alpha,*})$. Then

$$\sum_{n=0}^{\infty}\sum_{j=0}^{2^n-1}\left[2^{n(\alpha+2)}\int\limits_{R_{n,j}} N_{\varphi,\alpha}(w)\,dA(w)\right]^{p/2}<\infty.$$

Since, thanks to Schwarz's Lemma, $N_{\varphi,\beta}(w)\leqslant N_{\varphi,\alpha}(w)(1-|w|^2)^{\beta-\alpha}$, we have

$$N_{\varphi,\beta}(w)\leqslant\left(2.2^{-n}\right)^{\beta-\alpha}N_{\varphi,\alpha}(w)\quad\text{for } w\in R_{n,j}.$$

It follows that

$$\sum_{n=0}^{\infty}\sum_{j=0}^{2^n-1}\left[2^{n(\beta+2)}\int\limits_{R_{n,j}} N_{\varphi,\beta}(w)\,dA(w)\right]^{p/2}<\infty,$$

and that proves Corollary 3.2. □

It is known [16] that composition operators on H^2 separate Schatten classes, but the difficulty is that we must not only control the shape of $\varphi(\partial\mathbb{D})$, but also the parametrization $t\mapsto\varphi(e^{it})$, even if φ is univalent. In the case of the Dirichlet space, this difficulty disappears, because only the areas come into play, and we can easily prove the following result.

Theorem 3.3. *The composition operators on $\mathcal{D}_*$ separate Schatten classes, in the following sense. Let $0<p_1<\infty$. Then, there exists a symbol φ such that:*

$$C_\varphi \in \Bigl(\bigcap_{p>p_1} S_p(\mathcal{D}_*)\Bigr) \setminus S_{p_1}(\mathcal{D}_*).$$

Similarly, there exists a symbol φ such that:

$$C_\varphi \in S_{p_1}(\mathcal{D}_*) \setminus \Bigl(\bigcup_{p<p_1} S_p(\mathcal{D}_*)\Bigr).$$

In particular, for every $0 < p_1 < p_2 < \infty$, there exists φ such that $C_\varphi \in S_{p_2}(\mathcal{D}_*) \setminus S_{p_1}(\mathcal{D}_*)$.

Proof. Let $(h_n)_{n\geqslant 1}$, with $0 < h_n < 1$, be a sequence of real numbers with limit 0 to be adjusted, and J the Jordan curve formed by the segment $[0, 1]$ and the north and (truncated) north-east sides of the curvilinear rectangles

$$\bigl\{1 - 2^{-n} \leqslant |z| < 1 - 2^{-n-1}\bigr\} \times \bigl\{0 \leqslant \arg z < 2^{-n} h_n\bigr\}.$$

Let Ω_0 be the interior of J and $\Omega = \Omega_0 \cup D(0, 1/8)$. Let $\varphi : \mathbb{D} \to \Omega$ be a Riemann map such that $\varphi(0) = 0$. Since φ is univalent and bounded, it defines a symbol on $\mathcal{D}_*$, and the necessary and sufficient condition (3.6) for membership in $S_p(\mathcal{D}_*)$ reads:

$$\sum_{n=0}^{\infty} \bigl[4^n 4^{-n} h_n\bigr]^{p/2} = \sum_{n=0}^{\infty} {h_n}^{p/2} < \infty. \tag{3.7}$$

Indeed, it is clear that, for fixed n, the Hastings–Luecking windows $R_{n,j}$ satisfy:

$$R_{n,0} \cap \Omega \neq \emptyset; \qquad R_{n,j} \cap \Omega = \emptyset \quad \text{for } 1 \leqslant j < 2^n.$$

Therefore, only the Hastings–Luecking windows $R_{n,0}$ matter. Since:

$$A(R_{n,0} \cap \Omega) = \iint_{1-2^{-n} \leqslant r < 1-2^{-n-1},\ 0 \leqslant \theta < 2^{-n} h_n} r\, dr\, d\theta \approx 4^{-n} h_n,$$

we can test the criterion (3.7). Now, it is enough to take $h_n = (n + 1)^{-2/p_1}$ to get:

$$C_\varphi \in \Bigl(\bigcap_{p>p_1} S_p(\mathcal{D}_*)\Bigr) \setminus S_{p_1}(\mathcal{D}_*).$$

Similarly, the choice $h_n = (n + 1)^{-2/p_1} [\log(n + 2)]^{-4/p_1}$, gives a symbol φ such that:

$$C_\varphi \in S_{p_1}(\mathcal{D}_*) \setminus \Bigl(\bigcup_{p<p_1} S_p(\mathcal{D}_*)\Bigr).$$

This ends the proof. □

T. Carroll and C. Cowen [6] proved, but only for $\alpha > 0$, that there exist compact composition operators on $\mathcal{D}_\alpha$ which are in no Schatten class (see also [13]). In the next result, we shall see that this still true for $\alpha = 0$.

Theorem 3.4. *There exists a Schur function φ such that C_φ is compact on $\mathcal{D}_*$, but in no Schatten class $S_p(\mathcal{D}_*)$.*

Proof. It suffices to use the proof of Theorem 3.3 and to take, instead of the above h_n, $h_n = 1/\ln(n+2)$. □

For the next application, which will be used in Section 4, we need to recall the definition of the cusp map χ, introduced in [20], and later used, with a slightly different definition in [24]. Actually, we have to modify it slightly again in order to have $\chi(0) = 0$. We first define:

$$\chi_0(z) = \frac{(\frac{z-i}{iz-1})^{1/2} - i}{-i(\frac{z-i}{iz-1})^{1/2} + 1};$$

we note that $\chi_0(1) = 0$, $\chi(-1) = 1$, $\chi_0(i) = -i$, $\chi_0(-i) = i$, and $\chi_0(0) = \sqrt{2} - 1$. Then we set:

$$\chi_1(z) = \log \chi_0(z), \qquad \chi_2(z) = -\frac{2}{\pi}\chi_1(z) + 1, \qquad \chi_3(z) = \frac{a}{\chi_2(z)},$$

and finally:

$$\chi(z) = 1 - \chi_3(z),$$

where:

$$a = 1 - \frac{2}{\pi}\log(\sqrt{2} - 1) \in (1, 2) \tag{3.8}$$

is chosen in order that $\chi(0) = 0$. The image Ω of the (univalent) cusp map is formed by the intersection of the inside of the disk $D(1 - \frac{a}{2}, \frac{a}{2})$ and the outside of the two disks $D(1 + \frac{ia}{2}, \frac{a}{2})$ and $D(1 - \frac{ia}{2}, \frac{a}{2})$.

Corollary 3.5. *If χ is the cusp map, then C_χ belongs to all Schatten classes $S_p(\mathcal{D}_*)$, $p > 0$.*

Proof. Since χ is univalent, $\chi(0) = 0$, and $\Omega = \chi(\mathbb{D})$ has finite area, we have $\chi \in \mathcal{D}_*$. A little elementary geometry shows that, for some constant C, we have:

$$w \in \Omega, \quad 0 < h < 1 \quad \text{and} \quad |w| \geqslant 1 - h \quad \Longrightarrow \quad |\Im\mathrm{m}\, w| \leqslant Ch^2. \tag{3.9}$$

It follows (changing C if necessary) that $R_{n,j} \cap \Omega$ is contained in a rectangle of sizes 2^{-n} and $C4^{-n}$ and with area $C8^{-n}$. Hence, for a given n, at most C of the Hastings–Luecking windows $R_{n,j}$ can intersect Ω. Therefore, the series in Theorem 3.1 reduces, up to constants, to the series:

$$\sum_{n=0}^{\infty}(4^n 8^{-n})^{p/2} = \sum_{n=0}^{\infty} 2^{-np/2},$$

which converges for every $p > 0$. □

4. Logarithmic capacity and set of contact points

In view of the result of [11] mentioned in the introduction, if $\operatorname{Cap} K > 0$, there is no hope to find a symbol φ such that $E_\varphi = K$ and C_φ is Hilbert–Schmidt on $\mathcal{D}_*$. But as was later proved in [10], $\operatorname{Cap} K > 0$ is the only obstruction. We can improve on the results from [10] as follows: our composition operator is not only Hilbert–Schmidt, but in any Schatten class; moreover, we can replace $E_\varphi = K$ by $E_\varphi = E_\varphi(1) = K$.

Theorem 4.1. *For every compact set K of the unit circle $\mathbb{T}$ with logarithmic capacity* $\operatorname{Cap} K = 0$, *there exists a Schur function φ with the following properties*:

1) $\varphi \in A(\mathbb{D}) \cap \mathcal{D}_* := A$, *the* "*Dirichlet algebra*";
2) $E_\varphi = E_\varphi(1) = K$;
3) $C_\varphi \in \bigcap_{p>0} S_p(\mathcal{D}_*)$.

In fact, the approximation numbers of C_φ satisfy $a_n(C_\varphi) \leqslant a \exp(-b\sqrt{n})$.

This theorem actually results of the particular following case and the properties of the cusp map seen in Section 3.2.

Theorem 4.2. *For every compact set $K \subseteq \partial\mathbb{D}$ of logarithmic capacity* $\operatorname{Cap} K = 0$, *there exists a Schur function $q \in A(\mathbb{D}) \cap \mathcal{D}_*$ which peaks on K and such that the composition operator $C_q : \mathcal{D}_* \to \mathcal{D}_*$ is bounded* (*and even Hilbert–Schmidt*).

Recall that a function $q \in A(\mathbb{D})$, the disk algebra, is said to *peak* on a compact subset $K \subseteq \partial\mathbb{D}$ (and is called a *peaking function*) if:

$$q(z) = 1 \quad \text{if } z \in K; \qquad \left|q(z)\right| < 1 \quad \text{if } z \in \overline{\mathbb{D}} \setminus K. \tag{4.1}$$

Proof of Theorem 4.1. We simply take for φ the composed map $\varphi = \chi \circ q$, where χ is the cusp map and q our peaking function. Recall that $\chi \in A(\mathbb{D})$ and that χ peaks on $\{1\}$. We take advantage of this fact by composing with q, for which $C_q : \mathcal{D}_* \to \mathcal{D}_*$ is bounded as well as C_χ (since χ is univalent). We clearly have $\varphi \in A(\mathbb{D})$, $\varphi(z) = \chi(1) = 1$ for $z \in K$, and $|\varphi(z)| < 1$ for $z \notin K$, since then $|q(z)| < 1$. Therefore $E_\varphi(1) = K$. Moreover, C_φ being bounded on $\mathcal{D}_*$, we have in particular $\varphi = C_\varphi(z) \in \mathcal{D}_*$. Since $C_\varphi = C_q \circ C_\chi$, we get 3), by Corollary 3.5.

In [21], we prove that $a_n(C_\chi) \leqslant a \exp(-b\sqrt{n})$. Since $a_n(C_\varphi) \leqslant \|C_q\| a_n(C_\chi)$, by the ideal property of approximation numbers, this ends the proof of Theorem 4.1. □

In turn, the proof of Theorem 4.2 relies on the following crucial lemma.

Lemma 4.3. *Let $K \subseteq \partial\mathbb{D}$ be a compact set such that* $\operatorname{Cap} K = 0$. *Then, there exists a function $U : \overline{\mathbb{D}} \to \mathbb{R}^+ \cup \{\infty\}$, such that*:

1) $U(z) = \infty$ *if and only if* $z \in K$;
2) $U \geqslant 1$ *on* $\overline{\mathbb{D}}$;
3) U *is continuous on* $\overline{\mathbb{D}} \setminus K$, *harmonic in* $\mathbb{D}$ *and* $\int_{\mathbb{D}} |\nabla U|^2 \, dA < \infty$;
4) $\lim_{z \to K,\, z \in \overline{\mathbb{D}}} U(z) = \infty$;
5) *the conjugate function* $V = \tilde{U}$ *is continuous on* $\overline{\mathbb{D}} \setminus K$.

Proof of Theorem 4.2. Taking this lemma for granted, let us end the proof of the theorem. We set $f = U + iV$, $a = \mathrm{e}^{-1/f(0)}$ and $q = \varphi_a \circ \mathrm{e}^{-1/f}$, where $\varphi_a(z) = \frac{z-a}{1-\bar{a}z}$. In view of the third and fourth items of the lemma, we have $q \in A(\mathbb{D})$. Since $U \geqslant 1$, Lemma 2.4 shows that C_q is Hilbert–Schmidt on $\mathcal{D}_*$. Moreover, for $z \in K$, one has $f(z) = \infty$ and hence $q(z) = 1$ since $\varphi_a(1) = 1$ because $a \in \mathbb{R}$ (since $f(0) = U(0)$). On the other hand, when $z \notin K$, one has $|f(z)| < \infty$ and hence $|q(z)| < 1$. Therefore q peaks on K. □

Proof of Lemma 4.3. This proof is strongly influenced by that of Theorem III, p. 47, in [14], whose construction goes back to L. Carleson [5]. Let:

$$L(z) = \log\left(\frac{\mathrm{e}}{1-z}\right) = P(z) + iQ(z), \tag{4.2}$$

with

$$P(z) = \log \frac{\mathrm{e}}{|1-z|} \quad \text{and} \quad Q(z) = -\arg(1-z), \qquad \bigl|Q(z)\bigr| \leqslant \frac{\pi}{2}, \quad z \in \overline{\mathbb{D}} \setminus \{1\},$$

and write:

$$P(z) \sim \sum_{n \in \mathbb{Z}} \gamma_n z^n,$$

with

$$\gamma_n = 1/\bigl(2|n|\bigr) \quad \text{if } n \neq 0, \quad \text{and} \quad \gamma_0 = 1.$$

For $0 < \varepsilon < 1/2$, let $K_\varepsilon = \{z \in \mathbb{T};\ \operatorname{dist}(z, K) \leqslant \varepsilon\}$, μ_ε its equilibrium measure, and U_ε the logarithmic potential of μ_ε, that is:

$$U_\varepsilon(z) = \int_{K_\varepsilon} \log \frac{\mathrm{e}}{|z-w|} \, d\mu_\varepsilon(w),$$

that we could as well write (since $K_\varepsilon \subseteq \mathbb{T}$):

$$U_\varepsilon(z) = \int_{K_\varepsilon} P(z\bar{w}) \, d\mu_\varepsilon(w).$$

Let us set:

$$f_\varepsilon(z) = \int_{K_\varepsilon} L(z\bar{w}) \, d\mu_\varepsilon(w) = U_\varepsilon(z) + iV_\varepsilon(z), \tag{4.3}$$

with

$$V_\varepsilon(z) = \int_{K_\varepsilon} Q(z\bar{w})\, d\mu_\varepsilon(w).$$

Then, if I_ε is the energy of μ_ε, one has (see [29], Section 4) $I_\varepsilon = 1 + \sum_{n=1}^{\infty} \frac{|\widehat{\mu_\varepsilon}(n)|^2}{n}$, where $\widehat{\mu_\varepsilon}(n) = \int_{\mathbb{T}} \bar{w}^n\, d\mu_\varepsilon(w)$ is the n-th Fourier coefficient of μ_ε, and:

$$f_\varepsilon \in \mathcal{D} \quad \text{and} \quad \|f_\varepsilon\|_{\mathcal{D}}^2 = I_\varepsilon. \tag{4.4}$$

Note that $\|f_\varepsilon\|_{\mathcal{D}} \geqslant 1$.

We claim that there exist $\delta > 0$ and $0 < r < 1$ such that:

$$z \in \overline{\mathbb{D}} \quad \text{and} \quad \operatorname{dist}(z, K) \leqslant \delta \quad \Longrightarrow \quad U_\varepsilon(rz) \geqslant I_\varepsilon/2. \tag{4.5}$$

Indeed, let $P_a(t) = \frac{1-|a|^2}{|e^{it}-a|^2}$ be the Poisson kernel at $a \in \mathbb{D}$. Since U_ε is harmonic in $\mathbb{D}$ and integrable on $\mathbb{T}$ (see [7], Proposition 19.5.2), one has, for every $z \in \mathbb{D}$:

$$U_\varepsilon(z) = \int_{-\pi}^{\pi} U_\varepsilon\big(e^{it}\big) P_z(t) \frac{dt}{2\pi}. \tag{4.6}$$

Let now $\delta \leqslant \varepsilon/4$, to be adjusted later, and take $1-\delta \leqslant r < 1$. Suppose that $\operatorname{dist}(z, K) \leqslant \delta$, with $z \in \overline{\mathbb{D}}$, and let $u \in K$ such that $|z-u| \leqslant \varepsilon/4$. Note that then $|rz-u| \leqslant (1-r) + |z-u| \leqslant \varepsilon/2$. It follows from (4.6) that:

$$I_\varepsilon - U_\varepsilon(rz) = \int_{-\pi}^{\pi} \big[I_\varepsilon - U_\varepsilon\big(e^{it}\big)\big] P_{rz}(t) \frac{dt}{2\pi}$$

(it is useful to recall that $U_\varepsilon(z) \leqslant I_\varepsilon$ for every $z \in \mathbb{C}$). Set:

$$J_1 = \int_{|e^{it}-rz| \leqslant \varepsilon/2} \big[I_\varepsilon - U_\varepsilon\big(e^{it}\big)\big] P_{rz}(t) \frac{dt}{2\pi}$$

and

$$J_2 = \int_{|e^{it}-rz| > \varepsilon/2} \big[I_\varepsilon - U_\varepsilon\big(e^{it}\big)\big] P_{rz}(t) \frac{dt}{2\pi}.$$

For the integral J_1, we have:

$$\big|e^{it} - u\big| \leqslant \big|e^{it} - rz\big| + |rz - u| \leqslant \varepsilon;$$

therefore $e^{it} \in K_\varepsilon$. Since $U_\varepsilon = I_\varepsilon$ Lebesgue-almost everywhere on K_ε, by Frostman's Theorem, we get $J_1 = 0$.

For the integral J_2, we have:

$$P_{rz}(t) \leqslant \frac{2(1 - r|z|)}{(\varepsilon/2)^2} \leqslant 2\frac{(1-r) + r(1-|z|)}{(\varepsilon/2)^2} \leqslant \frac{4\delta}{(\varepsilon/2)^2} = \frac{16\delta}{\varepsilon^2};$$

hence (since $U_\varepsilon(e^{it}) \geqslant 0$):

$$J_2 \leqslant \frac{16\delta}{\varepsilon^2} I_\varepsilon.$$

Therefore, if we choose $0 < \delta \leqslant \varepsilon^2/32$, we get:

$$0 \leqslant I_\varepsilon - U_\varepsilon(rz) \leqslant I_\varepsilon/2,$$

which gives (4.5). □

Now, as $\operatorname{Cap} K = 0$, we know from (1.9) that $\lim_{\varepsilon \to 0^+} I_\varepsilon = \infty$, and we can adjust a sequence $\varepsilon_j \to 0^+$ so that:

$$I_{\varepsilon_j} \geqslant 4j^6. \tag{4.7}$$

Using (4.5), we find two sequences $(\delta_j)_j$ and $(r_j)_j$, with $0 < \delta_j \to 0$ and $1 > r_j \to 1$, such that, for every $j \geqslant 1$,

$$z \in \overline{\mathbb{D}} \quad \text{and} \quad \operatorname{dist}(z, K) \leqslant \delta_j \quad \Longrightarrow \quad U_{\varepsilon_j}(r_j z) \geqslant I_{\varepsilon_j}/2. \tag{4.8}$$

Finally, let us set:

$$f_j(z) = f_{\varepsilon_j}(r_j z) \tag{4.9}$$

and

$$f = U + iV = 1 + \sum_{j=1}^{\infty} j^{-2} \frac{f_j}{\|f_j\|_{\mathcal{D}}}. \tag{4.10}$$

The series defining f is absolutely convergent in $\mathcal{D}$. Note that $f(0)$ is real.

We now have:

1) f is continuous on $\overline{\mathbb{D}} \setminus K$.

Indeed, let $z \in \overline{\mathbb{D}} \setminus K$. Then, $\operatorname{dist}(z, K) > 0$ and there exists a neighborhood ω of z in $\overline{\mathbb{D}}$, an integer $j_0 = j_0(z)$ and a positive number $\delta > 0$ such that:

$$w \in \omega \quad \text{and} \quad j \geqslant j_0 \quad \Longrightarrow \quad \operatorname{dist}(r_j w, K_{\varepsilon_j}) \geqslant \delta.$$

We then have, for $w \in \omega$ and $j \geqslant j_0$:

$$\begin{aligned} \left|f_{\varepsilon_j}(w)\right| &= \left| \int_{K_{\varepsilon_j}} \log \frac{\mathrm{e}}{r_j w - u} \, d\mu_{\varepsilon_j}(u) \right| \\ &\leqslant \int_{K_{\varepsilon_j}} \left(\log \frac{\mathrm{e}}{|r_j w - u|} + \frac{\pi}{2} \right) d\mu_{\varepsilon_j}(u) \leqslant \log \frac{\mathrm{e}}{\delta} + \frac{\pi}{2} := C, \end{aligned}$$

since μ_{ε_j} is a probability measure supported by K_{ε_j}. Therefore, the series defining f is normally convergent on ω since its general term is dominated by $j^{-2}C$ on ω. Since the functions f_j are continuous on $\overline{\mathbb{D}}$, this shows that f is continuous at z.

2) $U(z) := \mathfrak{Re}\, f(z) \geqslant 1$.

This is obvious since, for every $z \in \overline{\mathbb{D}}$,

$$U_\varepsilon(z) := \mathfrak{Re}\, f_\varepsilon(z) = \int_{K_\varepsilon} \log \frac{\mathrm{e}}{|z - u|} \, d\mu_\varepsilon(u) \geqslant 0.$$

3) $\lim_{z \to K, z \in \overline{\mathbb{D}}} U(z) = \infty$.

Indeed, let $A > 0$. Take an integer $j \geqslant A$ and suppose that $\mathrm{dist}(z, K) \leqslant \delta_j$. Then, using the positivity of the U_{ε_k}'s as well as (4.4), (4.7) and (4.8), we have:

$$U(z) \geqslant j^{-2} \frac{U_{\varepsilon_j}(r_j z)}{\|f_{\varepsilon_j}\|_{\mathcal{D}}} \geqslant j^{-2} \frac{I_{\varepsilon_j}/2}{\sqrt{I_{\varepsilon_j}}} \geqslant j \geqslant A.$$

This ends the proof of our claims, and of Lemma 4.3. □

To end this paper, let us mention the following version of the classical Rudin–Carleson Theorem. Though it is not the main subject of this paper, it has the same flavor as Theorem 4.2. We do not give a proof, but only mention that it can be obtained by mixing the proofs of Theorems III.E.2 and III.E.6 in [33] (see pp. 181–187).

Theorem 4.4. *Let K be a compact subset of $\mathbb{T}$ with* $\mathrm{Cap}\, K = 0$. *Given any continuous strictly positive function $s \in C(\mathbb{T})$ equal to* 1 *on K, we can find, for every $h \in C(K)$ and every $\varepsilon > 0$, a function $f \in A(\mathbb{D}) \cap \mathcal{D}$ such that $f_{|K} = h$ and*:

$$\left|f(\theta)\right| \leqslant (1 + \varepsilon)\|h\|_\infty s(\theta), \quad \forall \theta \in \mathbb{T}; \qquad \|f\|_{\mathcal{D}} \leqslant (1 + \varepsilon)\|h\|_\infty.$$

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