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Bull. Sci. math. 128 (2004) 789–801

BULLETIN DES SCIENCES MATHEMATIQUES

www.elsevier.com/locate/bulsci

Some new rich subspaces of C. Applications

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Received 26 April 2004; accepted 10 May 2004

Available online 2 August 2004

Abstract

In this paper, we are interested in a class of subspaces of C, introduced by Bourgain [Studia Math. 77 (1984) 245–253]. Wojtaszczyk called them rich in his monograph [Banach Spaces for Analysts, Cambridge Univ. Press, 1991]. We give some new examples of such spaces: this allows us to recover previous results of Godefroy–Saab and Kysliakov on spaces with reflexive annihilator in a very simple way. We construct some other examples of rich spaces, hence having property (V) of Pełczyński and Dunford–Pettis property. We also recover the results due to Bourgain and Saccone saying that spaces of uniformly convergent Fourier series share these properties, by only using the main result of [Studia Math. 77 (1984) 245–253] and some very elementary arguments. We generalize too these results.

MSC: 42A55; 43A46; 46B20; 46E15

Keywords: Rich spaces; Property (V) of Pełczyński; Dunford–Pettis property; Reflexive annihilator; Lacunary sets; Spaces of uniformly convergent Fourier series

Let $C(S)$ be the space of continuous functions over a compact S. We are interested in a class of subspaces of $C(S)$ that appeared in the work of Bourgain [2]. In his monograph [17], Wojtaszczyk emphasizes the general principle contained in the paper of Bourgain and call this class of spaces “rich spaces” (be cautious with this terminology: there are different notions of rich space in analysis: this one is not linked a priori with the Daugavet property). More precisely, we have the following definition.

Definition 0.1. Let X be a subspace of $C(S)$. The space X is said to be a rich subspace of $C(S)$ if there exists a probability measure ν on S such that for every $h \in C(S)$ and every

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0007-4497/$ – see front matter
doi:10.1016/j.bulsci.2004.05.003

sequence (x_n) in the unit ball of X such that $\|x_n\|_{L^1(\nu)}$ converges to 0, the distance between hx_n and X tends to zero.

In [2] and [17], the following examples are given: $C(S)$, the ball algebra $A(\mathbb{B}_d)$ (where $\mathbb{B}_d$ is the unit ball of $\mathbb{C}^d$) as subspace of $C(\mathbb{S}^d)$ (where $\mathbb{S}_d$ is the unit sphere of $\mathbb{C}^d$). As a special case, this includes the disk algebra $A(\mathbb{D})$ viewed as a subspace of $C(\mathbb{T})$.

We are interested in giving some new examples and some applications.

This class of spaces appeared in order to produce some new examples of spaces having the Dunford–Pettis property. This is a consequence of the general following result [17].

Let X be a rich subspace of $C(S)$ and K a bounded subset of the dual space X^*. Then the following conditions are equivalent.

(i) K is not relatively weakly compact.
(ii) There exists a sequence in K equivalent to the unit vector basis of ℓ^1.
(iii) There exists a weakly unconditionaly series $\sum x_n$ in X such that $\inf_n \sup\{|k(x_n)|;\ k \in K\} > 0$.
(iv) There exists a weakly null sequence (x_n) in X such that $\inf_n \sup\{|k(x_n)|;\ k \in K\} > 0$.
(v) There exists $C > 0$ such that for every n, K contains $x_1^{(n)}, \dots, x_n^{(n)}$ such that the Banach–Mazur distance between span$(x_1^{(n)}, \dots, x_n^{(n)})$ and ℓ_n^1 is less than C.

With this result, it is trivial to see that if X is a rich subspace of $C(S)$, then X has the property (V) of Pełczyński and Dunford–Pettis property. Actually, X^* has the Dunford–Pettis property as well (see [2], or [17] ex.III.D.22). Let us also recall the two following definitions (see [4] for more information on these notions).

Definition 0.2. Let X be a Banach space. X has the Dunford–Pettis property if for every weakly null sequence (x_n) in X and every weakly null sequence (x_n^*) in X^*, then $x_n^*(x_n)$ tends to zero.

Equivalently, for every Banach space Y and every operator $T : X \to Y$ which is weakly compact, T maps a weakly Cauchy sequence in X into a norm Cauchy sequence.

Definition 0.3. Let X be a Banach space. X has the property (V) of Pełczyński if, for every non-relatively weakly compact bounded set $K \subset X^*$, there exists a weakly unconditionaly series $\sum x_n$ in X such that $\inf_n \sup\{|k(x_n)|;\ k \in K\} > 0$.

Equivalently, for every Banach space Y and every operator $T : X \to Y$ which is not weakly compact, there exists a subspace X_o of X isomorphic to c_o such that $T_{|X_o}$ is an isomorphic embedding.

We are going to give some examples in the setting of harmonic analysis: let G be an infinite metrizable compact abelian group, equipped with its normalized Haar measure dx, and Γ its dual group (discrete and countable). For example, when G is the unit circle of the complex plane, then Γ will be identified with $\mathbb{Z}$ by $p \mapsto e_p$, where $e_p(x) = e^{2i\pi px}$. The space of complex regular Borel measures over G, equipped with the norm of total variation will be denoted by $M(G)$. If $\mu \in M(G)$, its Fourier transform at the point γ is defined by $\hat{\mu}(\gamma) = \int_G \gamma(-x)\,d\mu(x)$. As usual $C(G)$ is the space of continuous functions on G equipped with the supremum norm and $\mathcal{P}(G)$ is the space of trigonometric polynomials.

For $B \subset M(G)$ and $\Lambda \subset \Gamma$, set:

$$B_\Lambda = \big\{ f \in B \mid \forall \gamma \notin \Lambda,\ \hat{f}(\gamma) = 0 \big\}.$$

B_Λ is the set of elements of B whose spectrum is contained in Λ.

We already mentioned that the disk algebra is a rich subspace of $C(\mathbb{T})$. The argument given in [17] and [2] uses the Cauchy kernel. We give here an alternative elementary argument which only uses the fact that the Fourier coefficients with negative index vanish ($A(\mathbb{D})$ naturally identifies with $C_{\mathbb{N}}(\mathbb{T})$). We need the following elementary remark, that we shall use too in Section 3.

Lemma 0.4. *Let X be a subspace of $C(S)$ and let $D \subset C(S)$ span a dense subspace of $C(S)$. We suppose that there exists a probability measure ν on S with the following property: for every $h \in D$ and every sequence (x_n) in the unit ball of X such that $\|x_n\|_{L^1(\nu)}$ converges to 0, the distance between hx_n and X tends to zero.*

Then X is a rich subspace of $C(S)$.

The disk algebra is a rich subspace of $C(\mathbb{T})$: we use the preceding lemma with $D = \{e_p \mid p \in \mathbb{Z}\}$. The probability measure ν is the Haar measure on the torus. Fixing $p \in \mathbb{Z}$ and a sequence (x_n) in the unit ball of $A(\mathbb{D})$ such that $\|x_n\|_{L^1(\nu)}$ converges to 0, we have: $e_p x_n \in A(\mathbb{D})$ if $p \geqslant 0$ so that $\|e_p x_n\|_{C/A} = 0$. If $p < 0$, $x_n = a_n + b_n$ where $a_n = \sum_{k=0}^{-p} \hat{x}_n(k) e_k$. We have then $e_p b_n \in A(\mathbb{D})$ and $\|e_p a_n\|_{C/A} \leqslant \sum_{k=0}^{-p} |\hat{x}_n(k)|$, which converges to zero since $\|x_n\|_1$ does.

This argument will be used again in Section 3 to obtain in an elementary way the fact that the space U_+ of analytic uniformly convergent Fourier series is rich.

1. Spaces with reflexive annihilator

Theorem 1.1. *Every subspace of $C(S)$ with reflexive annihilator is rich.*

Proof. Let X be a subspace of $C(S)$ with reflexive annihilator. We first use a theorem of Rosenthal (see [12] or [5] Theorem 15.11, p. 315): as the dual of $C(S)/X$ is $X^\perp$, hence reflexive, $C(S)/X$ is itself reflexive, so that there exists a probability measure σ over S such that $C(S)/X$ appears in a natural fashion as a quotient of $L^r(\sigma)$ (for some finite $r \geqslant 2$). Equivalently, $X^\perp$ is isomorphic to a subspace of $L^s(\sigma)$ (with $s \in]1, 2]$).

Now, let $(x_n)_n$ be sequence of norm 1 elements of X, such that $\|x_n\|_{L^1(\sigma)}$ tends to zero. Then, for every $h \in C$, we have

$$\inf\big\{\|hx_n + \delta\|;\ \delta \in X\big\} = \|hx_n\|_{C(S)/X} = \sup\big\{|\zeta(hx_n)|;\ \zeta \in X^\perp, \|\zeta\| = 1\big\}.$$

Now, there exists $c > 0$ such that for every $\zeta \in X^\perp$, there is a function $\varphi_\zeta \in L^s(\sigma)$ with $\|\varphi_\zeta\|_s \leqslant c\|\zeta\|$ and $\zeta(f) = \int_S \varphi_\zeta f \, d\sigma$, for any $f \in C(S)$. We then have, via Hölder's inequality, for every functional $\zeta \in X^\perp$ with norm 1,

$$\left|\zeta(hx_n)\right| \leqslant \int_S |\varphi_\zeta h x_n| \, d\sigma \leqslant \|\varphi_\zeta\|_s.\|h\|_\infty.\|x_n\|_r \leqslant c\|h\|_\infty.\|x_n\|_1^\theta.\|x_n\|_\infty^{1-\theta}$$

where $\theta = \frac{1}{r} \in]0, 1[$.

So that $\inf\{\|hx_n + \delta\|;\ \delta \in X\} \leqslant c\|h\|_\infty.\|x_n\|_1^\theta$, which tends to zero. □

In the setting of harmonic analysis, this is linked to the following notion.

Definition 1.2. Let $0 < p < \infty$ and let A be a subset of Γ. A is a $\Lambda(p)$ set if there exists $q \in]0, p[$ such that $L_A^p(G) = L_A^q(G)$.

This implies that for all $r \in]0, p[$: $L_A^p(G) = L_A^r(G)$.

Remark. For $p > 1$, this is equivalent to saying that every measure $\mu \in M_A(G)$ actually lies in $L_A^p(G)$.

As a particular case of the preceding theorem, we have

Corollary 1.3. *Let E be a $\Lambda(1)$ subset of Γ. Then $C_{E^c}(G)$ is a rich subspace of $C(G)$. Moreover, the measure ν can be chosen to be the Haar measure over G.*

Proof. It is a direct consequence of the preceding proof and of the fact that if E is a $\Lambda(1)$ set, then $M_E(G)$ is isomorphic to a subspace of $L^s(G, dx)$ for some $s \in]1, 2]$. □

With the results recalled in the introduction, we recover

Corollary 1.4 [7]. *Every subspace of $C(S)$ with reflexive annihilator has the property (V) of Pełczyński.*

Corollary 1.5. *Every subspace of $C(S)$ with reflexive annihilator, and its dual, have both the Dunford–Pettis property.*

This last result implies a simple proof of the following result of Kysliakov.

Corollary 1.6 [9]. *Let Y be a reflexive subspace of $L^1(\mu)$. Then $L^1(\mu)/Y$ has the Dunford–Pettis property.*

Proof. First, Y can be viewed as a reflexive subspace of $M(S)$. Moreover, we can write

$$M(S)/Y = L^1(\mu)/Y \oplus_1 M_{\text{sing}}(S)$$

where $M_{\text{sing}}(S)$ is the space of singular measure, with respect to μ.

Now, let T be weakly compact operator from $L^1(\mu)/Y$ to some Banach space Z and (f_j) a sequence weakly convergent to zero in $L^1(\mu)/Y$. We first observe that (f_j) also weakly convergent to zero in $M(S)/Y$; indeed, $(M(S)/Y)^*$ is a subset of $(L^1(\mu)/Y)^*$.

We consider the operator $\tilde{T}$ from $M(S)/Y$ to Z defined in the obvious way by $\tilde{T}(h+m) = T(h)$, where $h \in L^1(\mu)/Y$ and $m \in M_{\text{sing}}(S)$. We easily check that $\tilde{T}$ is weakly compact. As $M(S)/Y$ has the Dunford–Pettis property, we deduce that $\tilde{T}$ is completely continuous (i.e., maps weakly Cauchy sequence to norm Cauchy sequence) and that $\tilde{T}(f_j) = T(f_j)$ is norm convergent to zero. □

The proof of Theorem 1.1 can actually be extended to more general Orlicz spaces, which interpolate. For instance, we have the following result with the functions: $\psi_q(x) = \exp(x^q) - 1$, where $q > 1$ and $\varphi_p(x) = x(\ln x)^{1/p}$, where $p^{-1} + q^{-1} = 1$. The spaces L^{ψ_q} and L^{φ_p} are in duality with $\langle f, g\rangle = \int fg$. This is actually a special case of Theorem 1.1 (as $X^\perp$ is a fortiori reflexive) but the point is that we do not require the Rosenthal theorem in the proof.

Theorem 1.7. *Let X be a closed subspace of $C(S)$ such that $X^\perp \subset L^{\varphi_p}(S, \sigma)$, for some probability measure σ. Then X is rich.*

Proof. The proof is exactly the same (with the corresponding interpolation inequality) and we leave the details to the reader. Of course, X is rich with associated measure σ. Note that if a sequence (x_n) of functions in the unit ball of $C(S)$ converges almost everywhere to 0 then $\|x_n\|_{\psi_q}$ tends to zero. □

2. Examples in harmonic analysis. Applications

We begin with a very easy lemma, which leads to the construction of some new examples from old ones.

Lemma 2.1. *Let X_1 and X_2 be two rich subspaces of $C(S)$. We suppose that $(X_1 \cap X_2)^\perp$ is the direct sum of $X_1^\perp$ and $X_2^\perp$.*

Then $X_1 \cap X_2$ is a rich subspace of $C(S)$.

Moreover, the measure ν can be chosen as $(\nu_1 + \nu_2)/2$, where ν_i is associated to X_i.

Proof. There exist two probability measures ν_1 and ν_2 over S verifying: for $i \in \{1, 2\}$, for every $h \in C(S)$ and every sequence (x_n) in the unit ball of X_i such that $\|x_n\|_{L^1(\nu_i)}$ converges to 0, $\|hx_n\|_{C(S)/X_i}$ converges to 0.

By our assumption, there exists $C > 0$ such that every $\mu \in (X_1 \cap X_2)^\perp$ admits a decomposition:

$$\mu = \mu_1 + \mu_2 \quad \text{where } \mu_1 \in X_1^\perp,\ \mu_2 \in X_2^\perp \text{ with } \max\bigl(\|\mu_1\|, \|\mu_2\|\bigr) \leqslant C\|\mu\|.$$

Now, set $\nu = \frac{1}{2}(\nu_1 + \nu_2)$. This is a probability measure over S. We fix $h \in C(S)$ and a sequence (x_n) in the unit ball of $X_1 \cap X_2$ such that $\|x_n\|_{L^1(\nu)}$ converges to 0.

We have

$$\begin{aligned}\|hx_n\|_{C(S)/(X_1\cap X_2)} &= \sup_{\substack{\mu\in(X_1\cap X_2)^\perp\\ \|\mu\|\leqslant 1}} \left|\mu(hx_n)\right| \\ &\leqslant \sup_{\substack{\mu\in(X_1\cap X_2)^\perp\\ \|\mu\|\leqslant 1}} \left(\left|\mu_1(hx_n)\right| + \left|\mu_2(hx_n)\right|\right) \\ &\leqslant C\|hx_n\|_{C(S)/X_1} + C\|hx_n\|_{C(S)/X_2}.\end{aligned}$$

As $2\nu \geqslant \nu_i$, the sequence $\|x_n\|_{L^1(\nu_i)}$ tends to zero for each $i \in \{1,2\}$. We conclude that both $\|hx_n\|_{C(S)/X_1}$ and $\|hx_n\|_{C(S)/X_2}$ tend to zero (as X_2 and X_1 are rich). The result follows. □

Now, we give, in the setting of harmonic analysis on the torus, an example where the assumption of the preceding lemma is fulfilled.

Lemma 2.2. *Let $E \subset \mathbb{N}^*$ be a $\Lambda(1)$ subset of $\mathbb{Z}$. Then*

$$M_{\mathbb{Z}^-\cup E} = M_{\mathbb{Z}^-} \oplus M_E.$$

Proof. It is easy via the Kolmogorov theorem and contained in [15]. □

We deduce from the preceding results a new class of rich sets, hence obtaining some new spaces having both property (V) of Pełczyński and Dunford–Pettis property.

Theorem 2.3. *Let $E \subset \mathbb{N}$ be a $\Lambda(1)$ subset of $\mathbb{Z}$.*

Then the space $C_{\mathbb{N}\setminus E}(\mathbb{T})$ is a rich subspace of $C(\mathbb{T})$. Moreover, the measure ν can be chosen to be the Haar measure over $\mathbb{T}$.

Proof. Apply the two preceding lemmas and the fact that both the disk algebra and $C_{E^c}(\mathbb{T})$ (see Section 1) are rich subspaces of $C(\mathbb{T})$. The claim about the choice of the measure ν is proved by the fact that for both the disk algebra and $C_{E^c}(\mathbb{T})$, the measure can be chosen as the Haar measure over $\mathbb{T}$. □

We immediately obtain:

Corollary 2.4. *Let $E \subset \mathbb{N}$ be a $\Lambda(1)$ subset of $\mathbb{Z}$.*

Then $C_{\mathbb{N}\setminus E}(\mathbb{T})$ has the property (V) of Pełczyński and $M(\mathbb{T})/M_{\mathbb{Z}^-\cup E}(\mathbb{T})$ has the Dunford–Pettis property.

From this corollary, we are able to slightly improve a previous result of the author [10, Corollary 6.11] about the union of Riesz sets (we obtained the following result for $\Lambda \subset \mathbb{N}$ such that $L^\infty_\Lambda(\mathbb{T})$ does not contain an isomorphic copy of c_0). We recall that the following problem is open: "assume that $C_\Lambda(\mathbb{T})$ does not contain an isomorphic copy of c_0. Does it imply that $L^\infty_\Lambda(\mathbb{T})$ neither does?". On an other hand, the following result is interesting because it is known (see [11]) that there are $\Lambda(1)$ subsets E of $\mathbb{Z}$ such that $C_E(\mathbb{T})$ contains an isomorphic copy of c_0.

The result deals with the following notion.

Definition 2.5. A subset Λ of Γ is a Riesz set if $M_\Lambda(G) = L^1_\Lambda(G)$. More precisely, this means that, given any μ in $M_\Lambda(G)$, there is some h in $L^1(G)$ such that, for all γ in Γ, $\hat{h}(\gamma) = \hat{\mu}(\gamma)$.

More about Riesz sets can be found in [6].

Corollary 2.6. *Let $E \subset \mathbb{N}$ be a $\Lambda(1)$ subset of $\mathbb{Z}$ and $\Lambda \subset \mathbb{N}$ such that $C_\Lambda(\mathbb{T})$ does not contain an isomorphic copy of c_0.*

Then $\mathbb{Z}^- \cup E \cup \Lambda$ is a Riesz subset of $\mathbb{Z}$.

Proof. We use a general principle (see [10, Theorem 6.4]). We reproduce here quickly the argument. Let $\mu \in M_{\mathbb{Z}^- \cup E \cup \Lambda}(\mathbb{T})$. We consider the convolution operator T_μ by μ from $C_{\mathbb{N}\setminus E}(\mathbb{T})$ to $C_\Lambda(\mathbb{T})$. It is bounded by $\|\mu\|$. It is a weakly compact operator because otherwise: (as $C_{\mathbb{N}\setminus E}(\mathbb{T})$ has the property (V) of Pełczyński) T_μ would induce an isomorphism from a copy of c_0, contained in $C_{\mathbb{N}\setminus E}(\mathbb{T})$ onto a copy of c_0, contained in $C_\Lambda(\mathbb{T})$. By assumption, this is impossible.

Now from the weak compactness of T_μ, we obtain that T_μ is bounded from $L^\infty_{\mathbb{N}\setminus E}(\mathbb{T})$ to $C_\Lambda(\mathbb{T})$. As $\mathbb{Z}^- \cup E$ is a Riesz set (see [15]), an application of a theorem of Heard [8] gives the existence of $h \in L^1(\mathbb{T})$ such that $\hat{h} = \hat{\mu}$ on $\mathbb{N} \setminus E$. We then conclude that $\mu - h \in M_{\mathbb{N}\setminus E}(\mathbb{T}) = L^1_{\mathbb{N}\setminus E}(\mathbb{T})$. Hence, $\mu = h + (\mu - h) \in L^1(\mathbb{T})$. □

Using the techniques of Corollary 1.5, we obtain the following new example of a space with the Dunford–Pettis property.

Corollary 2.7. *Let $E \subset \mathbb{N}$ be a $\Lambda(1)$ subset of $\mathbb{Z}$.*

Then $L^1(\mathbb{T})/L^1_{\mathbb{Z}^- \cup E}(\mathbb{T})$ has the Dunford–Pettis property.

Proof. It suffices to reproduce the argument given in the proof of Corollary 1.6. The only point is the fact that

$$M(\mathbb{T})/M_{\mathbb{Z}^- \cup E}(\mathbb{T}) = L^1(\mathbb{T})/L^1_{\mathbb{Z}^- \cup E}(\mathbb{T}) \oplus_1 M_{\text{sing}}(\mathbb{T}).$$

This is due to the fact that $\mathbb{Z}^- \cup E$ is Riesz (and even Shapiro) (see [6]). □

3. Spaces of uniformly convergent Fourier series

We are going to show that the classical (and some less classical) spaces of uniformly convergent Fourier series are rich spaces. Actually, it turns out that the consequences of some of these results were already contained in [3] (via a result of Vinogradov, relying on the Carleson theorem), [13] (via some results on tight spaces) for the space U_+ and [14] for the space U. Nevertheless, we give here different arguments: we only show (in a very elementary manner) that these spaces are rich so that the geometric properties of these spaces ((V), DP, ...) are nothing but a direct and simple consequence of the strong general result of Bourgain recalled at the beginning of the paper [2,17].

After this, we generalize these results by removing a $\Lambda(2)$ set from the spectrum of U and U_+.

We recall that U denotes the space of uniformly convergent Fourier series on the torus, normed by $\|f\|_U = \sup_{n\in\mathbb{N}} \|\sum_{k=-n}^{n} \hat{f}(k)e_k\|_\infty$. More generally, for every $\Lambda \subset \mathbb{Z}$, we denote by $U_\Lambda = \{f \in U \mid \forall n \notin \Lambda,\ \hat{f}(n) = 0\}$. The space U_+ of analytic uniformly convergent Fourier series naturally identifies with $U_{\mathbb{N}}$. We recall the classical fact that the spaces U_Λ (or more generally the closed subspaces of U) are naturally isometrically embedded in a $C(K)$ space: to see this, writing $K_0 = \{0\} \cup \{1/n \mid n \in \mathbb{N}^*\}$ and $K = \mathbb{T} \times K_0$, we map $f \in U_\Lambda$ to $F_f(x,s)$ with $F_f(.,0) = f$ and $F_f(.,1/n) = \sum_{k=-n}^{n} \hat{f}(k)e_k$.

We shall use the following general principle.

Proposition 3.1. *Let X be a (closed) subspace of U. We suppose that for every $p \in \mathbb{Z}$ and every sequence $(x_n)_{n\in\mathbb{N}}$ in the unit ball of X such that $\|x_n\|_{L^1(\mathbb{T},dx)}$ converges to 0, we have that $\|e_p x_n\|_{U/X}$ converges to 0.*

Then X is a rich subspace of $C(K)$.

Proof. First, we notice that $D = \{e_p \otimes g \mid p \in \mathbb{Z},\ g \in C(K_0)\}$ spans the space $\mathcal{P}(\mathbb{T}) \otimes C(K_0)$, which is dense in $C(\mathbb{T}) \otimes C(K_0)$, which is itself dense in $C(K)$. So, by Lemma 0.4, it is sufficient to consider the case $h = e_p \otimes g$, for $p \in \mathbb{Z}$ and $g \in C(K_0)$. The probability measure ν will be simply $dx \otimes \delta_0$: note that, for $f \in U$, viewed too as a function of $C(\mathbb{T})$, we have $\|f\|_{L^1(K,\nu)} = \|f\|_{L^1(\mathbb{T},dx)}$. Let us check that this choice of measure works for X.

Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in the unit ball of X such that $\|x_n\|_{L^1(\nu)}$ converges to 0. Then we first notice that $e_p x_n \in U$ so that

$$\|g e_p x_n\|_{C(K)/X} \leqslant \big\|\big(g - g(0)\big) \otimes (e_p x_n)\big\|_{C(K)} + \big|g(0)\big|.\|e_p x_n\|_{U/X}.$$

Now fixing $\varepsilon > 0$, we have by continuity of g an m_0 such that for every $m > m_0$: $|g(1/m) - g(0)| \leqslant \varepsilon$. We have then

$$\begin{aligned}\big\|\big(g - g(0)\big)e_p x_n\big\|_{C(K)} &= \sup_{m,t}\big|\big(g(1/m) - g(0)\big)S_m(e_p x_n)(t)\big| \\ &\leqslant \max\Big(\sup_{m\leqslant m_0} 2\|g\|_\infty \big\|S_m(e_p x_n)\big\|_\infty, \sup_{m>m_0} \varepsilon\|e_p x_n\|_U\Big) \\ &\leqslant \max\left(2\|g\|_\infty \sup_{m\leqslant m_0} \sum_{k=-m-p}^{m+p} \big|\hat{x}_n(k)\big|, \varepsilon\big(2|p| + 1 + \|x_n\|_X\big)\right) \\ &\leqslant \max\Big(2\|g\|_\infty(2m_0+1) \max_{|k|\leqslant m_0+|p|} \big|\hat{x}_n(k)\big|, \varepsilon\big(2|p|+2\big)\Big).\end{aligned}$$

The hypothesis on the x_n exactly means that $\|x_n\|_{L^1(dx)} \to 0$ and this implies that for any $k \in \mathbb{Z}$, $\hat{x}_n(k) \to 0$. We then have for n sufficiently large

$$\big\|\big(g - g(0)\big)e_p x_n\big\|_{C(K)/X} \leqslant 2\varepsilon \max\big(\|g\|_\infty, |p| + 1\big).$$

On the other hand, the term $\|e_p x_n\|_{U/X}$ converges to 0 by hypothesis.

We conclude that for n sufficiently large $\|g e_p x_n\|_{C(K)/X} \leqslant \varepsilon + 2\varepsilon \max(\|g\|_\infty, |p| + 1)$. □

Now, we can state the following theorem and its immediate corollary.

Theorem 3.2. *The spaces U and U_+ are rich subspaces of $C(K)$.*

Corollary 3.3 [3,13,14]. *The spaces U and U_+ have properties (V) and Dunford–Pettis; their duals too have DP.*

Proof of Theorem 3.2. The case $X = U$ is a trivial consequence of the previous proposition.

Now, for $X = U_+$, we mix the preceding argument with the one given for the disk algebra. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in the unit ball of U_+ such that $\|x_n\|_{L^1(\nu)}$ converges to 0 and let us fix $p \in \mathbb{Z}$. We separate two cases: if $p \geqslant 0$ then $e_p x_n \in U_+$ so that $\|e_p x_n\|_{U/U_+} = 0$.

If $p < 0$, then we write $x_n = a_n + b_n$ where $a_n = \sum_{k=0}^{-p} \hat{x}_n(k)e_k$. We obviously have that $e_p b_n \in U_+$ (i.e., $\|e_p b_n\|_{U/U_+} = 0$) and $\|e_p a_n\|_{U/U_+} \leqslant \sum_{k=0}^{-p} |\hat{x}_n(k)|$, which converges to zero since $\|x_n\|_1$ does. The previous proposition gives the result. □

Now, we are going to construct some new examples, mixing the preceding results and the ones of Section 1.

Theorem 3.4. *Let E be a $\Lambda(2)$ subset of $\mathbb{Z}$. Then the space U_{E^c} is a rich subspace of $C(K)$.*

Let E be a $\Lambda(2)$ subset of $\mathbb{N}$. Then the space $U_{\mathbb{N}\setminus E}$ is a rich subspace of $C(K)$.

As an immediate corollary, we obtain that

Corollary 3.5. *Let E be a $\Lambda(2)$ subset of $\mathbb{Z}$. Then the space U_{E^c} has the property (V) and its dual has Dunford–Pettis property.*

Let E be a $\Lambda(2)$ subset of $\mathbb{N}$. Then the space $U_{\mathbb{N}\setminus E}$ has property (V) and its dual has the Dunford–Pettis property.

Proof of Theorem 3.4. First, we recall the following fact due to Vinogradov (Th IV [16]): let E be a $\Lambda(2)$ subset of $\mathbb{N}$: then there exists $C > 0$ such that for every $a \in \ell^2(E)$, there is $f \in U_+$ such that $\hat{f} = a$ on E and $\|f\|_U \leqslant C\|a\|_2$. This is the key ingredient for the proof. Note that this implies the same property for a $\Lambda(2)$ subset E of $\mathbb{Z}$ and U.

Let us prove the first point of the theorem. We claim that $(U_{E^c})^\perp$ (in U^*) is equal to $M_E(\mathbb{T})$ with equivalence of norms. Indeed, let P be a trigonometric polynomial. With the previous result of Vinogradov, we have $C > 0$ (depending on E only) and $f \in U$ such that $\hat{f} = \hat{P}$ on E and $\|f\|_U \leqslant C\|\hat{P}_{|E}\|_2 \leqslant C\|P\|_\infty$. For every $\alpha \in (U_{E^c})^\perp$ (with norm 1 say), we have

$$\left|\alpha(P)\right| = \left|\alpha(f)\right| \leqslant \|\alpha\|.\|f\|_U \leqslant C\|P\|_\infty.$$

By density, α defines a continuous linear functional on $C(\mathbb{T})$, with norm less than C, which is zero on E^c. This proves the claim.

The claim is equivalent to $\|f\|_{U/U_{E^c}} \leqslant C\|f\|_{C/C_{E^c}}$ for every $f \in U$. Corollary 1.3 (a $\Lambda(2)$ set is a fortiori a $\Lambda(1)$ set) and Proposition 3.1 give the result.

Let us prove the second point of the theorem. Fix $\alpha \in (U_{\mathbb{N}\setminus E})^{\perp}$ with norm 1. We claim that there is a decomposition $\alpha = a + b$, with $a \in (U_+)^{\perp}$, $b \in M_{\mathbb{Z}^- \cup E}(\mathbb{T})$, where $\|a\|_{U^*} \leqslant C + 1$ and $\|b\|_M \leqslant C$ (the constant C is the one given by the result of Vinogradov). Indeed, let P be an analytic trigonometric polynomial. We have $|\alpha(P)| \leqslant C\|P\|_\infty$ (using the same trick as in the proof of the first point) so that α defines a continuous linear functional on the disk algebra, with norm less than C, vanishing on $C_{\mathbb{N}\setminus E}(\mathbb{T})$. Then, via Hahn–Banach and the Riesz representation theorem, we obtain $b \in M_{\mathbb{Z}^- \cup E}(\mathbb{T})$, with norm less than C, which coincides with α on E. Writing $a = \alpha - b$, it is easy to check that $a \in (U_+)^{\perp}$ and that $\|a\|_{U^*} \leqslant C + 1$. This proves the claim.

Now, we use Proposition 3.1: we fix $p \in \mathbb{Z}$ and a sequence $(x_n)_{n\in\mathbb{N}}$ in the unit ball of $U_{\mathbb{N}\setminus E}$ such that $\|x_n\|_{L^1(\mathbb{T},dx)}$ converges to 0. We compute

$$\begin{aligned}
\|e_p x_n\|_{U/U_{\mathbb{N}\setminus E}} &= \sup_{\substack{\alpha \in (U_{\mathbb{N}\setminus E})^{\perp} \\ \|\alpha\|=1}} \left|\alpha(e_p x_n)\right| \\
&\leqslant \sup_{\substack{a \in (U_+)^{\perp} \\ \|a\| \leqslant C+1}} \left|a(e_p x_n)\right| + \sup_{\substack{b \in M_{\mathbb{Z}^- \cup E} \\ \|b\| \leqslant C}} \left|b(e_p x_n)\right| \\
&\leqslant (C+1)\|e_p x_n\|_{U/U_+} + C\|e_p x_n\|_{C/C_{\mathbb{N}\setminus E}}.
\end{aligned}$$

Using Theorem 3.2 (note that $x_n \in U_{\mathbb{N}\setminus E} \subset U_+$) and Theorem 2.3, this ends the proof (it is worth pointing out that in both Theorem 2.3 and 3.2, we always work with the Haar measure over the torus). □

These techniques can also be applied for the space of asymmetric uniformly convergent Fourier series:

$$U_{\text{as}} = \left\{ f \in C(\mathbb{T}) \mid \lim_{n,m \to +\infty} \sum_{k=-n}^{m} \hat{f}(k) e_k = f \text{ in } C(\mathbb{T}) \right\}$$

equipped with the norm $\|f\|_{U_{\text{as}}} = \sup_{n,m\in\mathbb{N}} \|\sum_{k=-n}^{m} \hat{f}(k) e_k\|_\infty$.

This space is isometrically isomorphic to a subspace of $C(\mathbb{T} \times K_0 \times K_0)$, mapping f to F_f, where

$$F_f(t, 1/n, 1/m) = \sum_{k=-n}^{m} \hat{f}(k) e_k(t), \qquad F_f(t, 1/n, 0) = \sum_{k=-n}^{+\infty} \hat{f}(k) e_k(t),$$

$$F_f(t, 0, 1/m) = \sum_{k=-\infty}^{m} \hat{f}(k) e_k(t)$$

and $F_f(t, 0, 0) = f(t)$.

This is along the same lines as for the case of U to prove that U_{as} is a rich subspace of $C(\mathbb{T} \times K_0 \times K_0)$ and we leave the details to the reader. Of course, results corresponding to Proposition 3.1 and Theorem 3.4 could be proved.

4. General remarks on rich spaces

First we are going to explain why the choice of the measure σ in the previous examples was not made at random. We begin with a reformulation of the definition.

Proposition 4.1. *Let X be a subspace of $C(S)$. Let σ be a probability measure on S.*

Then X is a rich subspace of $C(S)$ (associated with σ) if and only if for every $\varepsilon > 0$ and every $f \in C(S)$, there exists $k > 0$ such that for every $x \in X$,

$$\|fx\|_{C(S)/X} \leqslant k\|x\|_{L^1(\sigma)} + \varepsilon\|x\|_\infty.$$

Proof. Easy. □

For the following result, remind the definitions at the beginning of the paper, in the setting of harmonic analysis.

Proposition 4.2. *Let X be a translation invariant subspace of $C(G)$.*

Then X is a rich subspace of $C(G)$, associated with a probability measure σ if and only if X is a rich subspace of $C(G)$, associated with the Haar measure.

Proof. Obviously, we only have to prove the fact that we always can choose the Haar measure. Indeed, using the previous proposition, we have that for every $\varepsilon > 0$ and every $f \in C(G)$, there exists $k > 0$ such that for every $x \in X$,

$$\|fx\|_{C(S)/X} \leqslant k\|x\|_{L^1(\sigma)} + \varepsilon\|x\|_\infty.$$

By a standard Baire argument, the same k may be chosen on an (non-empty) open subset V (hence with positive Haar measure) i.e. for every $\varepsilon > 0$ and every $f \in C(G)$, there exists $k > 0$ and a (non-empty) open subset V of G such that for every $x \in X$ and every $g \in V$

$$\|f_g x\|_{C(S)/X} \leqslant k\|x\|_{L^1(\sigma)} + \varepsilon\|x\|_\infty.$$

Applying this inequality to f_g and x_g, for any $g \in V$, we obtain (because X is translation invariant):

$$\|fx\|_{C(S)/X} = \|f_g x_g\|_{C(S)/X} \leqslant k\|x_g\|_{L^1(\sigma)} + \varepsilon\|x\|_\infty.$$

Now, integrating ($|V|$ denotes the Haar measure of V), we get

$$\|fx\|_{C(S)/X} \leqslant |V|^{-1}k\int_V \|x_g\|_{L^1(\sigma)}\,dg + \varepsilon\|x\|_\infty.$$

Using Fubini, we notice that

$$\int_V \|x_g\|_{L^1(\sigma)}\,dg = \int_G\int_V \bigl|x(gs)\bigr|\,dg\,d\sigma(s) \leqslant \int_G\int_G \bigl|x(gs)\bigr|\,dg\,d\sigma(s)$$
$$= \int_G \|x\|_{L^1(dg)}\,d\sigma(s) = \|x\|_{L^1(dg)}.$$

We finally have

$$\|fx\|_{C(S)/X} \leqslant |V|^{-1}k\|x\|_{L^1(dg)} + \varepsilon\|x\|_\infty = k'\|x\|_{L^1(dg)} + \varepsilon\|x\|_\infty$$

where k' depends only on ε and f.

By the previous proposition, we have the result. □

Remark 4.3. It was observed by Bayart [1] that the bidisk algebra $A(\mathbb{D}^2)$ is not a rich subspace of $C(\mathbb{T}^2)$, with the Haar measure. The previous proposition shows that, actually, $A(\mathbb{D}^2)$ is not a rich subspace of $C(\mathbb{T}^2)$ (for any measure σ). Nevertheless, the bidisk algebra could be a rich subspace of some other space $C(S)$.

On the other hand, it was proved by Bourgain that polydisk algebras have both the property (V) and the Dunford–Pettis property.

This last result cannot be improved to the case of infinitely many variables. Via the Bohr point of view, the polydisk algebra on $\mathbb{T}^\infty$ is isomorphic to the space of Dirichlet series. The Bohr inequality can then be interpreted as saying that ℓ^1 is complemented. Therefore, it has not the property (V) and cannot be rich.

To conclude, the following proposition is an easy consequence of the characterizations of non relatively weakly compact subset of X^*, when X is rich.

Proposition 4.4. *Let X be a rich subspace of $C(S)$. Let X_0 be a subspace of X (so that $X_0^\perp \subset X^*$).*

Then the following assertions are equivalent.

(i) *X/X_0 has finite cotype.*
(ii) *X/X_0 does not contain a subspace isomorphic to c_0.*
(iii) *X/X_0 is reflexive.*
(iv) *$X_0^\perp$ has a nontrivial type.*

Proof. (iv) ⇒ (i) ⇒ (ii) and (iii) ⇒ (ii) are obvious or standard.

(iv) ⇔ (iii): the unit ball of $X_0^\perp$ does not uniformly contain the ℓ_n^1 if and only if it is relatively weakly compact (see (v) in the introduction). This is equivalent to $X_0^\perp = (X/X_0)^*$ reflexive, and to X/X_0 reflexive.

(ii) ⇒ (iii): considering the canonical surjection $p: X \to X/X_0$. As X has the property (V) and X/X_0 does not contain a subspace isomorphic to c_0, the map p is weakly compact so the unit ball of X/X_0 is relatively weakly compact. □

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