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Zeros of the derivatives of the Riemann zeta function on $\mathrm{Re}\, s < 1/2$

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ABSTRACT

The Riemann hypothesis is equivalent to nonvanishing of $\zeta'(s)$ in the strip $0 < \mathrm{Re}\, s < 1/2$ and Levinson and Montgomery proved that the number of zeros of $\zeta'(s)$ in the region $\mathrm{Re}\, s < \sigma$, $T < \mathrm{Im}\, s < 2T$ is less than $T^{1+(\sigma-1/2)/4} w(T)^2 \log T$ for $\sigma \leqslant 1/2 - w(T)/\log T$, where the function $w(T)$ satisfies $w(T) \to \infty$ as $T \to \infty$. In this paper, we generalize it to the higher derivatives $\zeta^{(k)}(s)$ $(k > 1)$.

1. Introduction

The Riemann zeta function $\zeta(s)$ is defined by the Dirichlet series $\sum_{n=1}^{\infty} n^{-s}$ in $\mathrm{Re}\, s > 1$ and extended analytically to $\mathbb{C}$ except one simple pole at 1. Due to the Euler product $\prod_p (1 - p^{-s})^{-1}$ and the functional equation $\zeta(s) = \chi(s)\zeta(1-s)$, where $\chi(s) = 2^s \pi^{s-1} \sin(\pi s/2) \Gamma(1-s)$, $\zeta(s)$ has the trivial zeros at negative even integers $-2, -4, \ldots$ and all the other zeros are in the critical strip $0 < \mathrm{Re}\, s < 1$. We expect that all the nontrivial zeros of $\zeta(s)$ are in the critical line $\mathrm{Re}\, s = 1/2$, which is known as the Riemann hypothesis, but no one can prove it. Instead we count the number of

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zeros $\beta + i\gamma$ of $\zeta(s)$ satisfying $\beta > \sigma$ and $T < \gamma < 2T$, which is denoted by $N(\sigma, T, 2T)$. By the symmetries of zeros, we only consider the case $\sigma \geqslant 1/2$. Note that RH implies $N(\sigma, T, 2T) = 0$ in such case. There were intensive studies to an upper bound for $N(\sigma, T, 2T)$ of the type $T^{A(\sigma)}(\log T)^{B(\sigma)}$ for some functions $A(\sigma)$ and $B(\sigma)$. For the details, see Chapter 11 of [1]. In this paper, we are interested in a bound of the type

$$N(\sigma, T, 2T) \ll T^{1+c(1/2-\sigma)} \log T \tag{1.1}$$

uniformly for $\sigma \geqslant 1/2$. This type of result is still the best possible upper bound for σ near 1/2. Selberg [7] obtained $c = 1/4$ of (1.1) and later Jutila [2] obtained $c = 1 - \epsilon$ for any $\epsilon > 0$. Eq. (1.1) for $c = 2$ is known as the density hypothesis.

Next we study the first derivative $\zeta'(s)$. It has trivial zeros $\rho'_n \in (-2n-2, -2n)$ for $n = 1, 2, \ldots$ and no other zeros in the left-half plane $\operatorname{Re} s \leqslant 0$. On the other hand, it is nonvanishing on $\operatorname{Re} s > \sigma_1$ for some $2 < \sigma_1 < 3$, while it has an infinity of zeros in every strip between $\operatorname{Re} s = 1/2$ and $\operatorname{Re} s = \sigma_1$. Speiser [8] discovered that RH is equivalent to nonvanishing of $\zeta'(s)$ in the vertical strip $0 < \operatorname{Re} s < 1/2$. A certain connection between densities of zeros of $\zeta(s)$ and $\zeta'(s)$ is crucial in Levinson's observation [5] when he proved that at least one third of zeros of $\zeta(s)$ are on the critical line, which is a significant improvement of Selberg's [7] (small) positive proportion. If we can extend Levinson's observation to the higher derivatives, it may be possible to show that almost all zeros of $\zeta(s)$ on the critical line. Concerning this interesting point of view, see Section 3.2 of [3].

Let $N_1^-(\sigma, T, 2T)$ denote the number of zeros $\rho' = \beta' + i\gamma'$ of $\zeta'(s)$ satisfying $\beta' < \sigma$ and $T < \gamma' < 2T$. Then Levinson and Montgomery proved the following theorem, which is Corollary to Theorem 6 in [6].

Theorem A. *If $\sigma = 1/2 - w(T)/\log T$ where $w(T) \to \infty$ as $T \to \infty$, then*

$$N_1^-(\sigma, T, 2T) \ll T^{1+(\sigma-1/2)/4} w(T)^2 \log T.$$

Note that RH can be rephrased by $N_1^-(1/2, T, 2T) = 0$ for any $T > 0$. In this sense, Theorem A has a similar nature to (1.1). The main purpose of this paper is generalizing Theorem A to the higher derivatives of the Riemann zeta function.

The zero distribution of $\zeta^{(k)}$ for $k > 1$ is similar to $\zeta'(s)$. Spira [9] proved that there is an α_k so that $\zeta^{(k)}(s)$ has only real zeros for $\operatorname{Re} s \leqslant \alpha_k$ and exactly one real zero in each open interval $(-1-2n, 1-2n)$ for $1 - 2n \leqslant \alpha_k$. By a general theorem of Dirichlet series, $\zeta^{(k)}(s)$ is nonvanishing on $\operatorname{Re} s > \sigma_k$ for some $\sigma_k > 1$. There is not a statement equivalent to RH in terms of $\zeta^{(k)}(s)$, but Levinson and Montgomery [6] proved that RH implies $\zeta^{(k)}(s)$ has at most a finite number of non-real zeros in $\operatorname{Re} s < 1/2$ for $k > 1$. This condition is best possible, since $\zeta''(s)$ has a zero near $-0.36 + 3.59i$.

Motivated from Theorem A, we define $N_k^-(\sigma, T, 2T)$ by the number of zeros $\rho^{(k)} = \beta^{(k)} + i\gamma^{(k)}$ of $\zeta^{(k)}(s)$ satisfying $\beta^{(k)} < \sigma$ and $T < \gamma^{(k)} < 2T$ and prove the following theorem.

Theorem 1. *Let $k > 1$ be an integer and let A and c_0 be positive real numbers with $0 < c_0 < 1/2$. Then we have*

$$N_k^-(\sigma, T, 2T) \ll T^{1+c_0(\sigma-1/2)} \frac{(\log T)^3 \log_3 T}{\log_2 T}$$

for $\sigma \leqslant 1/2 - A\log_2 T/\log T$ as $T \to \infty$.

Theorem 1 is in fact a consequence of the following theorem.

Theorem 2. *Let $k > 1$ be an integer. Let c, c_1 and c_2 be real numbers satisfying $0 < c < 1$, $c_1 > 1$ and $c_2 > c_1/(c_1 - 1)$. Then we have*

$$\sum_{\substack{\beta^{(k)} < 1-\sigma \\ T < \gamma^{(k)} < 2T}} \bigl(1 - \sigma - \beta^{(k)}\bigr) \leqslant \sum_{\substack{\beta > \sigma \\ T < \gamma < 2T}} (\beta - \sigma) + O\bigl(T^{1+(1/2-\sigma)c/(2c_1)} \log T\bigr)$$

for $\sigma \geqslant 1/2 + c_2 \log_3 T/\log_2 T$ as $T \to \infty$. If $1/2 < \sigma \leqslant 1/2 + c_2 \log_3 T/\log_2 T$, then there is an extra O-term

$$O\biggl(T^{1+(c/c_1)(1/2-\sigma)} \frac{(\log T)^2 \log_3 T}{\log_2 T}\biggr)$$

in the right hand side.

Note that Theorem 2 is not effective if $1/2 < \sigma < 1/2 + A\log_2 T/\log T$ for some A. We first state and prove the required lemmas in Section 2 and then prove the main theorems in Section 3.

2. Lemmas

The first two lemmas tell us that $|\zeta^{(k)}/\zeta(s)|$ is small for $k \geqslant 1$ if s is not too close to zeros ρ of $\zeta(s)$. We need several definitions for more precise statement.

Let $c_1 > 1$. We first define two sets

$$R_\rho(T) = \bigl\{x + iy \in \mathbb{C}\colon\ 0 \leqslant x - 1/2 \leqslant c_1(\beta - 1/2),\ |\gamma - y| \leqslant \log T \text{ and } T \leqslant y \leqslant 2T\bigr\}$$

and

$$R(T) = \bigcup_\rho R_\rho(T).$$

Since $R_\rho(T)$ contains a neighborhood of ρ, $R(T)$ contains all s near ρ. We expect that $|\zeta^{(k)}/\zeta(s)|$ is small for $s \notin R(T)$, since ρ is a pole of $\zeta^{(k)}/\zeta(s)$. In this sense $R(T)$ is an exceptional set. Moreover, we will need the fact that there are only small exceptions. To see this, we define

$$B_\sigma(T) = \{t \in [T, 2T]\colon\ \sigma + it \in R(T)\},$$

then it equals

$$= \bigcup_\rho \{t \in [T, 2T]\colon\ \sigma + it \in R_\rho(T)\} = [T, 2T] \cap \bigcup_\rho^* [\gamma - \log T, \gamma + \log T], \tag{2.1}$$

where the *-union is over all zeros ρ with $\beta \geqslant 1/2 + 1/c_1(\sigma - 1/2)$ and $T \leqslant \gamma \leqslant 2T$. Thus, we have

$$|B_\sigma(T)| \leqslant 2(\log T)N(1/2 + 1/c_1(\sigma - 1/2), T, 2T) \ll T^{1+c/c_1(1/2-\sigma)}(\log T)^2 \tag{2.2}$$

by (1.1) for $0 < c < 1$.

The first lemma is the case $k = 1$.

Lemma 1. *Let $t \in [T, 2T] \setminus B_\sigma(T)$ and $\sigma \geqslant 1/2 + \phi(T)/\log_2 T$, where $\phi(T)$ satisfies $\phi(T) = o(\log_2 T)$ and $\phi(T) \nearrow \infty$ as $T \to \infty$. Then we have*

$$\frac{\zeta'}{\zeta}(s) \ll (\log T)\left(e^{-2\phi(T)} + \phi(T)^{-2} e^{-c_3\phi(T)} \log_2 T\right),$$

where $c_3 = 1 - 1/c_1$.

Proof. We begin with the equation

$$\begin{aligned} \frac{\zeta'}{\zeta}(s) = &- \sum_{n<x^2} \frac{\Lambda_x(n)}{n^s} + \frac{x^{2(1-s)} - x^{1-s}}{(1-s)^2 \log x} \\ &+ \frac{1}{\log x} \sum_{n=1}^\infty \frac{x^{-2n-s} - x^{-2(2n+s)}}{(2n+s)^2} + \frac{1}{\log x} \sum_\rho \frac{x^{\rho-s} - x^{2(\rho-s)}}{(s-\rho)^2}, \end{aligned} \tag{2.3}$$

where $\Lambda_x(n)$ equals $\Lambda(n)$ for $1 \leqslant n \leqslant x$ and $\Lambda(n)\log(x^2/n)/\log x$ for $x \leqslant n \leqslant x^2$. The first sum in (2.3) is

$$\left|\sum_{n<x^2} \frac{\Lambda_x(n)}{n^s}\right| \leqslant \sum_{n<x^2} \frac{\Lambda(n)}{n^\sigma} \ll x^{2(1-\sigma)} \tag{2.4}$$

by the prime number theorem. The next two terms are easily bounded by $O(x^{2(1-\sigma)})$.

Now we consider the last sum of (2.3). We split it into two sums $\sum_{|\gamma-t|\geqslant h}$ and $\sum_{|\gamma-t|<h}$. First, we have

$$\sum_{|\gamma-t|\geqslant h}\frac{x^{\rho-s}-x^{2(\rho-s)}}{(s-\rho)^2}\ll x^{2(1-\sigma)}\sum_{|\gamma-t|\geqslant h}\frac{1}{|\gamma-t|^2}\ll x^{2(1-\sigma)}\sum_{n=1}^{\infty}\sum_{nh\leqslant|\gamma-t|\leqslant(n+1)h}\frac{1}{|\gamma-t|^2}$$
$$\ll x^{2(1-\sigma)}\sum_{n=1}^{\infty}\frac{1}{n^2h^2}h\log(t+nh)\ll x^{2(1-\sigma)}(\log T)/h. \tag{2.5}$$

In the second sum, $t\in[T,2T]\setminus B_\sigma(T)$ implies $\sigma-1/2\geqslant c_1(\beta-1/2)$. Then we have

$$\beta-\sigma<c_3(1/2-\sigma)<0$$

for $c_3=1-1/c_1>0$. Thus, we have

$$\sum_{|\gamma-t|\leqslant h}\frac{x^{\rho-s}-x^{2(\rho-s)}}{(s-\rho)^2}\ll x^{c_3(1/2-\sigma)}\sum_{|\gamma-t|\leqslant h}\frac{1}{(\sigma-\beta)^2+(\gamma-t)^2}$$
$$\ll x^{c_3(1/2-\sigma)}\sum_{1\leqslant n\leqslant h}\sum_{n-1\leqslant|\gamma-t|\leqslant n}\frac{1}{c_3^2(\sigma-1/2)^2+(\gamma-t)^2}$$
$$\ll x^{c_3(1/2-\sigma)}(\sigma-1/2)^{-2}\log T. \tag{2.6}$$

By (2.3)–(2.6) with $h=\log T$, we have

$$\frac{\zeta'}{\zeta}(s)\ll x^{2(1-\sigma)}+x^{c_3(1/2-\sigma)}(\sigma-1/2)^{-2}\frac{\log T}{\log x}.$$

By letting $x=\log T$, we have

$$\frac{\zeta'}{\zeta}(s)\ll(\log T)\big(e^{-2\phi(T)}+\phi(T)^{-2}e^{-c_3\phi(T)}\log_2 T\big)$$

for $\sigma-1/2\geqslant\phi(T)/\log_2 T$. This proves the lemma. □

The bound for $\zeta'/\zeta(s)$ in Lemma 1 depends on the choice of $\phi(T)$. By choosing $\phi(T)=(\log_3 T)/c_3$ in Lemma 1, we obtain the following corollary.

Corollary 1. *Let $\sigma\geqslant 1/2+(c_1/(c_1-1))\log_3 T/\log_2 T$ and $t\in[T,2T]\setminus B_\sigma(T)$. Then we have*

$$\frac{\zeta'}{\zeta}(s)=o\left(\frac{\log T}{(\log_3 T)^2}\right).$$

Lemma 2. *Fix a constant $c_2>c_1/(c_1-1)$. Let $\sigma\geqslant 1/2+c_2\log_3 T/\log_2 T$ and $t\in[T,2T]\setminus B_\sigma(T)$. Then we have*

$$\frac{\zeta^{(k)}}{\zeta}(s)=o\left(\frac{(\log T)^k}{(\log_3 T)^{2k}}\right) \tag{2.7}$$

for any $k=1,2,\ldots$.

Proof. We prove the lemma by induction. Corollary 1 proves the case $k = 1$. We assume (2.7) holds for $k = 1, \dots, n$. First observe the equation

$$\frac{d}{ds}\frac{\zeta'}{\zeta}(s) = \frac{\zeta''}{\zeta}(s) + \zeta'(s)\frac{d}{ds}\frac{1}{\zeta(s)} = \frac{\zeta''}{\zeta}(s) - \zeta'(s)\frac{\zeta'(s)}{(\zeta(s))^2} = \frac{\zeta''}{\zeta}(s) - \left(\frac{\zeta'(s)}{\zeta(s)}\right)^2.$$

By differentiating it $(n-1)$-times, we can inductively deduce the formula

$$\frac{d^n}{ds^n}\frac{\zeta'}{\zeta}(s) - \frac{\zeta^{(n+1)}}{\zeta}(s) = \sum_{\sum_{\ell=1}^{n} \ell r_\ell = n+1} c_{\mathbf{r}} \prod_{\ell=1}^{n} \left(\frac{\zeta^{(\ell)}}{\zeta}(s)\right)^{r_\ell}$$

for some $c_{\mathbf{r}} \in \mathbb{Z}$, where $\mathbf{r} = (r_1, \dots, r_n)$ and $r_\ell \in \mathbb{Z}_{\geqslant 0}$. By induction hypothesis, we have

$$\frac{d^n}{ds^n}\frac{\zeta'}{\zeta}(s) - \frac{\zeta^{(n+1)}}{\zeta}(s) = o\left(\frac{(\log T)^{n+1}}{(\log_3 T)^{2n+2}}\right).$$

Hence, it is enough to show that

$$\frac{d^n}{ds^n}\frac{\zeta'}{\zeta}(s) = o\left(\frac{(\log T)^{n+1}}{(\log_3 T)^{2n+2}}\right) \tag{2.8}$$

for $t \in [T, 2T] \setminus B_\sigma(T)$ and $\sigma \geqslant 1/2 + c_2 \log_3 T/\log_2 T$. Since $\zeta'/\zeta(z)$ has no pole in the disc $\{z \in \mathbb{C}\colon |z - s| \leqslant (\log_3 T)^2/\log T\}$, we apply Cauchy's differentiation formula

$$\frac{d^n}{ds^n}\frac{\zeta'}{\zeta}(s) = \frac{n!}{2\pi i}\int_{|z-s|=(\log_3 T)^2/\log T} \frac{\zeta'}{\zeta}(z)\frac{dz}{(z-s)^{n+1}}.$$

By this and Corollary 1, we complete the proof. □

The following technical lemma is required.

Lemma 3. *Let $\chi(s) = 2^s \pi^{s-1} \sin(\pi s/2)\Gamma(1-s)$, then*

$$\frac{\chi^{(k)}}{\chi}(\sigma + it) = \bigl(-\log|t|\bigr)^k + O\bigl(\bigl(\log|t|\bigr)^{k-1}\bigr) \tag{2.9}$$

for $k = 1, 2, \dots$ and $|t| \geqslant t_0$ on any fixed vertical strip $a \leqslant \sigma \leqslant b$. For each positive integer k there is $\tau_k > 0$ such that $\chi^{(k)}(s)$ has no zero or pole in the region $|t| \geqslant \tau_k$ and $a \leqslant \sigma \leqslant b$.

See Lemma 1 in [4] for a proof of the first part and the second part easily follows. Note that the function $\chi(s)$ satisfies the functional equation $\zeta(s) = \chi(s)\zeta(1-s)$.

3. Proof of theorems

3.1. Proof of Theorem 2

Define a function $J_k(s)$ by

$$J_k(s) = \zeta^{(k)}(1-s)/\chi^{(k)}(1-s). \tag{3.1}$$

By Lemma 3, $J_k(s)$ has its only zeros $1-\rho^{(k)} = 1-\beta^{(k)} - i\gamma^{(k)}$ in $|\operatorname{Im} s| > \tau_k$, thus we let $T > \tau_k$. We want to apply Littlewood's lemma to $J_k(s)/\zeta(s)$ in a similar way of the proof of Theorem 6 in [6]. For $\sigma > 1/2$ we decompose $B_\sigma(T)$ by a disjoint union of closed intervals, say

$$B_\sigma(T) = \bigcup_{j=0}^{J} [t_j, t_j']$$

with an increasing order $t_j < t_{j+1}$ for $j = 0, 1, \ldots, J-1$. Let C_j be the rectangle with vertices $\sigma + it_j$, $\sigma + it_j'$, $\sigma_k + it_j'$ and $\sigma_k = it_j$, where σ_k is any real number such that $\sigma_k > (1+c_1)/2$ and $\zeta^{(k)}(s)$ has no zero on $\operatorname{Re} s \geqslant \sigma_k$. Here, c_1 is the constant in the definition of $R_\rho(T)$. By applying Littlewood's lemma to the function $J_k(s)/\zeta(s)$ and the contour C_j for each $j = 0, 1, 2, \ldots, J$, we have

$$\sum_{\substack{\beta^{(k)} < 1-\sigma \\ t_j < \gamma^{(k)} < t_j'}} \left(1 - \sigma - \beta^{(k)}\right) - \sum_{\substack{\beta > \sigma \\ t_j < \gamma < t_j'}} (\beta - \sigma) = V_{1,j} - V_{2,j} - H_{1,j} + H_{2,j}, \tag{3.2}$$

where

$$V_{1,j} = \frac{1}{2\pi} \int_{t_j}^{t_j'} \log\left|\frac{J_k}{\zeta}(\sigma + it)\right| dt,$$

$$V_{2,j} = \frac{1}{2\pi} \int_{t_j}^{t_j'} \log\left|\frac{J_k}{\zeta}(\sigma_k + it)\right| dt,$$

and

$$H_{1,j} = \frac{1}{2\pi} \int_{\sigma}^{\sigma_k} \arg \frac{J_k}{\zeta}(u + it_j)\, du,$$

$$H_{2,j} = \frac{1}{2\pi} \int_{\sigma}^{\sigma_k} \arg \frac{J_k}{\zeta}\left(u + it_j'\right) du.$$

To calculate the integrals $V_{i,j}$ and $H_{i,j}$, we need an expression

$$J_k(s) = \zeta(s) + \sum_{\ell=1}^{k} \binom{k}{\ell} (-1)^\ell \frac{\chi^{(k-\ell)}}{\chi^{(k)}} (1-s) \zeta^{(\ell)}(s), \tag{3.3}$$

which is the k-th derivative of $\zeta(1-s) = \chi(1-s)\zeta(s)$. First, we observe that

$$\frac{J_k}{\zeta}(s) = 1 + O\left(\frac{1}{\log T}\right)$$

for $\operatorname{Re} s = \sigma_k$ and so

$$V_{2,j} = O\left(\frac{t_j' - t_j}{\log T}\right).$$

Next, we consider $H_{1,j}$ and $H_{2,j}$. The contours for $H_{1,0}$ and $H_{2,J}$ may be included in $R(T)$. Hence, we use the trivial bound

$$\arg \frac{J_k}{\zeta}(s) = O(\log T) \tag{3.4}$$

for $\operatorname{Re} s > 1/2$, which may be proved by applying Lemma 9.4 [10] to $J_k(s)$ and $\zeta(s)$, to obtain

$$H_{1,0} = O(\log T),$$
$$H_{2,J} = O(\log T).$$

To get bounds of the other $H_{i,j}$, we use

$$\arg \frac{J_k}{\zeta}(x+iy) = o(1) \tag{3.5}$$

for $x + iy \notin R(T)$, $x \geqslant 1/2 + c_2 \log_3 T / \log_2 T$ and $y \in [T, 2T]$, which is a consequence of

$$\frac{J_k}{\zeta}(x+iy) = 1 + o(1) \tag{3.6}$$

obtained by (3.3), Lemma 2 and Lemma 3. If $\sigma \geqslant 1/2 + c_2 \log_3 T / \log_2 T$, then (3.5) implies

$$H_{i,j} = o(1)$$

for $(i,j) \neq (1,0), (2,J)$. If $1/2 < \sigma \leqslant 1/2 + c_2 \log_3 T / \log_2 T$, we need the trivial bound (3.4) together with (3.6). By splitting the integral into two parts $[\sigma, 1/2 + c_2 \log_3 T / \log_2 T]$ and $[1/2 + c_2 \log_3 T / \log_2 T, \sigma_k]$, we obtain

$$H_{i,j} = O\left(\frac{\log T \log_3 T}{\log_2 T}\right).$$

By adding (3.2) for all $j = 0, 1, \dots, J$, we have

$$\sum_{\substack{\beta^{(k)} < 1-\sigma \\ T < \gamma^{(k)} < 2T}} \left(1 - \sigma - \beta^{(k)}\right) - \sum_{\substack{\beta > \sigma \\ T < \gamma < 2T}} (\beta - \sigma)$$

$$= \frac{1}{2\pi} \sum_{j \leqslant J} \int_{t_j}^{t'_j} \log\left|\frac{J_k}{\zeta}(\sigma + it)\right| dt + O\left(\frac{\sum_{j \leqslant J}(t'_j - t_j)}{\log T}\right) + O(\log T) \tag{3.7}$$

for $\sigma \geqslant 1/2 + c_2 \log_3 T/\log_2 T$. If $1/2 < \sigma \leqslant 1/2 + c_2 \log_3 T/\log_2 T$, then we have an extra O-term

$$O\left(\frac{J \log T \log_3 T}{\log_2 T}\right) = O\left(T^{1+c/c_1(1/2-\sigma)} \frac{(\log T)^2 \log_3 T}{\log_2 T}\right) \tag{3.8}$$

on the right hand side of (3.7).

Now, we consider the sum of integrals in (3.7). Since the intervals $[t_j, t'_j]$ are disjoint, we have

$$\sum_{j \leqslant J} \int_{t_j}^{t'_j} \log\left|\frac{J_k}{\zeta}(\sigma + it)\right| dt = \int_{\bigcup [t_j, t'_j]} \log\left|\frac{J_k}{\zeta}(\sigma + it)\right| dt.$$

By (3.3), Lemma 3 and the inequality

$$\left|1 + \sum_{\ell \leqslant k} z_\ell\right| \leqslant 1 + \sum_{\ell \leqslant k} |z_\ell| \leqslant \prod_{\ell \leqslant k} \left(1 + |z_\ell|\right) \leqslant \exp\left(A_k \sum_{\ell \leqslant k} |z_\ell|^{1/4\ell}\right)$$

for some $A_k > 0$, we have

$$\int_{\bigcup [t_j, t'_j]} \log\left|\frac{J_k}{\zeta}(\sigma + it)\right| dt \leqslant A_k \int_{\bigcup [t_j, t'_j]} \sum_{\ell=1}^{k} \frac{1}{(\log T)^{1/4}} \left|\frac{\zeta^{(\ell)}}{\zeta}(\sigma + it)\right|^{1/4\ell} dt.$$

By the Cauchy–Schwarz inequality, it is smaller than

$$\ll (\log T)^{-1/4} \sum_{\ell=1}^{k} \sqrt{\sum_{j \leqslant J} (t'_j - t_j)} \left(\int_T^{2T} \left|\frac{\zeta^{(\ell)}}{\zeta}(\sigma + it)\right|^{1/2\ell} dt\right)^{1/2}. \tag{3.9}$$

We can deduce

$$\int_T^{2T} \left| \frac{\zeta^{(\ell)}}{\zeta}(\sigma + it) \right|^{1/2\ell} dt = O(T\sqrt{\log T}) \tag{3.10}$$

by modifying the proof of Claim in [4]. Then we have

$$\sum_{j \leqslant J} \int_{t_j}^{t_j'} \log \left| \frac{J_k}{\zeta}(\sigma + it) \right| dt \ll T^{1+(1/2-\sigma)c/(2c_1)} \log T \tag{3.11}$$

by (2.2), (3.9) and (3.10). Therefore, by (2.2), (3.7) and (3.11) we have

$$\sum_{\substack{\beta^{(k)} < 1-\sigma \\ T < \gamma^{(k)} < 2T}} \left(1 - \sigma - \beta^{(k)}\right) \leqslant \sum_{\substack{\beta > \sigma \\ T < \gamma < 2T}} (\beta - \sigma) + O\left(T^{1+(1/2-\sigma)c/(2c_1)} \log T\right) \tag{3.12}$$

for $\sigma > 1/2 + c_2 \log_3 T / \log_2 T$. If $1/2 < \sigma < 1/2 + c_2 \log_3 T / \log_2 T$, then we have an extra O-term

$$O\left(T^{1+c/c_1(1/2-\sigma)} \frac{(\log T)^2 \log_3 T}{\log_2 T}\right)$$

in the right hand side of (3.12), which comes from (3.8).

3.2. *Proof of Theorem 1*

By letting $c_0 = c/(2c_1)$ in Theorem 2, we have

$$\sum_{\substack{\beta^{(k)} < \sigma + 1/\log T \\ T < \gamma^{(k)} < 2T}} \left(\sigma + 1/\log T - \beta^{(k)}\right) \ll T^{1+c_0(\sigma-1/2)} \frac{(\log T)^2 \log_3 T}{\log_2 T}$$

for $\sigma \leqslant 1/2 - A \log_2 T / \log T$. Since

$$\sum_{\substack{\beta^{(k)} < \sigma + 1/\log T \\ T < \gamma^{(k)} < 2T}} \left(\sigma + 1/\log T - \beta^{(k)}\right) \geqslant \sum_{\substack{\beta^{(k)} < \sigma \\ T < \gamma^{(k)} < 2T}} 1/\log T = N_k^-(\sigma, T, 2T)/\log T,$$

we have

$$N_k^-(\sigma, T, 2T) \ll T^{1+c_0(\sigma-1/2)} \frac{(\log T)^3 \log_3 T}{\log_2 T}$$

for $\sigma \leqslant 1/2 - A \log_2 T / \log T$.

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