INITIAL COEFFICIENTS OF BI-UNIVALENT FUNCTIONS

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Abstract. An analytic function f defined on the open unit disk $\mathbb{D} = \{z : |z| < 1\}$ is bi-univalent if the function f and its inverse f^{-1} are univalent in $\mathbb{D}$. Estimates for the initial coefficients of bi-univalent functions f are investigated when f and f^{-1} respectively belong to some subclasses of univalent functions. Some earlier results are shown to be special cases of our results.

1. Introduction

Let $\mathcal{S}$ be the class of all univalent analytic functions f in the open unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ and normalized by the conditions $f(0) = 0$ and $f'(0) = 1$. For $f \in \mathcal{S}$, it is well known that the nth coefficient is bounded by n. The bounds for the coefficients give information about the geometric properties of these functions. Indeed, the bound for the second coefficient of functions in the class $\mathcal{S}$ gives rise to growth, distortion, covering theorems for univalent functions. In view of the influence of the second coefficient in the geometric properties of univalent functions, it is important to know the bounds for the (initial) coefficients of functions belonging to various subclasses of univalent functions. In this paper, we investigate this coefficient problem for certain subclasses of bi-univalent functions.

Recall that the Koebe one-quarter theorem [8] ensures that the image of $\mathbb{D}$ under every univalent function $f \in \mathcal{S}$ contains a disk of radius 1/4. Thus every univalent function f has an inverse f^{-1} satisfying $f^{-1}(f(z)) = z$, $(z \in \mathbb{D})$ and

$$f(f^{-1}(w)) = w, \quad (|w| < r_0(f), r_0(f) \geq 1/4).$$

A function $f \in \mathcal{S}$ is *bi-univalent* in $\mathbb{D}$ if both f and f^{-1} are univalent in $\mathbb{D}$. Let σ denote the class of bi-univalent functions defined in the unit disk $\mathbb{D}$. Lewin [12] investigated this class σ and obtained the bound for the second coefficient of the bi-univalent functions.

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Several authors subsequently studied similar problems in this direction (see [7, 15]). A function $f \in \sigma$ is bi-starlike or strongly bi-starlike or bi-convex of order α if f and f^{-1} are both starlike, strongly starlike or convex of order α, respectively. Brannan and Taha [6] obtained estimates for the initial coefficients of bi-starlike,strongly bi-starlike and bi-convex functions. Bounds for the initial coefficients of several classes of functions were also investigated in [2–5, 9, 11, 14, 16–20].

An analytic function f is *subordinate* to an analytic function g, written $f(z) \prec g(z)$, if there is an analytic function $w : \mathbb{D} \to \mathbb{D}$ with $w(0) = 0$ satisfying $f(z) = g(w(z))$. Ma and Minda [13] unified various subclasses of starlike ($\mathcal{S}^*$) and convex functions ($\mathcal{C}$) by requiring that either of the quantity $zf'(z)/f(z)$ or $1 + zf''(z)/f'(z)$ is subordinate to a more general superordinate function φ with positive real part in the unit disk $\mathbb{D}$, $\varphi(0) = 1$, $\varphi'(0) > 0$, φ maps $\mathbb{D}$ onto a region starlike with respect to 1 and symmetric with respect to the real axis. The class $\mathcal{S}^*(\varphi)$ of Ma-Minda starlike functions with respect to φ consists of functions $f \in \mathcal{S}$ satisfying the subordination $zf'(z)/f(z) \prec \varphi(z)$. Similarly, the class $\mathcal{C}(\varphi)$ of Ma-Minda convex functions consists of functions $f \in \mathcal{S}$ satisfying the subordination $1+zf''(z)/f'(z) \prec \varphi(z)$. Ma and Minda investigated growth and distortion properties of functions in $\mathcal{S}^*(\varphi)$ and $\mathcal{C}(\varphi)$ as well as Fekete-Szegö inequalities for $\mathcal{S}^*(\varphi)$ and $\mathcal{C}(\varphi)$. Their proof of Fekete-Szegö inequalities requires the univalence of φ. Ali *et al.* [3] have investigated Fekete-Szegö problems for various other classes and their proof does not require the univalence or starlikeness of φ. In particular, their results are valid even if one just assume the function φ to have a series expansion of the form $\varphi(z) = 1 + B_1 z + B_2 z^2 + \cdots$, $B_1 > 0$. So in this paper, we assume that φ has series expansion $\varphi(z) = 1 + B_1 z + B_2 z^2 + \cdots$, B_1, B_2 are real and $B_1 > 0$. A function f is Ma-Minda bi-starlike or Ma-Minda bi-convex if both f and f^{-1} are respectively Ma-Minda starlike or convex. Motivated by the Fekete-Szegö problem for the classes of Ma-Minda starlike and Ma-Minda convex functions [13], Ali *et al.* [1] recently obtained estimates of the initial coefficients for bi-univalent Ma-Minda starlike and Ma-Minda convex functions.

The present work is motivated by the results of Kędzierawski [10] who considered functions f belonging to certain subclasses of univalent functions while its inverse f^{-1} belongs to some other subclasses of univalent functions. Among other results, he obtained the following coefficient estimates.

Theorem 1.1. [10] *Let $f \in \sigma$ with Taylor series $f(z) = z + a_2 z^2 + \cdots$ and $g = f^{-1}$. Then*

$$|a_2| \leq \begin{cases} 1.5894 & \text{if } f \in \mathcal{S},\ g \in \mathcal{S}, \\ 2 & \text{if } f \in \mathcal{S}^*,\ g \in \mathcal{S}^*, \\ 1.507 & \text{if } f \in \mathcal{S}^*,\ g \in \mathcal{S}, \\ 1.224 & \text{if } f \in \mathcal{C},\ g \in \mathcal{S}. \end{cases}$$

We need the following classes investigated in [1–3].

Definition 1.1. *Let $\varphi : \mathbb{D} \to \mathbb{C}$ be analytic and $\varphi(z) = 1 + B_1 z + B_2 z^2 + \cdots$ with $B_1 > 0$ and $B_2 \in \mathbb{R}$. For $\alpha \geq 0$, let*

$$\mathcal{M}(\alpha, \varphi) := \left\{ f \in \mathcal{S} \,:\, (1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right) \prec \varphi(z) \right\},$$

$$\mathcal{L}(\alpha, \varphi) := \left\{ f \in \mathcal{S} \,:\, \left(\frac{zf'(z)}{f(z)}\right)^{\alpha} \left(1 + \frac{zf''(z)}{f'(z)}\right)^{1-\alpha} \prec \varphi(z) \right\},$$

$$\mathcal{P}(\alpha, \varphi) := \left\{ f \in \mathcal{S} \,:\, \frac{zf'(z)}{f(z)} + \alpha\frac{z^2 f''(z)}{f(z)} \prec \varphi(z) \right\}.$$

In this paper, we obtain the estimates for the second and third coefficients of functions f when

(1) $f \in \mathcal{P}(\alpha, \varphi)$ and $g := f^{-1} \in \mathcal{P}(\beta, \psi)$, or $g \in \mathcal{M}(\beta, \psi)$, or $g \in \mathcal{L}(\beta, \psi)$,
(2) $f \in \mathcal{M}(\alpha, \varphi)$ and $g \in \mathcal{M}(\beta, \psi)$, or $g \in \mathcal{L}(\beta, \psi)$,
(3) $f \in \mathcal{L}(\alpha, \varphi)$ and $g \in \mathcal{L}(\beta, \psi)$.

2. Coefficient Estimates

In the sequel, it is assumed that φ and ψ are analytic functions of the form

(2.1) $$\varphi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots, \quad (B_1 > 0)$$

and

(2.2) $$\psi(z) = 1 + D_1 z + D_2 z^2 + D_3 z^3 + \cdots, \quad (D_1 > 0).$$

Theorem 2.1. *Let $f \in \sigma$ and $g = f^{-1}$. If $f \in \mathcal{P}(\alpha, \varphi)$, $g \in \mathcal{P}(\beta, \psi)$ and f of the form*

(2.3) $$f(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$

then

$$|a_2| \leq \frac{B_1D_1\sqrt{B_1(1+3\beta)+D_1(1+3\alpha)}}{\sqrt{|\sigma B_1^2D_1^2-(1+2\alpha)^2(1+3\beta)(B_2-B_1)D_1^2-(1+2\beta)^2(1+3\alpha)(D_2-D_1)B_1^2|}} \tag{2.4}$$

and

$$2\sigma|a_3| \leq B_1(3+10\beta)+D_1(1+2\alpha)+(3+10\beta)|B_2-B_1|+\frac{(1+2\beta)^2B_1^2|D_2-D_1|}{D_1^2(1+2\alpha)} \tag{2.5}$$

where $\sigma := 2+7\alpha+7\beta+24\alpha\beta$.

Proof. Since $f \in \mathcal{P}(\alpha,\varphi)$ and $g \in \mathcal{P}(\beta,\psi)$, $g = f^{-1}$. There exist analytic functions $u, v : \mathbb{D} \to \mathbb{D}$, with $u(0) = v(0) = 0$, satisfying

$$\frac{zf'(z)}{f(z)}+\frac{\alpha z^2f''(z)}{f(z)} = \varphi(u(z)) \quad \text{and} \quad \frac{wg'(w)}{g(w)}+\frac{\beta w^2g''(w)}{g(w)} = \psi(v(w)). \tag{2.6}$$

Define the functions p_1 and p_2 by

$$p_1(z) := \frac{1+u(z)}{1-u(z)} = 1+c_1z+c_2z^2+\cdots \quad \text{and} \quad p_2(z) := \frac{1+v(z)}{1-v(z)} = 1+b_1z+b_2z^2+\cdots,$$

or, equivalently,

$$u(z) = \frac{p_1(z)-1}{p_1(z)+1} = \frac{1}{2}\left(c_1z+\left(c_2-\frac{c_1^2}{2}\right)z^2+\cdots\right) \tag{2.7}$$

and

$$v(z) = \frac{p_2(z)-1}{p_2(z)+1} = \frac{1}{2}\left(b_1z+\left(b_2-\frac{b_1^2}{2}\right)z^2+\cdots\right). \tag{2.8}$$

Then p_1 and p_2 are analytic in $\mathbb{D}$ with $p_1(0) = 1 = p_2(0)$. Since $u, v : \mathbb{D} \to \mathbb{D}$, the functions p_1 and p_2 have positive real part in $\mathbb{D}$, and $|b_i| \leq 2$ and $|c_i| \leq 2$. In view of (2.6), (2.7) and (2.8), clearly

$$\frac{zf'(z)}{f(z)}+\frac{\alpha z^2f''(z)}{f(z)} = \varphi\left(\frac{p_1(z)-1}{p_1(z)+1}\right) \quad \text{and} \quad \frac{wg'(w)}{g(w)}+\frac{\beta w^2g''(w)}{g(w)} = \psi\left(\frac{p_2(w)-1}{p_2(w)+1}\right). \tag{2.9}$$

Using (2.7) and (2.8) together with (2.1) and (2.2), it is evident that

$$\varphi\left(\frac{p_1(z)-1}{p_1(z)+1}\right) = 1+\frac{1}{2}B_1c_1z+\left(\frac{1}{2}B_1\left(c_2-\frac{c_1^2}{2}\right)+\frac{1}{4}B_2c_1^2\right)z^2+\cdots \tag{2.10}$$

and

$$\psi\left(\frac{p_2(w)-1}{p_2(w)+1}\right) = 1+\frac{1}{2}D_1b_1w+\left(\frac{1}{2}D_1\left(b_2-\frac{b_1^2}{2}\right)+\frac{1}{4}D_2b_1^2\right)w^2+\cdots. \tag{2.11}$$

Since f has the Maclaurin series given by (2.3), a computation shows that its inverse $g = f^{-1}$ has the expansion

$$g(w) = f^{-1}(w) = w - a_2w^2 + (2a_2^2 - a_3)w^3 + \cdots .$$

Since

$$\frac{zf'(z)}{f(z)} + \frac{\alpha z^2 f''(z)}{f(z)} = 1 + a_2(1+2\alpha)z + (2(1+3\alpha)a_3 - (1+2\alpha)a_2^2)z^2 + \cdots$$

and

$$\frac{wg'(w)}{g(w)} + \frac{\beta w^2 g''(w)}{g(w)} = 1 - (1+2\beta)a_2w + ((3+10\beta)a_2^2 - 2(1+3\beta)a_3)w^2 + \cdots ,$$

it follows from (2.9), (2.10) and (2.11) that

$$a_2(1+2\alpha) = \frac{1}{2}B_1c_1, \tag{2.12}$$

$$2(1+3\alpha)a_3 - (1+2\alpha)a_2^2 = \frac{1}{2}B_1\left(c_2 - \frac{c_1^2}{2}\right) + \frac{1}{4}B_2c_1^2, \tag{2.13}$$

$$-(1+2\beta)a_2 = \frac{1}{2}D_1b_1 \tag{2.14}$$

and

$$(3+10\beta)a_2^2 - 2(1+3\beta)a_3 = \frac{1}{2}D_1\left(b_2 - \frac{b_1^2}{2}\right) + \frac{1}{4}D_2b_1^2. \tag{2.15}$$

It follows from (2.12) and (2.14) that

$$b_1 = -\frac{B_1(1+2\beta)}{D_1(1+2\alpha)}c_1. \tag{2.16}$$

Equations (2.12), (2.13), (2.15) and (2.16) lead to

$$a_2^2 = \frac{B_1^2D_1^2[B_1(1+3\beta)c_2 + D_1(1+3\alpha)b_2]}{2[\sigma B_1^2D_1^2 - (1+2\alpha)^2(1+3\beta)(B_2-B_1)D_1^2 - (1+2\beta)^2(1+3\alpha)(D_2-D_1)B_1^2]},$$

which, in view of the well-known inequalities $|b_2| \leq 2$ and $|c_2| \leq 2$ for functions with positive real part, gives us the desired estimate on $|a_2|$ as asserted in (2.4).

By using (2.13), (2.15) and (2.16) lead to

$$\begin{aligned}2\sigma a_3 = {} & \frac{1}{2}\left[B_1(3+10\beta)c_2 + D_1(1+2\alpha)b_2\right] \\ & + \frac{c_1^2}{4}\left[(3+10\beta)(B_2-B_1) + \frac{(1+2\beta)^2B_1^2(D_2-D_1)}{D_1^2(1+2\alpha)}\right],\end{aligned}$$

and this yields the estimate given in (2.5). □

Remark 2.1. When $\alpha = \beta = 0$ and $B_1 = D_1 = 2$, then (2.4) reduces to Theorem 1.1. When $\beta = \alpha$ and $\psi = \varphi$, Theorem 2.1 reduces to [1, Theorem 2.2].

Theorem 2.2. *Let $f \in \sigma$ and $g = f^{-1}$. If $f \in \mathcal{P}(\alpha, \varphi)$ and $g \in \mathcal{M}(\beta, \psi)$, then*

$$|a_2| \leq \frac{B_1 D_1 \sqrt{B_1(1+2\beta) + D_1(1+3\alpha)}}{\sqrt{|\sigma B_1^2 D_1^2 - (1+2\alpha)^2(1+2\beta)(B_2 - B_1)D_1^2 - (1+\beta)^2(1+3\alpha)(D_2 - D_1)B_1^2|}} \tag{2.17}$$

and

$$2\sigma|a_3| \leq B_1(3+5\beta) + D_1(1+2\alpha) + (3+5\beta)|B_2 - B_1| + \frac{(1+\beta)^2 B_1^2 |D_2 - D_1|}{D_1^2(1+2\alpha)} \tag{2.18}$$

where $\sigma := 2 + 7\alpha + 3\beta + 11\alpha\beta$.

Proof. Let $f \in \mathcal{P}(\alpha, \varphi)$ and $g \in \mathcal{M}(\beta, \psi)$, $g = f^{-1}$. There exist analytic functions $u, v : \mathbb{D} \to \mathbb{D}$, with $u(0) = v(0) = 0$, such that

$$\frac{zf'(z)}{f(z)} + \frac{\alpha z^2 f''(z)}{f(z)} = \varphi(u(z)) \quad \text{and} \quad (1-\beta)\frac{wg'(w)}{g(w)} + \beta\left(1 + \frac{wg''(w)}{g'(w)}\right) = \psi(v(w)), \tag{2.19}$$

Since

$$\frac{zf'(z)}{f(z)} + \frac{\alpha z^2 f''(z)}{f(z)} = 1 + a_2(1+2\alpha)z + (2(1+3\alpha)a_3 - (1+2\alpha)a_2^2)z^2 + \cdots$$

and

$$(1-\beta)\frac{wg'(w)}{g(w)} + \beta\left(1 + \frac{wg''(w)}{g'(w)}\right) = 1 - (1+\beta)a_2 w + ((3+5\beta)a_2^2 - 2(1+2\beta)a_3)w^2 + \cdots,$$

then (2.10), (2.11) and (2.19) yield

$$a_2(1+2\alpha) = \frac{1}{2}B_1 c_1, \tag{2.20}$$

$$2(1+3\alpha)a_3 - (1+2\alpha)a_2^2 = \frac{1}{2}B_1\left(c_2 - \frac{c_1^2}{2}\right) + \frac{1}{4}B_2 c_1^2, \tag{2.21}$$

$$-(1+\beta)a_2 = \frac{1}{2}D_1 b_1 \tag{2.22}$$

and

$$(3+5\beta)a_2^2 - 2(1+2\beta)a_3 = \frac{1}{2}D_1\left(b_2 - \frac{b_1^2}{2}\right) + \frac{1}{4}D_2 b_1^2. \tag{2.23}$$

It follows from (2.20) and (2.22) that

$$b_1 = -\frac{B_1(1+\beta)}{D_1(1+2\alpha)}c_1. \tag{2.24}$$

Equations (2.20), (2.21), (2.23) and (2.24) lead to

$$a_2^2 = \frac{B_1^2 D_1^2 [B_1(1+2\beta)c_2 + D_1(1+3\alpha)b_2]}{2[\sigma B_1^2 D_1^2 - (1+2\alpha)^2(1+2\beta)(B_2-B_1)D_1^2 - (1+2\beta)^2(1+3\alpha)(D_2-D_1)B_1^2]},$$

which gives us the desired estimate on $|a_2|$ as asserted in (2.17) when $|b_2| \leq 2$ and $|c_2| \leq 2$.

Since (2.21), (2.23) and (2.24) lead to

$$\begin{aligned} 2\sigma a_3 &= \frac{1}{2}[B_1(3+5\beta)c_2 + D_1(1+2\alpha)b_2] \\ &\quad + \frac{c_1^2}{4}\left[(3+5\beta)(B_2-B_1) + \frac{(1+\beta)^2 B_1^2 (D_2-D_1)}{D_1^2(1+2\alpha)}\right], \end{aligned}$$

and this yields the estimate given in (2.18). □

Theorem 2.3. *Let $f \in \sigma$ and $g = f^{-1}$. If $f \in \mathcal{P}(\alpha,\varphi)$ and $g \in \mathcal{L}(\beta,\psi)$, then*

(2.25)

$$|a_2| \leq \frac{B_1 D_1 \sqrt{2[B_1(3-2\beta) + D_1(1+3\alpha)]}}{\sqrt{|\sigma B_1^2 D_1^2 - 2(1+2\alpha)^2(3-2\beta)(B_2-B_1)D_1^2 - 2(2-\beta)^2(1+3\alpha)(D_2-D_1)B_1^2|}}$$

and

$$\begin{aligned} |\sigma a_3| &\leq \frac{1}{2}B_1(\beta^2 - 11\beta + 16) + D_1(1+2\alpha) + \frac{1}{2}(\beta^2 - 11\beta + 16)|B_2 - B_1| \\ &\quad + \frac{(2-\beta)^2 B_1^2 |D_2 - D_1|}{D_1^2(1+2\alpha)} \end{aligned} \tag{2.26}$$

where $\sigma := 10 + 36\alpha - 7\beta - 25\alpha\beta + \beta^2 + 3\alpha\beta^2$.

Proof. Let $f \in \mathcal{P}(\alpha,\varphi)$ and $g \in \mathcal{L}(\beta,\psi)$, $g = f^{-1}$. Then there are analytic functions $u, v : \mathbb{D} \to \mathbb{D}$, with $u(0) = v(0) = 0$, satisfying

$$\frac{zf'(z)}{f(z)} + \frac{\alpha z^2 f''(z)}{f(z)} = \varphi(u(z)) \quad \text{and} \quad \left(\frac{wg'(w)}{g(w)}\right)^{\beta}\left(1 + \frac{wg''(w)}{g'(w)}\right)^{1-\beta} = \psi(v(w)), \tag{2.27}$$

Since

$$\frac{zf'(z)}{f(z)} + \frac{\alpha z^2 f''(z)}{f(z)} = 1 + a_2(1+2\alpha)z + (2(1+3\alpha)a_3 - (1+2\alpha)a_2^2)z^2 + \cdots$$

and

$$\begin{aligned} &\left(\frac{wg'(w)}{g(w)}\right)^{\beta}\left(1 + \frac{wg''(w)}{g'(w)}\right)^{1-\beta} \\ &= 1 - (2-\beta)a_2 w + \Big((8(1-\beta) + \frac{1}{2}\beta(\beta+5))a_2^2 - 2(3-2\beta)a_3\Big)w^2 + \cdots, \end{aligned}$$

then (2.10), (2.11) and (2.27) yield

$$a_2(1+2\alpha) = \frac{1}{2}B_1c_1, \tag{2.28}$$

$$2(1+3\alpha)a_3 - (1+2\alpha)a_2^2 = \frac{1}{2}B_1\left(c_2 - \frac{c_1^2}{2}\right) + \frac{1}{4}B_2c_1^2, \tag{2.29}$$

$$-(2-\beta)a_2 = \frac{1}{2}D_1b_1 \tag{2.30}$$

and

$$[8(1-\beta) + \frac{\beta}{2}(\beta+5)]a_2^2 - 2(3-2\beta)a_3 = \frac{1}{2}D_1\left(b_2 - \frac{b_1^2}{2}\right) + \frac{1}{4}D_2b_1^2. \tag{2.31}$$

It follows from (2.28) and (2.30) that

$$b_1 = -\frac{B_1(2-\beta)}{D_1(1+2\alpha)}c_1. \tag{2.32}$$

Equations (2.28), (2.29), (2.31) and (2.32) lead to

$$a_2^2 = \frac{B_1^2D_1^2[B_1(3-2\beta)c_2 + D_1(1+3\alpha)b_2]}{\sigma B_1^2D_1^2 - 2(1+2\alpha)^2(3-2\beta)(B_2-B_1)D_1^2 - 2(2-\beta)^2(1+3\alpha)(D_2-D_1)B_1^2},$$

which, in view of the well-known inequalities $|b_2| \le 2$ and $|c_2| \le 2$ for functions with positive real part, gives us the desired estimate on $|a_2|$ as asserted in (2.25).

By using (2.29), (2.31) and (2.32) lead to

$$\begin{aligned}\sigma a_3 = {} & \frac{B_1}{4}(\beta^2 - 11\beta + 16)c_2 + \frac{D_1}{2}(1+2\alpha)b_2 \\ & + \frac{c_1^2}{4}\left[\frac{1}{2}(\beta^2 - 11\beta + 16)(B_2 - B_1) + \frac{(2-\beta)^2B_1^2(D_2-D_1)}{D_1^2(1+2\alpha)}\right]\end{aligned}$$

and this yields the estimate given in (2.26). □

Theorem 2.4. *Let $f \in \sigma$ and $g = f^{-1}$. If $f \in \mathcal{M}(\alpha,\varphi)$, $g \in \mathcal{M}(\beta,\psi)$, then*

$$|a_2| \le \frac{B_1D_1\sqrt{B_1(1+2\beta) + D_1(1+2\alpha)}}{\sqrt{|\sigma B_1^2D_1^2 - (1+\alpha)^2(1+2\beta)(B_2-B_1)D_1^2 - (1+\beta)^2(1+2\alpha)(D_2-D_1)B_1^2|}} \tag{2.33}$$

and

$$2\sigma|a_3| \le B_1(3+5\beta) + D_1(1+3\alpha) + (3+5\beta)|B_2 - B_1| + \frac{(1+\beta)^2(1+3\alpha)B_1^2|D_2-D_1|}{D_1^2(1+\alpha)^2} \tag{2.34}$$

where $\sigma := 2 + 3\alpha + 3\beta + 4\alpha\beta$.

Proof. Let $f \in \mathcal{M}(\alpha,\varphi)$ and $g \in \mathcal{M}(\beta,\psi)$, $g = f^{-1}$. Then there are analytic functions $u, v : \mathbb{D} \to \mathbb{D}$, with $u(0) = v(0) = 0$, satisfying

(2.35)
$$(1-\alpha)\frac{zf'(z)}{f(z)}+\alpha\left(1+\frac{zf''(z)}{f'(z)}\right)=\varphi(u(z)),\quad (1-\beta)\frac{wg'(w)}{g(w)}+\beta\left(1+\frac{wg''(w)}{g'(w)}\right)=\psi(v(w)),$$

Since

$$(1-\alpha)\frac{zf'(z)}{f(z)}+\alpha\left(1+\frac{zf''(z)}{f'(z)}\right)=1+(1+\alpha)a_2z+(2(1+2\alpha)a_3-(1+3\alpha)a_2^2)z^2+\cdots$$

and

$$(1-\beta)\frac{wg'(w)}{g(w)}+\beta\left(1+\frac{wg''(w)}{g'(w)}\right)=1-(1+\beta)a_2w+((3+5\beta)a_2^2-2(1+2\beta)a_3)w^2+\cdots,$$

then (2.10), (2.11) and (2.35) yield

$$a_2(1+\alpha)=\frac{1}{2}B_1c_1, \tag{2.36}$$

$$2(1+2\alpha)a_3-(1+3\alpha)a_2^2=\frac{1}{2}B_1\left(c_2-\frac{c_1^2}{2}\right)+\frac{1}{4}B_2c_1^2, \tag{2.37}$$

$$-(1+\beta)a_2=\frac{1}{2}D_1b_1 \tag{2.38}$$

and

$$(3+5\beta)a_2^2-2(1+2\beta)a_3=\frac{1}{2}D_1\left(b_2-\frac{b_1^2}{2}\right)+\frac{1}{4}D_2b_1^2. \tag{2.39}$$

It follows from (2.36) and (2.38) that

$$b_1=-\frac{B_1(1+\beta)}{D_1(1+\alpha)}c_1. \tag{2.40}$$

Equations (2.36), (2.37), (2.39) and (2.40) lead to

$$a_2^2=\frac{B_1^2D_1^2[B_1(1+2\beta)c_2+D_1(1+2\alpha)b_2]}{2\sigma B_1^2D_1^2-2(1+\alpha)^2(1+2\beta)(B_2-B_1)D_1^2-2(1+\beta)^2(1+2\alpha)(D_2-D_1)B_1^2},$$

which, in view of the well-known inequalities $|b_2| \le 2$ and $|c_2| \le 2$ for functions with positive real part, gives us the desired estimate on $|a_2|$ as asserted in (2.33).

By using (2.37), (2.39) and (2.40) lead to

$$2\sigma a_3=\frac{B_1}{2}(3+5\beta)c_2+\frac{D_1}{2}(1+3\alpha)b_2+\frac{c_1^2}{4}\Big[(3+5\beta)(B_2-B_1) + \frac{(1+\beta)^2(1+3\alpha)B_1^2(D_2-D_1)}{D_1^2(1+\alpha)^2}\Big]$$

and this yields the estimate given in (2.34). □

Remark 2.2. When $\beta = \alpha$ and $\psi = \varphi$, Theorem 2.4 reduces to [1, Theorem 2.3].

Theorem 2.5. *Let $f \in \sigma$ and $g = f^{-1}$. If $f \in \mathcal{M}(\alpha, \varphi)$, $g \in \mathcal{L}(\beta, \psi)$, then*

$$|a_2| \leq \frac{B_1 D_1 \sqrt{2[B_1(3-2\beta) + D_1(1+2\alpha)]}}{\sqrt{|\sigma B_1^2 D_1^2 - 2(1+\alpha)^2(3-2\beta)(B_2 - B_1)D_1^2 - 2(2-\beta)^2(1+2\alpha)(D_2 - D_1)B_1^2|}} \tag{2.41}$$

and

$$\begin{aligned} |\sigma a_3| \leq \frac{B_1}{2}(\beta^2 - 11\beta + 16) + D_1(1+3\alpha) + \frac{1}{2}(\beta^2 - 11\beta + 16)|B_2 - B_1| \\ + \frac{(2-\beta)^2(1+3\alpha)B_1^2|D_2 - D_1|}{D_1^2(1+\alpha)^2} \end{aligned} \tag{2.42}$$

where $\sigma := 10 + 14\alpha - 7\beta + \beta^2 + 2\alpha\beta^2 - 10\alpha\beta$.

Proof. Let $f \in \mathcal{M}(\alpha, \varphi)$ and $g \in \mathcal{L}(\beta, \psi)$, $g = f^{-1}$. Then there are analytic functions $u, v : \mathbb{D} \to \mathbb{D}$, with $u(0) = v(0) = 0$, satisfying

$$(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right) = \varphi(u(z)), \quad \left(\frac{wg'(w)}{g(w)}\right)^{\beta}\left(1 + \frac{wg''(w)}{g'(w)}\right)^{1-\beta} = \psi(v(w)), \tag{2.43}$$

Since

$$(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right) = 1 + (1+\alpha)a_2 z + (2(1+2\alpha)a_3 - (1+3\alpha)a_2^2)z^2 + \cdots$$

and

$$\begin{aligned} &\left(\frac{wg'(w)}{g(w)}\right)^{\beta}\left(1 + \frac{wg''(w)}{g'(w)}\right)^{1-\beta} \\ &= 1 - (2-\beta)a_2 w + \left((8(1-\beta) + \frac{1}{2}\beta(\beta+5))a_2^2 - 2(3-2\beta)a_3\right)w^2 + \cdots, \end{aligned}$$

then (2.10), (2.11) and (2.43) yield

$$a_2(1+\alpha) = \frac{1}{2}B_1 c_1, \tag{2.44}$$

$$2(1+2\alpha)a_3 - (1+3\alpha)a_2^2 = \frac{1}{2}B_1\left(c_2 - \frac{c_1^2}{2}\right) + \frac{1}{4}B_2 c_1^2, \tag{2.45}$$

$$-(2-\beta)a_2 = \frac{1}{2}D_1 b_1 \tag{2.46}$$

and

$$[8(1-\beta)+\frac{\beta}{2}(\beta+5)]a_2^2-2(3-2\beta)a_3=\frac{1}{2}D_1\left(b_2-\frac{b_1^2}{2}\right)+\frac{1}{4}D_2b_1^2. \tag{2.47}$$

It follows from (2.44) and (2.46) that

$$b_1=-\frac{B_1(2-\beta)}{D_1(1+\alpha)}c_1. \tag{2.48}$$

Equations (2.44), (2.45), (2.47) and (2.48) lead to

$$a_2^2=\frac{B_1^2D_1^2[B_1(3-2\beta)c_2+D_1(1+2\alpha)b_2]}{\sigma B_1^2D_1^2-2(1+\alpha)^2(3-2\beta)(B_2-B_1)D_1^2-2(2-\beta)^2(1+2\alpha)(D_2-D_1)B_1^2},$$

which, in view of the well-known inequalities $|b_2|\leq 2$ and $|c_2|\leq 2$ for functions with positive real part, gives us the desired estimate on $|a_2|$ as asserted in (2.41).

By using (2.45), (2.47) and (2.48) lead to

$$\sigma a_3=\frac{B_1}{4}(\beta^2-11\beta+16)c_2+\frac{D_1}{2}(1+3\alpha)b_2+\frac{c_1^2}{4}\Big[(\beta^2-11\beta+16)(B_2-B_1)+\frac{(2-\beta)^2(1+3\alpha)B_1^2(D_2-D_1)}{D_1^2(1+\alpha)^2}\Big]$$

and this yields the estimate given in (2.42). □

Theorem 2.6. *Let $f\in\sigma$ and $g=f^{-1}$. If $f\in\mathcal{L}(\alpha,\varphi)$, $g\in\mathcal{L}(\beta,\psi)$, then*

$$|a_2|\leq\frac{B_1D_1\sqrt{2[B_1(3-2\beta)+D_1(3-2\alpha)]}}{\sqrt{|\sigma B_1^2D_1^2-2(2-\alpha)^2(3-2\beta)(B_2-B_1)D_1^2-2(2-\beta)^2(3-2\alpha)(D_2-D_1)B_1^2|}} \tag{2.49}$$

and

$$2|\sigma a_3|\leq B_1(\beta^2-11\beta+16)+D_1(8-5\alpha-\alpha^2)+(\beta^2-11\beta+16)|B_2-B_1|+\frac{(2-\beta)^2(\alpha^2+5\alpha-8)B_1^2|D_2-D_1|}{D_1^2(2-\alpha)^2} \tag{2.50}$$

where $\sigma:=24+3\alpha^2+3\beta^2-17\alpha-17\beta-2\beta\alpha^2-2\alpha\beta^2-12\alpha\beta$.

Proof. Let $f\in\mathcal{L}(\alpha,\varphi)$ and $g\in\mathcal{L}(\beta,\psi)$, $g=f^{-1}$. Then there are analytic functions $u,v:\mathbb{D}\to\mathbb{D}$, with $u(0)=v(0)=0$, satisfying

$$\left(\frac{zf'(z)}{f(z)}\right)^{\alpha}\left(1+\frac{zf''(z)}{f'(z)}\right)^{1-\alpha}=\varphi(u(z)),\quad \left(\frac{wg'(w)}{g(w)}\right)^{\beta}\left(1+\frac{wg''(w)}{g'(w)}\right)^{1-\beta}=\psi(v(w)), \tag{2.51}$$

Since

$$\left(\frac{zf'(z)}{f(z)}\right)^{\alpha}\left(1+\frac{zf''(z)}{f'(z)}\right)^{1-\alpha}$$
$$=1+(2-\alpha)a_2z+\left(2(3-2\alpha)a_3+\frac{(\alpha-2)^2-3(4-3\alpha)}{2}a_2^2\right)z^2+\cdots$$

and

$$\left(\frac{wg'(w)}{g(w)}\right)^{\beta}\left(1+\frac{wg''(w)}{g'(w)}\right)^{1-\beta}$$
$$=1-(2-\beta)a_2w+\Big((8(1-\beta)+\frac{1}{2}\beta(\beta+5))a_2^2-2(3-2\beta)a_3\Big)w^2+\cdots,$$

then (2.10), (2.11) and (2.51) yield

$$a_2(2-\alpha)=\frac{1}{2}B_1c_1, \tag{2.52}$$

$$2(3-2\alpha)a_3+\frac{1}{2}[(\alpha-2)^2-3(4-3\alpha)]a_2^2=\frac{1}{2}B_1\left(c_2-\frac{c_1^2}{2}\right)+\frac{1}{4}B_2c_1^2, \tag{2.53}$$

$$-(2-\beta)a_2=\frac{1}{2}D_1b_1 \tag{2.54}$$

and

$$[8(1-\beta)+\frac{\beta}{2}(\beta+5)]a_2^2-2(3-2\beta)a_3=\frac{1}{2}D_1\left(b_2-\frac{b_1^2}{2}\right)+\frac{1}{4}D_2b_1^2. \tag{2.55}$$

It follows from (2.52) and (2.54) that

$$b_1=-\frac{B_1(2-\beta)}{D_1(2-\alpha)}c_1. \tag{2.56}$$

Equations (2.52), (2.53), (2.55) and (2.56) lead to

$$a_2^2=\frac{B_1^2D_1^2[B_1(3-2\beta)c_2+D_1(3-2\alpha)b_2]}{\sigma B_1^2D_1^2-2(2-\alpha)^2(3-2\beta)(B_2-B_1)D_1^2-2(2-\beta)^2(3-2\alpha)(D_2-D_1)B_1^2},$$

which, in view of the well-known inequalities $|b_2|\leq 2$ and $|c_2|\leq 2$ for functions with positive real part, gives us the desired estimate on $|a_2|$ as asserted in (2.49).

By using (2.53), (2.55) and (2.56) lead to

$$2\sigma a_3=\frac{B_1}{2}(\beta^2-11\beta+16)c_2+\frac{D_1}{2}(8-5\alpha-\alpha^2)b_2+\frac{c_1^2}{4}\Big[(\beta^2-11\beta+16)(B_2-B_1)$$
$$+\frac{(2-\beta)^2(\alpha^2+5\alpha-8)B_1^2(D_2-D_1)}{D_1^2(2-\alpha)^2}\Big]$$

and this yields the estimate given in (2.50). □

Remark 2.3. When $\beta = \alpha$ and $\psi = \varphi$, Theorem 2.6 reduces to [1, Theorem 2.4].

References

[1] R. M. Ali, S. K. Lee, V. Ravichandran, and S. Supramaniam, Coefficient estimates for bi-univalent Ma-Minda starlike and convex functions, Appl. Math. Lett. **25** (2012), no. 3, 344–351.

[2] R. M. Ali, S. K. Lee, V. Ravichandran, and S. Supramaniam, The Fekete-Szegő coefficient functional for transforms of analytic functions, Bull. Iranian Math. Soc. **35** (2009), no. 2, 119–142, 276.

[3] R. M. Ali, V. Ravichandran and N. Seenivasagan, Coefficient bounds for p-valent functions, Appl. Math. Comput. **187** (2007), no. 1, 35–46.

[4] B. Bhowmik and S. Ponnusamy, Coefficient inequalities for concave and meromorphically starlike univalent functions, Ann. Polon. Math. **93** (2008), no. 2, 177–186.

[5] B. Bhowmik, S. Ponnusamy and K.J. Wirths, On the Fekete-Szegö problem for concave univalent functions, J. Math. Anal. Appl. **373** (2011), no. 2, 432–438.

[6] D. A. Brannan and T. S. Taha, On some classes of bi-univalent functions, Studia Univ. Babeş-Bolyai Math. **31** (1986), no. 2, 70–77.

[7] D. A. Brannan, J. Clunie and W. E. Kirwan, Coefficient estimates for a class of star-like functions, Canad. J. Math. **22** (1970), 476–485.

[8] P. L. Duren, *Univalent Functions*, Springer, New York, 1983.

[9] B. A. Frasin and M. K. Aouf, New subclasses of bi-univalent functions, Appl. Math. Lett. **24** (2011), no. 9, 1569–1573.

[10] A. W. Kędzierawski, Some remarks on bi-univalent functions, Ann. Univ. Mariae Curie-Skłodowska Sect. A **39** (1985), 77–81 (1988).

[11] S. S. Kumar, V. Kumar and V. Ravichandran, Estimates for the initial coefficients of bi-univalent functions, preprint.

[12] M. Lewin, On a coefficient problem for bi-univalent functions, Proc. Amer. Math. Soc. **18** (1967), 63–68.

[13] W. C. Ma and D. Minda, A unified treatment of some special classes of univalent functions, in *Proceedings of the Conference on Complex Analysis (Tianjin, 1992)*, 157–169, Conf. Proc. Lecture Notes Anal. I Int. Press, Cambridge, MA.

[14] A. K. Mishra and P. Gochhayat, Fekete-Szegö problem for a class defined by an integral operator, Kodai Math. J. **33** (2010), no. 2, 310–328.

[15] E. Netanyahu, The minimal distance of the image boundary from the origin and the second coefficient of a univalent function in $|z| < 1$, Arch. Rational Mech. Anal. **32** (1969), 100–112.

[16] T. N. Shanmugam, C. Ramachandran and V. Ravichandran, Fekete-Szegő problem for subclasses of starlike functions with respect to symmetric points, Bull. Korean Math. Soc. **43** (2006), no. 3, 589–598.

[17] H. M. Srivastava, Some inequalities and other results associated with certain subclasses of univalent and bi-univalent analytic functions, in *Nonlinear Analysis*, Springer Series on Optimization and Its Applications, Vol. 68, pp. 607-630, Springer, Berlin, New York and Heidelberg, 2012.

[18] H. M. Srivastava, A. K. Mishra and P. Gochhayat, Certain subclasses of analytic and bi-univalent functions, Appl. Math. Lett. **23** (2010), no. 10, 1188–1192.

[19] Q.-H. Xu, H.-G. Xiao, H.M. Srivastava, A certain general subclass of analytic and bi-univalent functions and associated coefficient estimate problems, App. Math. Comput. **218** (2012), no. 23, 11461–11465.

[20] Q.-H. Xu, H.-G. Xiao, H.M. Srivastava, Coefficient estimates for a certain subclass of analytic and bi-univalent functions, Appl. Math. Lett. **25** (2012), no. 6, 990–994.

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