

CORRECTIONS TO
Introduction to Smooth Manifolds

Version 3.0

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- **Page 4, second paragraph after Lemma 1.1:** Omit redundant “the.”
- **Page 11, Example 1.6:** In the third line above the second equation, change “for each j ” to “for each i .”
- **Page 12, Example 1.7, line 5:** Change “manifold” to “smooth manifold.”
- **Page 13, Example 1.11:** Just before and in the displayed equation, change $\varphi_j^\pm \circ (\varphi_i^\pm)^{-1}$ to $\varphi_i^\pm \circ (\varphi_j^\pm)^{-1}$ (twice).
- **Page 21, Problem 1-3:** Change the definition of $\tilde{\sigma}$ to $\tilde{\sigma}(x) = -\sigma(-x)$. (This is stereographic projection from the south pole.)
- **Page 24, 5th line below the heading:** “multiples” is misspelled.
- **Page 24, last paragraph before Exercise 2.1:** There is a subtle problem with the definition of smooth maps between manifolds given here, because this definition doesn’t obviously imply that smooth maps are continuous. Here’s how to fix it. Replace the third sentence of this paragraph by “We say F is a *smooth map* if for any $p \in M$, there exist charts (U, φ) containing p and (V, ψ) containing $F(p)$ such that $F(U) \subset V$ and the composite map $\psi \circ F \circ \varphi^{-1}$ is smooth from $\varphi(U)$ to $\psi(V)$. Note that this definition implies, in particular, that every smooth map is continuous: If $W \subset N$ is any open set, for each $p \in F^{-1}(W)$ we can choose a coordinate domain $V \subset W$ containing $F(p)$, and then the definition guarantees the existence of a coordinate domain U containing p such that $U \subset F^{-1}(V) \subset F^{-1}(W)$, which implies that $F^{-1}(W)$ is open.”
- **Page 25, Lemma 2.2:** Change the statement of this lemma to “*Let M, N be smooth manifolds and let $F: M \rightarrow N$ be any map. Show that F is smooth if and only if it is continuous and satisfies the following condition: Given any smooth atlases $\{(U_\alpha, \varphi_\alpha)\}$ and $\{(V_\beta, \psi_\beta)\}$ for M and N , respectively, each map $\psi_\beta \circ F \circ \varphi_\alpha^{-1}$ is smooth on its domain of definition.*”
- **Page 30, line 6:** Change “topology of \widetilde{M} ” to “topology of M .”
- **Page 31, Example 2.10(e), first line:** Change “complex” to “real.”
- **Page 36, Exercise 2.9:** Replace the first sentence of the exercise by the following: “Show that a cover $\{U_\alpha\}$ of X by precompact open sets is locally finite if and only if each U_α intersects U_β for only finitely many β .”
- **Page 39, line 5:** Insert a period after the word “manifold.”
- **Page 40, Problem 2-2:** The first sentence should read “Let $M = \overline{\mathbb{B}^n}, \dots$.”

- **Page 41, line 5 from bottom:** Change “abstract definition of” to “abstract definition of tangent vectors.”
- **Page 41, line 4 from bottom:** “showing” is misspelled.
- **Page 43, line 4:** Change \mathbb{S}^n to \mathbb{S}^{n-1} (twice).
- **Page 48, last displayed equation:** The derivative should be evaluated at $t = 0$:

$$\tilde{v}_a f = \left. \frac{d}{dt} \right|_{t=0} f(a + tv).$$

- **Page 51, line 15:** Insert “ $p \leftrightarrow \hat{p}$,” before the word “and.”
- **Page 54, two lines below the first displayed equation:** Insert “it” before “is customary.”
- **Page 57, four lines below the first displayed equation:** Delete “depending on context.”
- **Page 58, line 5:** Change (2.2) to (3.6).
- **Page 59, just below the commutative diagram:** Replace the first phrase after the diagram by “and for each $q \in U$, the restriction of Φ to E_q is a linear isomorphism from E_q to $\{q\} \times \mathbb{R}^k \cong \mathbb{R}^k$.”
- **Page 60, Exercise 3.6:** Move this exercise after the second paragraph on this page.
- **Page 60, last sentence before the heading “Vector Fields”:** Change “3.13” to “Lemma 3.12.”
- **Page 63, Lemma 3.17:** Both vector fields are mistakenly written as X in several places in this lemma and its proof. In fact, to be consistent with the surrounding text, they should have been called Y and Z . Replace the entire lemma and proof by:

Lemma 3.17. *Suppose $F: M \rightarrow N$ is a smooth map, $Y \in \mathcal{T}(M)$, and $Z \in \mathcal{T}(N)$. Then Y and Z are F -related if and only if for every smooth function f defined on an open subset of N ,*

$$Y(f \circ F) = (Zf) \circ F. \tag{3.7}$$

Proof. For any $p \in M$,

$$\begin{aligned} Y(f \circ F)(p) &= Y_p(f \circ F) \\ &= (F_* Y_p) f, \end{aligned}$$

while

$$\begin{aligned} (Zf) \circ F(p) &= (Zf)(F(p)) \\ &= Z_{F(p)} f. \end{aligned}$$

Thus (3.7) is true for all f if and only if $F_* Y_p = Z_{F(p)}$ for all p , i.e., if and only if Y and Z are F -related. \square

- **Page 64, Problem 3-2:** The third displayed equation should be

$$\alpha^{-1}(X_1, \dots, X_k) = j_{1*}X_1 + \dots + j_{k*}X_k.$$

- **Page 74, second line from bottom:** Delete the symbol γ_* .
- **Page 83, last displayed equation:** Should be changed to

$$f(q) - \tilde{f}(q) = \int_{p_0}^q \omega - \int_{p_1}^q \omega = \int_{p_0}^{p_1} \omega.$$

- **Page 86, Example 4.26, line 1:** Change “Example 4.7” to “Example 4.18.”
- **Page 95, part (f):** Change “ $(y - 2)^2 + z^2 + 1$ ” to “ $(y - 2)^2 + z^2 = 1$.”
- **Page 111, second line from bottom:** Change “ W is open” to “ $\pi(W)$ is open.”
- **Page 112, 5th line from bottom:** Change $q \in M$ to $q \in N$.
- **Page 117, second line under (5.10):** Change “observe that E has rank $k \dots$ ” to “observe that E has rank less than or equal to $k \dots$.”
- **Page 119, fourth line under the heading “Immersed Submanifolds”:** change the word “groups” to “subgroups.”
- **Page 120, 5th line after the subheading:** Insert missing right parenthesis after “topology.”
- **Page 121, line 7 from bottom:** Change “fo” to “for.”
- **Page 126, Problem 5-3:** Delete this problem. (The answer is already given in Example 5.2.)
- **Page 127, Problem 5-11:** Change the definition of S to

$$S = \{(x, y) : |x| = 1 \text{ and } |y| \leq 1, \text{ or } |y| = 1 \text{ and } |x| \leq 1\}.$$

- **Page 127, Problem 5-14:** Delete part (b).
- **Page 129, line 4 from bottom:** Change “in the sense \dots ” to “in a sense \dots .”
- **Page 130, second line from bottom:** Change $F(B_j)$ to $F(A \cap \overline{B} \cap B_j)$.
- **Page 133, proof of Theorem 6.9:** In the second paragraph of the proof, replace the first sentence by “For each i , let $\varphi_i \in C^\infty(M)$ be a bump function that is supported in W_i and identically equal to 1 on \overline{U}_i .”
- **Page 139, last displayed equation:** Change $\left(v^j \frac{\partial}{\partial x^j}\right)$ to $\left(v^j \frac{\partial}{\partial x^j} \Big|_x\right)$.
- **Page 142, three lines above the last displayed equation:** Change “a $\tilde{\delta}$ -approximation” to “ $\tilde{\delta}$ -close.”

- **Page 143, proof of Proposition 6.20, first line:** Change “ $H: M \times I \rightarrow M$ ” to “ $H: M \times I \rightarrow N$.”
- **Page 148, line 2:** Change “by continuity” to “by continuity of $\pi_G \circ \Theta^{-1}$.”
- **Page 154, paragraph 2, line 2:** Change “contained in $G_K = \{g \in G : (g \cdot K) \cap K \neq \emptyset\}$ ” to “contained in $G_{K'} = \{g \in G : (g \cdot K') \cap K' \neq \emptyset\}$, where $K' = K \cup \{p\}, \dots$.”
- **Page 154, paragraph below conditions (i) and (ii):** Change U to W (twice).
- **Page 167, Problem 7-7(c):** Add the hypothesis that $n > 1$.
- **Page 169, Problem 7-24:** Change $U(n)$ to $U(n + 1)$.
- **Page 176, proof of Proposition 8.3, lines 1, 2, and 11:** Change X to Y (three times).
- **Page 178, first full paragraph:** Add the following sentence at the end of this paragraph: “Applying this observation to $V = (V^*)^*$ and $W = (W^*)^*$ proves (b).”
- **Page 183, third display:** In the second line, change $T^{\sigma\tau}$ to $T^{\tau\sigma}$.
- **Page 183, first line after the third display:** Change “ $\eta = \sigma\tau$ ” to “ $\eta = \tau\sigma$.”
- **Page 191, Corollary 8.20:** This corollary, and the paragraph preceding it, should be moved to page 195, immediately following the proof of Proposition 8.26.
- **Page 192, first display:** Change dt to $d\varphi$ (three times).
- **Page 204, Exercise 9.1(d):** Change “independent” to “dependent.”
- **Page 206, last displayed equation:** Change $e^{123}(X, Y, X)$ to $e^{123}(X, Y, Z)$.
- **Page 220, Exercise 9.7:** Change the statement to “Let (V, ω) be a $2n$ -dimensional symplectic vector space, \dots .”
- **Page 222, line 6 from bottom:** Change “pullback” to “dual map.”
- **Page 222, line 5 from bottom:** Change $T_{(p,\eta)}(T^*M)$ to $T_{(p,\eta)}^*(T^*M)$.
- **Page 225, Problem 9-1:** In the last line, change $\det(v_1, \dots, v_n)$ to $|\det(v_1, \dots, v_n)|$.
- **Page 225, Problem 9-6(a):** Change the definition of the coordinates to “ $(x, y, z) = (\rho \sin \varphi \cos \theta, \rho \sin \varphi \sin \theta, \rho \cos \varphi)$ ” [insert missing factors of ρ].
- **Page 227, Problem 9-9:** Replace the first two sentences of the problem by the following: “Let (V, ω) be a symplectic vector space of dimension $2n$. Show that for every symplectic, isotropic, coisotropic, or Lagrangian subspace $S \subset V$, there exists a symplectic basis (A_i, B_i) for V with the following property:”
- **Page 229, 9th line from bottom:** Delete the redundant “which.”
- **Page 235, 3rd line from bottom:** Delete the word “locally.”

- **Page 239, second display:** Two occurrences of dx^i should be changed to dx^1 , so the equation reads:

$$\begin{aligned}(N \lrcorner \Omega)|_{\partial M} &= f \sum_{i=1}^n (-1)^{i-1} dx^i(N) dx^1|_{\partial M} \wedge \cdots \wedge \widehat{dx^i}|_{\partial M} \wedge \cdots \wedge dx^n|_{\partial M} \\ &= (-1)^{n-1} f dx^n(N) dx^1|_{\partial M} \wedge \cdots \wedge dx^{n-1}|_{\partial M}.\end{aligned}$$

- **Page 239, third line from bottom:** Change \mathbb{R}^n to \mathbb{R}^{n-1} .
- **Page 249, equation (10.6):** Change ω_i to ω_n (twice).
- **Page 251, third displayed equation:** Should be changed to

$$\int_{\gamma} df = \int_{[a,b]} \gamma^* df = \int_M df = f(\gamma(b)) - f(\gamma(a)).$$

- **Page 256, equation (10.10):** Change σ_1 and σ_0 to $\tilde{\sigma}_1$ and $\tilde{\sigma}_0$, respectively.
- **Pages 259–267:** Change every occurrence of $\langle \cdot, \cdot \rangle$ to $\langle \cdot, \cdot \rangle_g$.
- **Page 261, proof of Lemma 10.38, 7th line:** Change “Corollary 10.40” to “Proposition 10.37.”
- **Page 261, fourth display and the two sentences following it:** Change \widetilde{M} to S (four times).
- **Page 262, second line from bottom:** Change “domain with smooth boundary” to “regular domain.”
- **Page 263, second paragraph after the subheading:** Add the following sentence at the end of the paragraph: “Since β takes smooth sections to smooth sections, it also defines an isomorphism (which we denote by the same symbol) $\beta: \mathcal{T}(M) \rightarrow \mathcal{A}^2(M)$.”
- **Page 267, Problem 10-16:** In parts (b) and (c), change “connected” to “compact and connected.”
- **Page 267, Problem 10-17:** Add the phrase “(without boundary)” after “Riemannian manifold.”
- **Page 271, line 5:** Change “Example 4.23” to “Example 4.26,” and change “closed 1-form” to “1-form.”
- **Page 274, line above equation (11.3):** Interchange M and N .
- **Page 275, two lines above Case I:** Change “can be written as ... ” to “can be written locally as”
- **Page 275, Case I:** In the first line, delete the phrase “because $dt \wedge dt = 0$.”

- **Page 275, line above the last display:** Change this line to “On the other hand, because $dt \wedge dt = 0$,”.
- **Page 280, proof of Theorem 11.15, fourth line:** Change $I[a\omega]$ to $I[a\Omega]$.
- **Page 281, proof of Theorem 11.18, third line:** Change “ $\alpha: \widetilde{M} \rightarrow M$ ” to “ $\alpha: \widetilde{M} \rightarrow \widetilde{M}$.”
- **Page 284, line 5:** Change $y < R$ to $y < -R$, and change \widetilde{F} to E .
- **Page 289, line above equation (11.11):** Change $\mathcal{A}^p(V)$ to $\mathcal{A}^p(U)$.
- **Page 293, line 7:** Change $\sigma \circ F$ to $F \circ \sigma$.
- **Page 295, equation (11.18):** Change δ to ∂^* (twice).
- **Page 296, third line below the subheading:** Change “ p -form on M ” to “ p -form ω on M .”
- **Page 298, second line after equation (11.19):** Change “ $(p-1)$ -chain” to “ $(p+1)$ -chain.”
- **Page 299, proof of Lemma 11.32, last line:** Change this sentence to “This implies $\mathcal{J}(F^*[\omega])[\sigma] = \mathcal{J}[\omega][F \circ \sigma] = \mathcal{J}[\omega](F_*[\sigma]) = F^*(\mathcal{J}[\omega])[\sigma]$, which was to be proved.”
- **Page 299, proof of Lemma 11.33, fifth line:** Change “ $(p-1)$ -form” to “ p -form,” and change “ p -chain” to “smooth $(p-1)$ -chain.”
- **Page 299, line 3 from bottom:** Change the first $\mathcal{A}^p(U)$ to $\mathcal{A}^{p-1}(U)$, and change the second to $\mathcal{A}^{p-1}(V)$.
- **Page 299, line 2 from bottom:** Change “smooth simplices” to “smooth chains.”
- **Page 300, proof of Theorem 11.34, Step 1:** In the second line, change 11.27(c) to 11.27(b).
- **Page 300, last line:** Change “spanned” to “generated.” Also, change “0-simplex” to “singular 0-simplex.”
- **Page 302, last line before Step 5:** Replace the last sentence by “Finally, $U \cap V$ is de Rham because it is the disjoint union of the sets $B_m \cap B_{m+1}$, each of which has a finite de Rham cover consisting of sets of the form $U_\alpha \cap U_\beta$, where U_α and U_β are basis sets used to define B_m and B_{m+1} , respectively. Thus $U \cup V$ is de Rham by Step 3.”
- **Page 303, Problem 11-2(b):** In the displayed equation, change P_i to $P_i(\omega)$.
- **Page 304, Problem 11-4, line 4:** Change “A smooth submanifold” to “A smooth oriented submanifold.”
- **Page 304, Problem 11-4, line 6:** Assume $S \subset M$ is compact.
- **Page 304, Problem 11-4, line 9:** Change 1985 to 1982.
- **Page 309, fifth line after the first display:** Change “This the reason” to “This is the reason”

- **Page 313, two lines above Lemma 12.7:** Change “local” to “global.”
- **Page 313, second display:** Add the following condition before the two that are already listed:

$$\text{If } s \in \mathcal{D}_p, \text{ then } \mathcal{D}_{\theta(s,p)} = \{t \in \mathbb{R} : s + t \in \mathcal{D}_p\}.$$

- **Page 329, proof of Lemma 13.1:** Replace the first three sentences of the proof by the following: “First we prove that $[V, W]_p$ is a tangent vector, i.e., a linear derivation of $C^\infty(M)$ at p . It is obviously linear over \mathbb{R} , so only the product rule needs to be checked.”
- **Page 336, proof of Proposition 13.9, second line:** Change $(-t, p)$ to (t, p) .
- **Page 338, line below the first displayed equation:** Change W to S .
- **Page 338, second and fourth displayed equations:** Change $(0, (0, \dots, 0, x^{k+1}, \dots, x^n))$ to $(0, \dots, 0, x^{k+1}, \dots, x^n)$ (three times).
- **Page 338, fourth displayed equation:** Make the following substitution (four times):

$$\frac{\partial}{\partial x^i} \mapsto \frac{\partial}{\partial x^i} \Big|_{x_0}.$$

- **Page 338, second line after the fourth displayed equation:** Change $(V_i)_{\psi(x)} f$ to $(V_i)_{\psi(x_0)} f$.
- **Page 339, Proposition 13.11(e):** Change η to ω .
- **Page 340, last displayed equation:** Change $df(y)$ to $df(Y)$ in the first line.
- **Page 341, proof of Proposition 13.14:** In the second paragraph of the proof, change “Proposition 13.11(d)” to “Proposition 13.11(b).”
- **Page 342, last displayed equation:** In the second line of the display, change $YT_{ij} dx^i$ to $YT_{ij} dx^i \otimes dx^j$.
- **Page 343, proof of Lemma 13.17, last line:** Change $(\mathcal{L}_V W)$ to $(\mathcal{L}_X \tau)$.
- **Page 343, proof of Proposition 13.18, first line:** Change “ $\theta_t^* \tau = \tau$ for all t ” to “ $\theta_t^* \tau = \tau$ on the domain of θ_t for each t .”
- **Page 351, displayed equations:** Change H to X (three times).
- **Page 360, sentence before Proposition 14.6:** Change “consequences” to “consequence,” and change “lemma” to “proposition.”
- **Page 365, Example 14.11(g):** Change “z-axis” to “y-axis.”
- **Page 367, line 3:** Delete the parenthetical remark.
- **Page 367, subheading after the proof of Lemma 14.14:** *Frobenius* should be capitalized.

- **Page 374, second display:** In the last term of the equation, change V_{hg} to \tilde{V}_{hg} .
- **Page 376, line 9 from bottom:** Add missing right parenthesis in $\text{Lie}(\text{GL}(n, \mathbb{R}))$.
- **Page 388, last line:** Delete “ $\subset \mathfrak{gl}(n, \mathbb{R})$.”
- **Page 394, line 3:** Change $(\exp Y_i)^n$ to $(\exp Y_i)^{n_i}$.
- **Page 426, statement of Theorem A.20:** Change the first sentence to “... *such that the partial derivatives $\partial f / \partial x^i : U \times [a, b] \rightarrow \mathbb{R}$ are also continuous for $i = 1, \dots, n$.*”