PROBLEMS IN ELEMENTARY NUMBER THEORY

Hojoo Lee, Version 0.8 [2003/12/17]

God does arithmetic. **C. F. Gauss**

$1,\ -24,\ 252,\ -1472,\ 4830,\ -6048,\ -16744,\ 84480,$
$-113643,\ -115920,\ 534612,\ -370944,\ -577738,$
$401856,\ 1217160,\ 987136,\ -6905934,\ 2727432,\ 10661420,$
$-7109760,\ -4219488,\ -12830688,\ 18643272,\ 21288960,$
$-25499225,\ 13865712,\ -73279080,\ 24647168,\ \cdots$

Contents

1. Introduction

The heart of Mathematics is its problems. Paul Halmos

1. Introduction *Number Theory* is a beautiful branch of Mathematics. The purpose of this book is to present a collection of interesting questions in *Number Theory*. Many of the problems are mathematical competition problems all over the world including IMO, APMO, APMC, Putnam, etc. I have given sources of the problems at the end of the book. The book is available at

http://my.netian.com/∼ideahitme/eng.html

2. How You Can Help This is an unfinished manuscript. I would greatly appreciate hearing about any errors in the book, even minor ones. I also would like to hear about

a) challenging problems in *Elementary* Number Theory,
b) interesting problems concerned with the *History* of Number Theory,
c) *beautiful* results that are *easily* stated, and
d) *remarks* on the problems in the book.

You can send all comments to the author at **hojoolee@korea.com** .

3. Acknowledgments The author would like to thank the following people : *Alexander A. Zenkin*, *Arne Smeets*, *Curt A. Monash*, *Don Coppersmith*, *Edward F. Schaefer*, *Edwin Clark*, *George Baloglou*, *Ha Duy Hung*, *Keith Matthews*, *Leonid G. Fel*, *Orlando Doehring*, and *Tran Nam Dung*

2. Notations and Abbreviations

Notations

$\mathbf{Z}$ is the set of integers
$\mathbf{N}$ is the set of positive integers
$\mathbf{N}_0$ is the set of nonnegative integers
$m|n$ n is a multiple of m.
$\sum_{d|n} f(d) = \sum_{d\in D(n)} f(d)$ $(D(n) = \{d \in \mathbf{N} : d|n\})$
$[x]$ the greatest integer less than or equal to x
$\{x\}$ the fractional part of x $(\{x\} = x - [x])$
$\phi(n)$ the number of positive integers less than n that are relatively prime to n
$\pi(x)$ the number of primes p with $p \le x$

Abbreviations

AIME American Invitational Mathematics Examination
APMO Asian Pacific Mathemats Olympiads
IMO International Mathematical Olympiads

3. Divisibility Theory I

Why are numbers beautiful? It's like asking why is Beethoven's Ninth Symphony beautiful. If you don't see why, someone can't tell you. I know numbers are beautiful. If they aren't beautiful, nothing is. Paul Erdös

A 1. *(Kiran S. Kedlaya) Show that if x, y, z are positive integers, then $(xy+1)(yz+1)(zx+1)$ is a perfect square if and only if $xy+1$, $yz+1$, $zx+1$ are all perfect squares.*

A 2. *Find infinitely many triples (a, b, c) of positive integers such that a, b, c are in arithmetic progression and such that $ab+1$, $bc+1$, and $ca+1$ are perfect squares.*

A 3. *Let a and b be positive integers such that $ab+1$ divides a^2+b^2. Show that*

$$\frac{a^2+b^2}{ab+1}$$

is the square of an integer. [1]

A 4. *(Shailesh Shirali) If a, b, c are positive integers such that*

$$0 < a^2+b^2-abc \le c,$$

show that a^2+b^2-abc is a perfect square. [2]

A 5. *Let x and y be positive integers such that xy divides x^2+y^2+1. Show that* [3]

$$\frac{x^2+y^2+1}{xy} = 3.$$

A 6. *(R. K. Guy and R. J. Nowakowki) (i) Find infinitely many pairs of integers a, b with $1 < a < b$, so that ab exactly divides a^2+b^2-1. (ii) With a and b as in (i), what are the possible values of*

$$\frac{a^2+b^2-1}{ab}.$$

A 7. *Let n be a positive integer such that $2+2\sqrt{28n^2+1}$ is an integer. Show that $2+2\sqrt{28n^2+1}$ is the square of an integer.*

A 8. *The integers a and b have the property that for every nonnegative integer n the number of $2^n a+b$ is the square of an integer. Show that $a=0$.*

A 9. *Prove that among any ten consecutive positive integers at least one is relatively prime to the product of the others.*

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[1] See **J34**

[2] This is a generalization of **A3** ! Indeed, $a^2+b^2-abc=c$ implies that $\frac{a^2+b^2}{ab+1}=c\in\mathbf{N}$.

[3] See **J26**

A 10. *Let n be a positive integer with $n \geq 3$. Show that*

$$n^{n^{n^n}} - n^{n^n}$$

is divisible by 1989.

A 11. *Let a, b, c, d be integers. Show that the product*

$$(a-b)(a-c)(a-d)(b-c)(b-d)(c-d)$$

is divisible by 12. [4]

A 12. *Let k, m, n be natural numbers such that $m+k+1$ is a prime greater than $n+1$. Let $c_s = s(s+1)$. Prove that the product $(c_{m+1}-c_k)(c_{m+2}-c_k)\cdots(c_{m+n}-c_k)$ is divisible by the product $c_1c_2\cdots c_n$.*

A 13. *Show that for all prime numbers p*

$$Q(p) = \prod_{k=1}^{p-1} k^{2k-p-1}$$

is an integer.

A 14. *Let n be an integer with $n \geq 2$. Show that n does not divide $2^n - 1$.*

A 15. *Let $k \geq 2$ and $n_1, n_2, \cdots, n_k \geq 1$ be natural numbers having the property $n_2 | 2^{n_1} - 1$, $n_3 | 2^{n_2} - 1$, $\cdots$, $n_k | 2^{n_{k-1}} - 1$, $n_1 | 2^{n_k} - 1$. Show that $n_1 = n_2 = \cdots = n_k = 1$.*

A 16. *Determine if there exists a positive integer n such that n has exactly* 2000 *prime divisors and $2^n + 1$ is divisible by n.*

A 17. *Let m and n be natural numbers such that*

$$A = \frac{(m+3)^n + 1}{3m}.$$

is an integer. Prove that A is odd.

A 18. *Let m and n be natural numbers and let $mn+1$ be divisible by* 24*. Show that $m+n$ is divisible by* 24*, too.*

A 19. *Let $f(x) = x^3 + 17$. Prove that for each natural number $n \geq 2$, there is a natural number x for which $f(x)$ is divisible by 3^n but not 3^{n+1}.*

A 20. *Determine all positive integers n for which there exists an integer m so that $2^n - 1$ divides $m^2 + 9$.*

A 21. *Let n be a positive integer. Show that the product of n consecutive integers is divisible by $n!$*

A 22. *Prove that the number*

$$\sum_{k=0}^{n} \binom{2n+1}{2k+1} 2^{3k}$$

is not divisible by 5 *for any integer $n \geq 0$.*

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[4]There is a strong generalization. See **J1**

A 23. *(Wolstenholme's Theorem) Prove that if*

$$1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{p-1}$$

is expressed as a fraction, where $p \geq 5$ is a prime, then p^2 divides the numerator.

A 24. *If p is a prime number greater than 3 and $k = [\frac{2p}{3}]$. Prove that the sum*

$$\binom{p}{1}+\binom{p}{2}+\cdots+\binom{p}{k}$$

is divisible by p^2.

A 25. *Show that $\binom{2n}{n} | LCM[1, 2, \cdots, 2n]$ for all positive integers n.*

A 26. *Let m and n be arbitrary non-negative integers. Prove that*

$$\frac{(2m)!(2n)!}{m!n!(m+n)!}$$

is an integer. $(0! = 1)$.

A 27. *Show that the coefficients of a binomial expansion $(a+b)^n$ where n is a positive integer, are odd, if and only if n is of the form $2^k - 1$ for some positive integer k.*

A 28. *Prove that the expression*

$$\frac{gcd(m,n)}{n}\binom{n}{m}$$

is an integer for all pairs of positive integers $n \geq m \geq 1$.

A 29. *For which positive integers k, is it true that there are infinitely many pairs of positive integers (m, n) such that*

$$\frac{(m+n-k)!}{m!n!}$$

is an integer ?

A 30. *Show that if $n \geq 6$ is composite, then n divides $(n-1)!$.*

A 31. *Show that there exist infinitely many positive integers n such that $n^2 + 1$ divides $n!$.*

A 32. *Let p and q be natural numbers such that*

$$\frac{p}{q} = 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots-\frac{1}{1318}+\frac{1}{1319}.$$

Prove that p is divisible by 1979.

A 33. *Let $b > 1$, a and n be positive integers such that $b^n - 1$ divides a. Show that in base b, the number a has at least n non-zero digits.*

A 34. *Let $p_1, p_2, \cdots, p_n$ be distinct primes greater than 3. Show that $2^{p_1 p_2 \cdots p_n}+1$ has at least 4^n divisors.*

A 35. *Let $p \geq 5$ be a prime number. Prove that there exists an integer a with $1 \leq a \leq p-2$ such that neither $a^{p-1}-1$ nor $(a+1)^{p-1}-1$ is divisible by p^2.*

A 36. *An integer $n > 1$ and a prime p are such that n divides $p-1$, and p divides n^3-1. Show that $4p+3$ is the square of an integer.*

A 37. *Let n and q be integers with $n \geq 5$, $2 \leq q \leq n$. Prove that $q-1$ divides $\left[\frac{(n-1)!}{q}\right]$.*

A 38. *If n is a natural number, prove that the number $(n+1)(n+2)\cdots(n+10)$ is not a perfect square.*

A 39. *Let p be a prime with $p > 5$, and let $S = \{p - n^2 | n \in \mathbf{N}, n^2 < p\}$. Prove that S contains two elements a, b such that $1 < a < b$ and a divides b.*

A 40. *Let n be a positive integer. Prove that the following two statements are equivalent.*

- ◦ *n is not divisible by 4*
- ◦ *There exist $a, b \in \mathbf{Z}$ such that a^2+b^2+1 is divisible by n.*

A 41. *Determine the greatest common divisor of the elements of the set*

$$\{n^{13} - n | n \in \mathbf{Z}\}.$$

A 42. *Show that there are infinitely many composite n such that $3^{n-1}-2^{n-1}$ is divisible by n*

A 43. *Suppose that 2^n+1 is an odd prime for some positive integer n. Show that n must be a power of 2.*

A 44. *Suppose that p is a prime number and is greater than 3. Prove that $7^p - 6^p - 1$ is divisible by 43.*

A 45. *Suppose that $4^n + 2^n + 1$ is prime for some positive integer n. Show that n must be a power of 3.*

A 46. *Let b, m, n be positive integers $b > 1$ and m and n are different. Suppose that $b^m - 1$ and $b^n - 1$ have the same prime divisors. Show that $b+1$ must be a power of 2.*

A 47. *Show that a and b have the same parity if and only if there exist integers c and d such that $a^2+b^2+c^2+1 = d^2$.*

A 48. *Let n be a positive integer with $n > 1$. Prove that*

$$\frac{1}{2}+\cdots+\frac{1}{n}$$

is not an integer.

A 49. *Let n be a positive integer. Prove that*

$$\frac{1}{3}+\cdots+\frac{1}{2n+1}$$

is not an integer.

A 50. *Prove that there is no positive integer n such that, for $k = 1, 2, \cdots, 9$, the leftmost digit (in decimal notation) of $(n+k)!$ equals k.*

A 51. *Show that every integer $k > 1$ has a multiple less than k^4 whose decimal expansion has at most four distinct digits.*

A 52. *Let a, b, c and d be odd integers such that $0 < a < b < c < d$ and $ad = bc$. Prove that if $a + d = 2^k$ and $b + c = 2^m$ for some integers k and m, then $a = 1$.*

A 53. *Let d be any positive integer not equal to 2, 5, or 13. Show that one can find distinct a, b in the set $\{2, 5, 13, d\}$ such that $ab - 1$ is not a perfect square.*

A 54. *Suppose that x, y, z are positive integers with $xy = z^2 + 1$. Prove that there exist integers a, b, c, d such that $x = a^2 + b^2$, $y = c^2 + d^2$, $z = ac + bd$.*

A 55. *A natural number n is said to have the property P, if whenever n divides $a^n - 1$ for some integer a, n^2 also necessarily divides $a^n - 1$.*

(a) Show that every prime number n has propery P.
(b) Show that there are infinitely many composite numbers n that possess property P.

A 56. *Show that for every natural number n the product*

$$\left(4-\frac{2}{1}\right)\left(4-\frac{2}{2}\right)\left(4-\frac{2}{3}\right)\cdots\left(4-\frac{2}{n}\right)$$

is an integer.

A 57. *Let a, b, c be integers such that $a + b + c$ divides $a^2 + b^2 + c^2$. Prove that there are infinitely many positive integers n such that $a + b + c$ divides $a^n + b^n + c^n$.*

A 58. *Prove that for every positive integer n the following proposition holds : The number 7 is a divisor of $3^n + n^3$ if and olny if 7 is a divisor of $3^n n^3 + 1$.*

A 59. *Let $k \geq 14$ be an integer, and let p_k be the largest prime number which is strictly less than k. You may assume that $p_k \geq 3k/4$. Let n be a composite integer. Prove :*

(a) if $n = 2p_k$, then n does not divide $(n-k)!$
(b) if $n > 2p_k$, then n divides $(n-k)!$.

A 60. *Suppose that n has (at least) two essentially distinct representations as a sum of two squares. Specifically, let $n = s^2 + t^2 = u^2 + v^2$, where $s \geq t \geq 0$, $u \geq v \geq 0$, and $s > u$. If $d = gcd(su - tv, n)$, show that d is a propor divisor of n.*

A 61. *Let r and m be positive integers, and define*

$$P_r(m) = \prod_{n \neq m} \frac{n^r - m^r}{n^r + m^r}.$$

(a) Show that $P_1(m) = 0$ and that

$$P_3(m) = (-1)^m \frac{2}{3} (m!)^2 \prod_{n=1}^{m} \frac{n-m}{n^3+m^3}.$$

(b) Show that $P_2(m) = (-1)^m \frac{m\pi}{\sinh(m\pi)}$ and that, more generally, $P_{2s}(m)$ is given by

$$(-1)^{m+1} \frac{2^{\epsilon} m\pi}{s} (\sinh(m\pi))^{(-1)^s} \times$$
$$\prod_{j=1}^{n} \left(\cosh \left(2\pi m \sin \frac{j\pi}{2s} \right) - \cos \left(2\pi m \cos \frac{j\pi}{2s} \right) \right)^{(-1)^j},$$

where $\epsilon = \frac{1+(-1)^s}{2}$.

A 62. *Prove that there exist an infinite number of ordered pairs (a, b) of integers such that for every positive integer t, the number $at+b$ is a triangular number if and only if t is a triangular number. (The triangular numbers are the $t_n = n(n+1)/2$ with n in $\{0, 1, 2, \dots\}$.)*

A 63. *For any positive integer $n > 1$, let $p(n)$ be the greatest prime divisor of n. Prove that there are infinitely many positive integers n with $p(n) < p(n+1) < p(n+2)$.*

A 64. *Let $p(n)$ be the greatest odd divisor of n. Prove that*

$$\frac{1}{2^n} \sum_{k=1}^{2^n} \frac{p(k)}{k} > \frac{2}{3}.$$

A 65. *There is a large pile of cards. On each card one of the numbers 1, 2, $\cdots$, n is written. It is known that the sum of all numbers of all the cards is equal to $k \cdot n!$ for some integer k. Prove that it is possible to arrange cards into k stacks so that the sum of numbers written on the cards in each stack is equal to $n!$.*

A 66. *The last digit of the number $x^2 + xy + y^2$ is zero (where x and y are positive integers). Prove that two last digits of this numbers are zeros.*

A 67. *Clara computed the product of the first n positive integers and Valerid computed the product of the first m even positive integers, where $m \geq 2$. They got the same answer. Prove that one of them had made a mistake.*

4. Divisibility Theory II

Number theorists are like lotus-eaters - having tasted this food they can never give it up. Leopold Kronecker

B 1. *Determine all integers $n > 1$ such that*

$$\frac{2^n + 1}{n^2}$$

is an integer.

B 2. *Determine all pairs (n, p) of nonnegative integers such that*

- ◦ *p is a prime,*
- ◦ *$n < 2p$, and*
- ◦ *$(p-1)^n + 1$ is divisible by n^{p-1}.*

B 3. *Determine all pairs (n, p) of positive integers such that*

- ◦ *p is a prime, $n > 1$,*
- ◦ *$(p-1)^n + 1$ is divisible by n^{p-1}.* [5]

B 4. *Find an integer n, where $100 \le n \le 1997$, such that*

$$\frac{2^n + 2}{n}$$

is also an integer.

B 5. *Find all triples (a, b, c) such that $2^c - 1$ divides $2^a + 2^b + 1$.*

B 6. *Find all integers a, b, c with $1 < a < b < c$ such that*

$$(a-1)(b-1)(c-1) \quad \text{is a divisor of} \quad abc - 1.$$

B 7. *Find all positive integers, representable uniquely as*

$$\frac{x^2 + y}{xy + 1},$$

where x, y are positive integers.

B 8. *Determine all ordered pairs (m, n) of positive integers such that*

$$\frac{n^3 + 1}{mn - 1}$$

is an integer.

B 9. *Determine all pairs of integers (a, b) such that*

$$\frac{a^2}{2a^2b - b^3 + 1}$$

is a positive integer.

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[5]The anwser is $(n, p) = (2, 2), (3, 3)$. Note that this problem is a very *nice* generalization of the above two IMO problems B1 and B2 !

B 10. *Find all pairs of positive integers $m, n \geq 3$ for which there exist infinitely many positive integers a such that*

$$\frac{a^m + a - 1}{a^n + a^2 - 1}$$

is itself an integer.

B 11. *Determine all triples of positive integers (a, m, n) such that $a^m + 1$ divides $(a + 1)^n$.*

B 12. *Which integers are represented by $\frac{(x+y+z)^2}{xyz}$ where x, y, and z are positive integers?*

B 13. *Find all $n \in \mathbf{N}$ such that $[\sqrt{n}]|n$.*

B 14. *Determine all $n \in \mathbf{N}$ for which (i) n is not the square of any integer, and (ii) $[\sqrt{n}]^3$ divides n^2.*

B 15. *Find all $n \in \mathbf{N}$ such that $2^{n-1}|n!$.*

B 16. *Find all positive integers (x, n) such that $x^n + 2^n + 1$ is a divisor of $x^{n+1} + 2^{n+1} + 1$.*

B 17. *Find all positive integers n such that $3^n - 1$ is divisible by 2^n.*

B 18. *Find all positive integers n such that $9^n - 1$ is divisible by 7^n.*

B 19. *Determine all pairs (a, b) of integers for which $a^2 + b^2 + 3$ is divisible by ab.*

B 20. *Determine all pairs (x, y) of positive integers with $y|x^2+1$ and $x|y^3+1$.*

B 21. *Determine all pairs (a, b) of positive integers such that $ab^2 + b + 7$ divides $a^2b + a + b$.*

B 22. *Let a and b be positive integers. When $a^2 + b^2$ is divided by $a + b$, the quotient is q and the remainder is r. Find all pairs (a, b) such that $q^2 + r = 1977$.*

B 23. *Find the largest positive integer n such that n is divisible by all the positive integers less than $n^{1/3}$.*

B 24. *Find all $n \in \mathbf{N}$ such that $3^n - n$ is divisible by 17.*

B 25. *Suppose that a, b are natural numbers such that*

$$p = \frac{4}{b}\sqrt{\frac{2a - b}{2a + b}}$$

is a prime number. What is the maximum possible value of p?

B 26. *Find all positive integer N which have the following properties*

- *N has exactly 16 positive divisors $1 = d_1 < d_2 < \cdots < d_{15} < d_{16} = N$,*
- *The divisor with d_2 is equal to $(d_2 + d_4)d_6$.*

B 27. *Find all positive integers n that have exactly 16 positive integral divisors $d_1, d_2 \cdots, d_{16}$ such that $1 = d_1 < d_2 < \cdots < d_{16} = n$, $d_6 = 18$, and $d_9 - d_8 = 17$.*

B 28. *Suppose that n is a positive integer and let*

$$d_1 < d_2 < d_3 < d_4$$

be the four smallest positive integer divisors of n. Find all integers n such that

$$n = {d_1}^2 + {d_2}^2 + {d_3}^2 + {d_4}^2.$$

B 29. *Let $1 = d_1 < d_2 < \cdots < d_k = n$ be all different divisors of positive integer n written in ascending order. Determine all n such that ${d_7}^2 + {d_{10}}^2 = \left(\frac{n}{d_{22}}\right)^2$.*

B 30. *Let $n \geq 2$ be a positive integer, with divisors*

$$1 = d_1 < d_2 < \cdots < d_k = n \ .$$

Prove that

$$d_1 d_2 + d_2 d_3 + \cdots + d_{k-1} d_k$$

is always less than n^2, and determine when it is a divisor of n^2.

B 31. *Find all positive integers n such that*

(a) n has exactly 6 positive divisors $1 < d_1 < d_2 < d_3 < d_4 < n$,
(b) $1 + n = 5(d_1 + d_2 + d_3 + d_4)$.

B 32. *Find all composite numbers n, having the property : each divisor d of n $(d \neq 1, n)$ satisfies inequalties $n - 20 \leq d \leq n - 12$.*

B 33. *Determine all three-digit numbers N having the property that N is divisible by 11, and $\frac{N}{11}$ is equal to the sum of the squares of the digits of N.*

B 34. *When 4444^{4444} is written in decimal notation, the sum of its digits is A. Let B be the sum of the digits of A. Find the sum of the digits of B. (A and B are written in decimal notation.)*

B 35. *A wobbly number is a positive integer whose digits in base 10 are alternatively non-zero and zero the units digit being non-zero. Determine all positive integers which do not divide any wobbly number.*

B 36. *Let n be a composite natural number and p be a proper divisor of n. Find the binary representatoin of the smallest natural number N such that*

$$\frac{(1 + 2^p + 2^{n-p})N - 1}{2}$$

is an integer.

B 37. *Find the smallest positive integer n such that*

(i) n has exactly 144 distinct positive divisors, and
(ii) there are ten consecutive integers among the positive divisors of n.

B 38. *Determine the least possible value of the natural number n such that $n!$ ends in exactly 1987 zeros.*

B 39. *Find four positive integers, each not exceeding 70000 and each having more than 100 divisors.*

B 40. *For each integer $n > 1$, let $p(n)$ denote the largest prime factor of n. Determine all triples (x, y, z) of distinct positive integers satisfying*

(i) x, y, z are in arithmetic progression, and
(ii) $p(xyz) \leq 3$.

B 41. *Find all positive integers a and b such that*

$$\frac{a^2+b}{b^2-a} \quad \text{and} \quad \frac{b^2+a}{a^2-b}$$

are both integers.

B 42. *For each positive integer n, write the sum $\sum_{m=1}^{n} 1/m$ in the form p_n/q_n, where p_n and q_n are relatively prime positive integers. Determine all n such that 5 does not divide q_n.*

B 43. *Find all natural numbers n such that the number $n(n+1)(n+2)(n+3)$ has exactly three prime divisors.*

B 44. *Prove that there exist infinitely many pairs (a, b) of relatively prime positve integers such that*

$$\frac{a^2-5}{b} \text{ and } \frac{b^2-5}{a}$$

are both positive integers.

B 45. *Determine all triples (l, m, n) of distinct positive integers satisfying*

$$gcd(l,m)^2 = l+m, gcd(m,n)^2 = m+n, \text{ and } gcd(n,l)^2 = n+l.$$

5. Arithmetic in Z_n

Mathematics is the queen of the sciences and number theory is the queen of Mathematics. Johann Carl Friedrich Gauss

5.1. **Primitive Roots.**

C 1. *Let n be a positive integer. Show that there are infinitely many primes p such that the smallest positive primitive root of p is greater than n.*

C 2. *Let p be a prime with $p > 4\left(\frac{p-1}{\phi(p-1)}\right)^2 2^{2k}$, where k denotes the number of distinct prime divisors of $p-1$, and let M be an intger. Prove that the set of integers $M+1$, $M+2$, $\cdots$, $M+2\left[\frac{p-1}{\phi(p-1)}2^k\sqrt{p}\right]-1$ contains a primitive root to modulus p.*

C 3. *Show that for each odd prime p, there is an integer g such that $1 < g < p$ and g is a primitive root modulo p^n for every positive integer n.*

C 4. *Let g be a Fibonacci primitive root $(\mathrm{mod}\ \ p)$. i.e. g is a primitive root $(\mathrm{mod}\ \ p)$ satisfying $g^2 \equiv g+1 (\mathrm{mod}\ p)$. Prove that*

(a) $g-1$ is also a primitive root $(\mathrm{mod}\ \ p)$.
(b) if $p = 4k+3$, then $(g-1)^{2k+3} \equiv g-2 (\mathrm{mod}\ p)$ and deduce that $g-2$ is also a primitive root $(\mathrm{mod}\ \ p)$.

C 5. *If $g_1, \cdots, g_{\phi(p-1)}$ are the primitive roots mod p in the range $1 < g \le p-1$, prove that*

$$\sum_{i=1}^{\phi(p-1)} g_i \equiv \mu(p-1)(\mathrm{mod}\ p).$$

C 6. *Suppose that m does not have a primitive root. Show that*

$$a^{\frac{\phi(m)}{2}} \equiv -1\ (\mathrm{mod}\ m)$$

for every a relatively prime m.

C 7. *Suppose that $p > 3$ is prime. Prove that the products of the primitive roots of p between 1 and $p-1$ is congruent to 1 modulo p.*

5.2. **Qudratic Residues.**

C 8. *Find all positive integers n that are quadratic residues modulo all primes greater than n.*

C 9. *The positive integers a and b are such that the numbers $15a+16b$ and $16a-15b$ are both squares of positive integers. What is the least possible value that can be taken on by the smaller of these two squares?*

C 10. *Let p be an odd prime number. Show that the smallest positive quadratic nonresidue of p is smaller than $\sqrt{p}+1$.*

C 11. *Let M be an integer, and let p be a prime with $p > 25$. Show that the sequence M, $M+1$, $\cdots$, $M+3[\sqrt{p}]-1$ contains a qudratic non-residue to modulus p.*

C 12. *Let p be an odd prime and let Z_p denote (the field of) integers modulo p. How many elements are in the set*

$$\{x^2 : x \in Z_p\} \cap \{y^2+1 : y \in Z_p\}?$$

5.3. **Congruences.**

C 13. *If p is an odd prime, prove that*

$$\binom{k}{p} \equiv \left[\frac{k}{p}\right] (mod\, p).$$

C 14. *Suppose p is an odd prime. Prove that*

$$\sum_{j=0}^{p} \binom{p}{j}\binom{p+j}{j} \equiv 2^p + 1 \pmod{p^2}.$$

C 15. *(Morley) Show that*

$$(-1)^{\frac{p-1}{2}} \binom{p-1}{\frac{p-1}{2}} \equiv 4^{p-1} (mod\, p^3)$$

for all prime numbers p with $p \geq 5$.

C 16. *Let n be a positive integer. Prove that n is prime if and only if*

$$\binom{n-1}{k} \equiv (-1)^k (mod\, n)$$

for all $k \in \{0, 1, \cdots, n-1\}$.

C 17. *Prove that for $n \geq 2$,*

$$\overbrace{2^{2^{\cdot^{\cdot^{\cdot^{2}}}}}}^{n \text{ terms}} \equiv \overbrace{2^{2^{\cdot^{\cdot^{\cdot^{2}}}}}}^{n-1 \text{ terms}} \pmod{n}.$$

C 18. *Show that, for any fixed integer $n \geq 1$, the sequence*

$$2,\ 2^2,\ 2^{2^2},\ 2^{2^{2^2}}, \ldots (mod\, n)$$

is eventually constant.

C 19. *Somebody incorrectly remembered Fermat's little theorem as saying that the congruence $a^{n+1} \equiv a$ (mod n) holds for all a if n is prime. Describe the set of integers n for which this property is in fact true.*

C 20. *Characterize the set of positive integers n such that, for all integers a, the sequence a, a^2, a^3, $\cdots$ is periodic modulo n.*

C 21. *Show that there exists a composite number n such that $a^n \equiv a$ (mod n) for all $a \in \mathbf{Z}$.*

C 22. *Let p be a prime number of the form $4k+1$. Suppose that $2p+1$ is prime. Show that there is no $k \in \mathbf{N}$ with $k < 2p$ and $2^k \equiv 1 \ (mod\ 2p+1)$*

C 23. *During a break, n children at school sit in a circle around their teacher to play a game. The teacher walks clockwise close to the children and hands out candies to some of them according to the following rule. He selects one child and gives him a candy, then he skips the next child and gives a candy to the next one, then he skips 2 and gives a candy to the next one, then he skips 3, and so on. Determine the values of n for which eventually, perhaps after many rounds, all children will have at least one candy each.*

C 24. *Suppose that $m > 2$, and let P be the product of the positive integers less than m that are relatively prime to m. Show that $P \equiv -1 (mod\ m)$ if $m = 4$, p^n, or $2p^n$, where p is an odd prime, and $P \equiv 1 (mod\ m)$ otherwise.*

C 25. *Let Γ consist of all polynomials in x with integer coefficienst. For f and g in Γ and m a positive integer, let $f \equiv g \pmod{m}$ mean that every coefficient of $f - g$ is an integral multiple of m. Let n and p be positive integers with p prime. Given that f, g, h, r and s are in Γ with $rf + sg \equiv 1 \pmod{p}$ and $fg \equiv h \pmod{p}$, prove that there exist F and G in Γ with $F \equiv f \pmod{p}$, $G \equiv g \pmod{p}$, and $FG \equiv h \pmod{p^n}$.*

C 26. *Determine the number of integers $n \geq 2$ for which the congruence*

$$x^{25} \equiv x \ (mod\ n)$$

is true for all integers x.

C 27. *Let $n_1, \cdots, n_k$ and a be positive integers which satify the following condtions :*

i) for any $i \neq j$, $(n_i, n_j) = 1$,
ii) for any i, $a^{n_i} \equiv 1 (mod\ n_i)$,
iii) for any i, $n_i \not| a-1$.

Show that there exist at least $2^{k+1} - 2$ integers $x > 1$ with $a^x \equiv 1 (mod\ x)$.

C 28. *Determine all positive integers $n \geq 2$ that satisfy the following condition ; For all integers a, b relatively prime to n,*

$$a \equiv b \ (mod\ n) \Longleftrightarrow ab \equiv 1 \ (mod\ n).$$

C 29. *Determine all positive integers n such that $xy+1 \equiv 0 (mod\ n)$ implies that $x + y \equiv 0 (mod\ n)$.*

C 30. *Let p be a prime number. Determine the maximal degree of a polynomial $T(x)$ whose coefficients belong to $\{0, 1, \cdots, p-1\}$ whose degree is less than p, and which satisfies*

$$T(n) = T(m) \ (mod\ p) \Longrightarrow n = m \ (mod\ p)$$

for all integers n, m.

C 31. *Let $a_1, \cdots, a_k$ and $m_1, \cdots, m_k$ be integers $2 \le m_1$ and $2m_i \le m_{i+1}$ for $1 \le i \le k-1$. Show that there are infinitely many integers x which do not satisfy any of congruences*

$$x \equiv a_1 \ (mod\ m_1), x \equiv a_2 \ (mod\ m_2), \cdots, x \equiv a_k \ (mod\ m_k).$$

C 32. *Show that* 1994 *divides* $10^{900} - 2^{1000}$.

C 33. *Determine the last three digits of*

$$2003^{2002^{2001}}.$$

C 34. *Prove that* $1980^{1981^{1982}} + 1982^{1981^{1980}}$ *is divisible by* 1981^{1981}.

6. Primes and Composite Numbers

Wherever there is number, there is beauty. Proclus Diadochus

D 1. *Prove that the number* $512^3 + 675^3 + 720^3$ *is composite.* [6]

D 2. *Show that there are infinitely many primes.*

D 3. *Find all natural numbers* n *for which every natural number whose decimal representation has* $n-1$ *digits* 1 *and one digit* 7 *is prime.*

D 4. *Prove that there do not exist polynomials* P *and* Q *such that*

$$\pi(x) = \frac{P(x)}{Q(x)}$$

for all $x \in \mathbf{N}$.

D 5. *Show that there exist two consecutive integer squares such that there are at least* 1000 *primes between them.*

D 6. *Let* a, b, c, d *be integers with* $a > b > c > d > 0$. *Suppose that* $ac + bd = (b + d + a - c)(b + d - a + c)$. *Prove that* $ab + cd$ *is not prime.*

D 7. *Prove that for any prime* p *in the interval* $\left(n, \frac{4n}{3}\right]$, p *divides*

$$\sum_{j=0}^{n} \binom{n}{j}^4$$

D 8. *Let* a, b, *and* n *be positive integers with* $gcd(a, b) = 1$. *Without using the Dirichlet's theorem*[7], *show that there are infinitely many* $k \in \mathbf{N}$ *such that* $gcd(ak + b, n) = 1$.

D 9. *Without using the Dirichlet's theorem, show that there are infintely many primes ending in the digit* 9.

D 10. *Let* p *be an odd prime. Without using the Dirichlet's theorem, show that there are infintely many primes of the form* $2pk + 1$.

D 11. *Show that, for each* $r \geq 1$, *there are infintely many primes* $p \equiv 1 \ (mod\ 2^r)$.

D 12. *Prove that if* p *is a prime, then* $p^p - 1$ *has a prime factor that is congruent to* 1 *modulo* p.

D 13. *Let* p *be a prime number. Prove that there exists a prime number* q *such that for every integer* n, $n^p - p$ *is not divisible by* q.

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[6]Note that there is a hint at chapter 17.

[7]For any $a, b \in \mathbf{N}$ with $gcd(a, b) = 1$, there are infinitely many primes of the form $ak + b$.

D 14. *Let* $p_1 = 2, p_2 = 3, p_3 = 5, \cdots, p_n$ *be the first* n *prime numbers, where* $n \geq 3$. *Prove that*

$$\frac{1}{{p_1}^2} + \frac{1}{{p_2}^2} + \cdots + \frac{1}{{p_n}^2} + \frac{1}{p_1 p_2 \cdots p_n} < \frac{1}{2}.$$

D 15. *Let* p_n *be the* n*th prime :* $p_1 = 2, p_2 = 3, p_3 = 5, \cdots$. *Show that the infinite series*

$$\sum_{n=1}^{\infty} \frac{1}{p_n}$$

diverges.

D 16. *Prove that* $\log n \geq k \log 2$, *where* n *is a natural number and* k *is the number of distinct primes that divide* n.

D 17. *Find the smallest prime which is not the difference (in some order) of a power of* 2 *and a power of* 3.

D 18. *Find the sum of all distinct positive divisors of the number* 104060401.

D 19. *Prove that* 1280000401 *is composite.*

D 20. *Prove that* $\frac{5^{125}-1}{5^{25}-1}$ *is a composite number.*

D 21. *Find the factor of* $2^{33} - 2^{19} - 2^{17} - 1$ *that lies between* 1000 *and* 5000.

D 22. *Prove that for each positive integer* n *there exist* n *consecutive positive integers none of which is an integral power of a prime number.*

D 23. *Show that there exists a positive integer* k *such that* $k \cdot 2^n + 1$ *is composite for all* $n \in \mathbf{N}_0$.

D 24. *Show that for all integer* $k > 1$, *there are infinite many natural numbers* n *such that* $k \cdot 2^{2^n} + 1$*is composite.*

D 25. *Show that* $n^{\pi(2n)-\pi(n)} < 4^n$ *for all positive integer* n.

D 26. *Four integers are marked on a circle. On each step we simultaneously replace each number by the difference between this number and next number on the circle in a given direction (that is, the numbers* a, b, c, d *are replaced by* $a-b$, $b-c$, $c-d$, $d-a$*). Is it possible after* 1996 *such steps to have numbers* a, b, c, d *such that the numbers* $|bc-ad|$, $|ac-bd|$, $|ab-cd|$ *are primes ?*

D 27. *Let* s_n *denote the sum of the first* n *primes. Prove that for each* n *there exists an integer whose square lies between* s_n *and* s_{n+1}.

D 28. *Given an odd integer* $n > 3$, *let* k *and* t *be the smallest positive integers such that both* $kn+1$ *and* tn *are squares. Prove that* n *is prime if and only if both* k *and* t *are greater than* $\frac{n}{4}$

D 29. *Represent the number* $989 \cdot 1001 \cdot 1007 + 320$ *as the product of primes.*

D 30. *Suppose n and r are nonnegative integers such that no number of the form $n^2+r-k(k+1)$ $(k \in \mathbf{N})$ equals to -1 or a positive composite number. Show that $4n^2+4r+1$ is 1, 9 or prime.*

D 31. *Let $n \geq 5$ be an integer. Show that n is prime if and only if $n_i n_j \neq n_p n_q$ for every partition of n into 4 integers, $n = n_1+n_2+n_3+n_4$, and for each permutation (i,j,p,q) of $(1,2,3,4)$.*

D 32. *Prove that there are no positive integers a and b such that for all different primes p and q greater than 1000, the number $ap+bq$ is also prime.*

D 33. *Let p_n denote the nth prime number. For all $n \geq 6$, prove that*

$$\pi\left(\sqrt{p_1 p_2 \cdots p_n}\right) > 2n.$$

D 34. *There exists a block of 1000 consecutive positive integers containing no prime numbers, namely, $1001!+2$, $1001!+3$, $\cdots$, $1001!+1001$. Does there exist a block of 1000 consecutive positive integers containing exactly five prime numbers?*

7. Rational and Irrational Numbers

God made the integers, all else is the work of man. Leopold Kronecker

E 1. *Suppose that a rectangle with sides a and b is arbitrily cut into squares with sides $x_1, \cdots, x_n$. Show that $\frac{x_i}{a} \in \mathbf{Q}$ and $\frac{x_i}{b} \in \mathbf{Q}$ for all $i \in \{1, \cdots, n\}$.*

E 2. *Find the smallest positive integer n such that*

$$0 < n^{\frac{1}{4}} - [n^{\frac{1}{4}}] < 0.00001.$$

E 3. *Prove that for any positive integers a and b*

$$\left|a\sqrt{2} - b\right| > \frac{1}{2(a+b)}.$$

E 4. *Prove that there exist positive integers m and n such that*

$$\left|\frac{m^2}{n^3} - \sqrt{2001}\right| < \frac{1}{10^8}.$$

E 5. *Let a, b, c be integers, not all zero and each of absolute value less than one million. Prove that*

$$\left|a + b\sqrt{2} + c\sqrt{3}\right| > \frac{1}{10^{21}}.$$

E 6. *Let a, b, c be integers, not all equal to 0. Show that*

$$\frac{1}{4a^2 + 3b^2 + 2c^2} \le |{}^3\sqrt{4}a + {}^3\sqrt{2}b + c|.$$

E 7. *(Hurwitz) Prove that for any irrational number ξ, there are infinitely many rational numbers $\frac{m}{n}$ $((m, n) \in \mathbf{Z} \times \mathbf{N})$ such that*

$$\left|\xi - \frac{n}{m}\right| < \frac{1}{\sqrt{5}m^2}.$$

E 8. *Show that π is irrational.*

E 9. *Show that $e = \sum_{n=0}^{\infty} \frac{1}{n!}$ is irrational.*

E 10. *Show that $\cos \frac{\pi}{7}$ is irrational.*

E 11. *Show that $\frac{1}{\pi} \arccos\left(\frac{1}{\sqrt{2003}}\right)$ is irrational.*

E 12. *Show that $\cos 1^\circ$ is irrational.*

E 13. *An integer-sided triangle has angles $p\theta$ and $q\theta$, where p and q are relatively prime integers. Prove that $\cos\theta$ is irrational.*

E 14. *It is possible to show that $\csc \frac{3\pi}{29} - \csc \frac{10\pi}{29} = 1.999989433....$ Prove that there are no integers j, k, n with odd n satisfying $\csc \frac{j\pi}{n} - \csc \frac{k\pi}{n} = 2$.*

E 15. *For which angles θ, a rational number of degrees, is it the case that $\tan^2\theta + \tan^2 2\theta$ is irrational ?*

E 16. *Find all x and y which are rational multiples of π with $0 < x < y < \frac{\pi}{2}$ and $\tan x + \tan y = 2$.*

E 17. *Let a be a rational number with $0 < a < 1$ and suppose that $cos(3\pi\alpha) + 2cos(2\pi\alpha) = 0$. (Angle meauserments are in radians.) Prove that $a = \frac{2}{3}$.*

E 18. *Suppose $\tan\alpha = \frac{p}{q}$, where p and q are integers and $q \neq 0$. Prove the number $\tan\beta$ for which $\tan 2\beta = \tan 3\alpha$ is rational only when $p^2 + q^2$ is the square of an integer.*

E 19. *(K. Mahler, 1953) Prove that for any $p, q \in \mathbf{N}$ with $q > 1$ the following inequality holds.* [8]

$$\left|\pi - \frac{p}{q}\right| \geq q^{-42}$$

E 20. *(K. Alladi, M. Robinson, 1979) Suppose that $p, q \in \mathbf{N}$ satisfy the inequality $e(\sqrt{p+q} - \sqrt{q})^2 < 1$.*[9] *Show that the number $ln\left(1 + \frac{p}{q}\right)$ is irrational.*

E 21. *Prove that there cannot exist a positive rational number x such that*

$$x^{[x]} = \frac{9}{2}$$

holds.

E 22. *Let x, y, z non-zero real numbers such that xy, yz, zx are rational.*
(a) Show that the number $x^2 + y^2 + z^2$ is rational.
(b) If the number $x^3 + y^3 + z^3$ is also rational, show that x, y, z are rational.

E 23. *Show that the cube roots of three distinct primes cannot be terms in an arithmetic progression.*

E 24. *Let n be an integer greater than or equal to 3. Prove that there is a set of n points in the plane such that the distance between any two points is irrational and each set of three points determines a non-degenerate triangle with rational area.*

E 25. *If x is a positive rational number show that x can be uniquely expressed in the form*

$$x = a_1 + \frac{a_2}{2!} + \frac{a_3}{3!} + \cdots,$$

where a_1, a_2, $\cdots$ are integers, $0 \leq a_n \leq n-1$, for $n > 1$, and the series terminates. Show also that x can be expressed as the sum of reciprocals of different integers, each of which is greater than 10^6.

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[8] This is a deep theorem in *transcendental number theory*. Note that it follows from this result that π is irrational ! In fact, it's known that for sufficiently large q, the exponent 42 can be replaced by 30. Here is a similar result due to A. Baker : For any rationals $\frac{p}{q}$, one has $|ln\ 2 - \frac{p}{q}| \geq 10^{-100000} q^{-12.5}$. [AI, pp. 106]

[9] Here, $e = \sum_{n \geq 0} \frac{1}{n!}$.

E 26. *Find all polynomials W with real coefficients possessing the following property : if $x+y$ is a rational number, then $W(x)+W(y)$ is rational as well.*

E 27. *Prove that every positive rational number can be represented in the form*

$$\frac{a^3+b^3}{c^3+d^3}$$

for some positive integers a, b, c, d.

E 28. *The set S is a finite subset of $[0,1]$ with the following property : for all $s \in S$, there exist $a,\, b \in S \bigcup \{0,1\}$ with $a, b \neq x$ such that $x = \frac{a+b}{2}$. Prove that all the numbers in S are rational.*

E 29. *Let $S = \{x_0, x_1, \cdots, x_n\} \subset [0,1]$ be a finite set of real numbers with $x_0 = 0$ and $x_1 = 1$, such that every distance between pairs of elements occurs at least twice, except for the distance 1. Prove that all of the x_i are rational.*

E 30. *You are given three lists A, B, and C. List A contains the numbers of the form 10^k in base 10, with k any integer greater than or equal to 1. Lists B and C contain the same numbers translated into base 2 and 5 respectively:*

A	*B*	*C*
10	*1010*	*20*
100	*1100100*	*400*
1000	*1111101000*	*13000*
$\vdots$	$\vdots$	$\vdots$

Prove that for every integer $n > 1$, there is exactly one number in exactly one of the lists B or C that has exactly n digits.

E 31. *(Beatty) Prove that if α and β are positive irrational numbers satisfying $\frac{1}{\alpha} + \frac{1}{\beta} = 1$, then the sequences*

$$[\alpha], [2\alpha], [3\alpha], \cdots$$

and

$$[\beta], [2\beta], [3\beta], \cdots$$

together include every positive integer exactly once.

E 32. *For a positive real number α, define*

$$S(\alpha) = \{[n\alpha] | n = 1, 2, 3, \cdots\}.$$

Prove that $\mathbf{N}$ cannot be expressed as the disjoint union of three sets $S(\alpha)$, $S(\beta)$, and $S(\gamma)$.

E 33. *Let $f(x) = \prod_{n=1}^{\infty} \left(1 + \frac{x}{2^n}\right)$. Show that at the point $x = 1$, $f(x)$ and all its derivatives are irrational.*

E 34. *Let $\{a_n\}_{n \geq 1}$ be a sequence of positive numbers such that*

$$ {a_{n+1}}^2 = a_n + 1 \textit{ for all } n \in \mathbf{N}.$$

Show that the sequence contains an irrational number.

E 35. *Does there exist a circle and an infinte set of points on it such that the distance between any two points of the set is rational ?*

8. Diophantine Equations I

In the margin of his copy of Diophantus' Arithmetica, Pierre de Fermat wrote : *To divide a cube into two other cubes, a fourth power or in general any power whatever into two powers of the same denomination above the second is impossible, and I have assuredly found an admirable proof of this, but the margin is too narrow to contain it.*

F 1. *One of Euler's conjecture*[10] *was disproved in the* 1980*s by three American Mathematicians*[11] *when they showed that there is a positive integer* n *such that*

$$n^5 = 133^5 + 110^5 + 84^5 + 27^5.$$

Find the value of n.

F 2. *The number* 21982145917308330487013369 *is the thirteenth power of a positive integer. Which positive integer?*

F 3. *Does there exist a solution to the equation*

$$x^2 + y^2 + z^2 + u^2 + v^2 = xyzuv - 65$$

in integers x, y, z, u, v *greater than* 1998*?*

F 4. *Find all pairs* (x, y) *of positive rational numbers such that* $x^2+3y^2 = 1$.

F 5. *Find all pairs* (x, y) *of rational numbers such that* $y^2 = x^3 - 3x + 2$.

F 6. *Show that there are infinitely many pairs* (x, y) *of rational numbers such that* $x^3 + y^3 = 9$.

F 7. *Determine all pairs* (x, y) *of positive integers satisfying the equation*

$$(x + y)^2 - 2(xy)^2 = 1.$$

F 8. *Show that the equation*

$$x^3 + y^3 + z^3 + t^3 = 1999$$

has infinitely many integral solutions. [12]

F 9. *Determine all integers* a *for which the equation*

$$x^2 + axy + y^2 = 1$$

has infinitely many distinct integer solutions x, y.

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[10] In 1769, Euler, by generalising Fermat's Last Theorem, conjectured that "it is impossible to exhibit three fourth powers whose sum is a fourth power", "four fifth powers whose sum is a fifth power, and similarly for higher powers" [Rs]

[11] L. J. Lander, T. R. Parkin, J. L. Selfridge

[12] More generally, the following result is known : let n be an integer, then the equation $x^3 + y^3 + z^3 + w^3 = n$ has infinitely many integral solutions (x, y, z, w) if there can be found one solution $(x, y, z, w) = (a, b, c, d)$ with $(a + b)(c + d)$ negative and with either $a \neq b$ and $c \neq d$. [Eb2, pp.90]

F 10. *Prove that there are unique positive integers* a *and* n *such that*

$$a^{n+1} - (a+1)^n = 2001.$$

F 11. *Find all* $(x, y, n) \in \mathbf{N}^3$ *such that* $gcd(x, n+1) = 1 and\ x^n + 1 = y^{n+1}$.

F 12. *Find all* $(x, y, z) \in \mathbf{N}^3$ *such that* $x^4 - y^4 = z^2$.

F 13. *Find all pairs* (x, y) *of positive integers that satisfy the equation* [13]

$$y^2 = x^3 + 16.$$

F 14. *Show that the equation* $x^2 + y^5 = z^3$ *has infinitely many solutions in integers* x, y, z *for which* $xyz \neq 0$.

F 15. *Prove that there are no integers* x, y *satisfying* $x^2 = y^5 - 4$.

F 16. *Find all pairs* (a, b) *of different positive integers that satisfy the equation* $W(a) = W(b)$, *where* $W(x) = x^4 - 3x^3 + 5x^2 - 9x$.

F 17. *Find all positive integers* n *for which the equation*

$$a + b + c + d = n\sqrt{abcd}$$

has a solution in positive integers.

F 18. *Determine all positive integer solutions* (x, y, z, t) *of the equation*

$$(x+y)(y+z)(z+x) = xyzt$$

for which $gcd(x, y) = gcd(y, z) = gcd(z, x) = 1$.

F 19. *Find all* $(x, y, z, n) \in \mathbf{N}^4$ *such that* $x^3 + y^3 + z^3 = nx^2y^2z^2$.

F 20. *Determine all positive integers* n *for which the equation*

$$x^n + (2+x)^n + (2-x)^n = 0$$

has an integer as a solution.

F 21. *Prove that the equation*

$$6(6a^2 + 3b^2 + c^2) = 5n^2$$

has no solutions in integers except $a = b = c = n = 0$.

F 22. *Find all integers* (a, b, c, x, y, z) *such that*

$$a + b + c = xyz,\ x + y + z = abc,\ a \geq b \geq c \geq 1,\ x \geq y \geq z \geq 1.$$

F 23. *Find all* $(x, y, z) \in \mathbf{N}^3$ *such that* $x^3 + y^3 + z^3 = x + y + z = 3$.

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[13]It's known that there are infinitely many integers k so that the equation $y^2 = x^3 + k$ has no integral solutions. For example, if k has the form $k = (4n-1)^3 - 4m^2$, where m and n are integers such that no prime $p \equiv -1$ (mod 4) divides m, then the equation $y^2 = x^3 + k$ has no integral solutions. For a proof, see [Tma, pp. 191].

F 24. *Prove that if n is a positive integer such that the equation*

$$x^3 - 3xy^2 + y^3 = n.$$

has a solution in integers (x, y), then it has at least three such solutions.
Show that the equation has no solutions in integers when $n = 2891$.

F 25. *What is the smallest positive integer t such that there exist integers $x_1, x_2, \cdots, x_t$ with*

$${x_1}^3 + {x_2}^3 + \cdots + {x_t}^3 = 2002^{2002} \quad ?$$

F 26. *Solve in integers the following equation*

$$n^{2002} = m(m+n)(m+2n)\cdots(m+2001n).$$

F 27. *Prove that there exist infinitely many positive integers n such that $p = nr$, where p and r are respectively the semiperimeter and the inradius of a triangle with integer side lengths.*

F 28. *Let a, b, c be positive integers such that a and b are relatively prime and c is relatively prime either to a and b. Prove that there exist infinitely many triples (x, y, z) of distinct positive integers x, y, z such that*

$$x^a + y^b = z^c.$$

F 29. *Find all pairs of integers (x, y) satisfying the equality*

$$y(x^2 + 36) + x(y^2 - 36) + y^2(y - 12) = 0$$

F 30. *Let a, b, c be given integers $a > 0$, $ac - b^2 = P = P_1 P_2 \cdots P_n$, where $P_1, \cdots, P_n$ are (distinct) prime numbers. Let $M(n)$ denote the number of pairs of integers (x, y) for which $ax^2 + bxy + cy^2 = n$. Prove that $M(n)$ is finite and $M(n) = M(p^k \cdot n)$ for every inegers $k \geq 0$.*

F 31. *Determine integer solutions of the system*

$$2uv - xy = 16,$$
$$xv - yu = 12.$$

F 32. *Let n be a natural number. Solve in whole numbers the equation $x^n + y^n = (x - y)^{n+1}$.*

F 33. *Does there exist an integer such that its cube is equal to $3n^2 + 3n + 7$, where n is integer?*

F 34. *Are there integers m and n such that $5m^2 - 6mn + 7n^2 = 1985$?*

F 35. *Find all cubic polynomials $x^3 + ax^2 + bx + c$ admitting the rational numbers a, b and c as roots.*

F 36. *Prove that the equation $a^2 + b^2 = c^2 + 3$ has infinitely many integer solutions (a, b, c).*

F 37. *Prove that for each positive integer n there exist odd positive integers x_n and y_n such that ${x_n}^2 + 7{y_n}^2 = 2^n$.*

9. Diophantine Equations II

The positive integers stand there, a continual and inevitable challenge to the curiousity of every healthy mind. Godfrey Harold Hardy

G 1. *Given that*

$$34! = 95232799cd96041408476186096435ab000000,$$

determine the digits a, b, c, d.

G 2. *Prove that the equation* $(x_1-x_2)(x_2-x_3)(x_3-x_4)(x_4-x_5)(x_5-x_6)(x_6-x_7)(x_7-x_1) = (x_1-x_3)(x_2-x_4)(x_3-x_5)(x_4-x_6)(x_5-x_7)(x_6-x_1)(x_7-x_2)$ *has a solution in natural numbers where all* x_i *are different.*

G 3. *(P. Erdös) Show that the equation* $\binom{n}{k} = m^l$ *has no integral solution with* $l \geq 2$ *and* $4 \leq k \leq n-4$.

G 4. *Solve in positive integers the equation* $10^a + 2^b - 3^c = 1997$.

G 5. *Solve the equation* $28^x = 19^y + 87^z$, *where* x, y, z *are integers.*

G 6. *Show that the equation* $x^7 + y^7 = 1998^z$ *has no solution in posive integers.*

G 7. *Solve the equation* $2^x - 5 = 11^y$ *in positive integers.*

G 8. *Find all positive integers* x, y *such that* $7^x - 3^y = 4$.

G 9. *Show that* $|12^m - 5^n| \geq 7$ *for all* $m, n \in \mathbf{N}$.

G 10. *Show that there is no positive integer* k *for which the equation*

$$(n-1)! + 1 = n^k$$

is true when n *is greater than* 5.

G 11. *Determine all integers* a *and* b *such that*

$$(19a+b)^{18} + (a+b)^{18} + (19b+a)^{18}$$

is a positive square.

G 12. *Let* b *be a positive integer. Determine all* 200*-tuple integers of non-negative integers* $(a_1, a_2, \cdots, a_{2002})$ *satisfying*

$$\sum_{j=1}^{n} {a_j}^{a_j} = 2002b^b.$$

G 13. *Is there a positive integers* m *such that the equation*

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{abc} = \frac{m}{a+b+c}$$

has infintely many solutions in positive integers a, b, c *?*

G 14. *Consider the system*

$$x + y = z + u$$
$$2xy = zu$$

Find the greatest value of the real constant m *such that* $m \le \frac{x}{y}$ *for any positive integer solution* (x, y, z, u) *of the system, with* $x \ge y$.

G 15. *Determine all positive rational number* $r \neq 1$ *such that* $r^{\frac{1}{r-1}}$ *is rational.*

G 16. *Show that the equation* $\{x^3\} + \{y^3\} = \{z^3\}$ *has infinitely many rational non-integer solutions.*

G 17. *Let* n *be a positive integer. Prove that the equation*

$$x + y + \frac{1}{x} + \frac{1}{y} = 3n$$

does not have solutions in positive rational numbers.

G 18. *Find all pairs* (x, y) *of positive rational numbers such that* $x^y = y^x$

G 19. *Find all pairs* (a, b) *of positive integers that satisfy the equation*

$$a^{b^2} = b^a.$$

G 20. *Find all pairs* (a, b) *of positive integers that satisfy the equation*

$$a^{a^a} = b^b.$$

G 21. *Let* x, a, b *be positive integers such that* $x^{a+b} = a^b b$. *Prove that* $a = x$ *and* $b = x^x$.

G 22. *Find all pairs* (m, n) *of integers that satisfy the equation*

$$(m - n)^2 = \frac{4mn}{m + n - 1}$$

G 23. *Find all pairwise relatively prime positive integers* l, m, n *such that*

$$(l + m + n)\left(\frac{1}{l} + \frac{1}{m} + \frac{1}{n}\right)$$

is an integer.

G 24. *Let* x, y, z *be integers with* $z > 1$. *Show that*

$$(x + 1)^2 + (x + 2)^2 + \cdots + (x + 99)^2 \neq y^z.$$

G 25. *Find all values of the positive integers* m *and* n *for which*

$$1! + 2! + 3! + \cdots + n! = m^2$$

G 26. *Prove that if* a, b, c, d *are integers such that* $d = (a + 2^{\frac{1}{3}} b + 2^{\frac{2}{3}} c)^2$ *then* d *is a perfect square (i. e. is the square of an integer).*

G 27. *Find a pair of relatively prime four digit natural numbers* A *and* B *such that for all natural numbers* m *and* n, $|A^m - B^n| \ge 400$.

G 28. *Find all solutions in positive integers* a, b, c *to the equation*

$$a!b! = a! + b! + c!.$$

G 29. *Find positive integers* a *and* b *such that*

$$(\sqrt[3]{a} + \sqrt[3]{b} - 1)^2 = 49 + 20\sqrt[3]{6}.$$

G 30. *For what positive numbers* a *is*

$$\sqrt[3]{2 + \sqrt{a}} + \sqrt[3]{2 - \sqrt{a}}$$

an integer ?

G 31. *Find all integer solutions to* $2(x^5 + y^5 + 1) = 5xy(x^2 + y^2 + 1)$.

G 32. *A triangle with integer sides is called Heronian if its are is an integer. Does there exist a Heronian triangle whose sides are the arithmetic, geometric and harmonic means of two positive integers ?*

10. Functions in Number Theory

Gauss once said "Mathematics is the queen of the sciences and number theory is the queen of mathematics." If this be true we may add that the Disauistiones is the Magna Charta of number theory. M. Cantor

10.1. **Floor Function and Fractional Part Function.**

H 1. *Let α be the positive root of the equation $x^2 = 1991x + 1$. For natural numbers m, n define*

$$m * n = mn + [\alpha m][\alpha n],$$

where [x] is the greatest integer not exceeding x. Prove that for all natural numbers p, q, r,

$$(p * q) * r = p * (q * r).$$

H 2. *Show that $[\sqrt{n} + \sqrt{n+1}] = [\sqrt{4n+1}] = [\sqrt{4n+2}] = [\sqrt{4n+3}]$ for all positive integer n.*

H 3. *Find all real numbers α for which the equality*

$$[\sqrt{n} + \sqrt{n+\alpha}] = [\sqrt{4n+1}]$$

holds for all positive integer n.[14]

H 4. *Prove that for all positive integer n,*

$$[\sqrt{n} + \sqrt{n+1} + \sqrt{n+2}] = [\sqrt{9n+8}].$$

H 5. *Prove that $[n^{\frac{1}{3}} + (n+1)^{\frac{1}{3}}] = [(8n+3)^{\frac{1}{3}}]$ for every positive integer n.*

H 6. *Prove that $[n^{\frac{1}{3}} + (n+1)^{\frac{1}{3}} + (n+2)^{\frac{1}{3}}] = [(27n+26)^{\frac{1}{3}}]$ for all positive integer n.*

H 7. *Show that for all positive integers m and n,*

$$gcd(m,n) = m + n - mn + 2\sum_{k=0}^{m-1}\left[\frac{kn}{m}\right].$$

H 8. *Show that for all primes p,*

$$\sum_{k=1}^{p-1}\left[\frac{k^3}{p}\right] = \frac{(p+1)(p-1)(p-2)}{4}.$$

H 9. *Let p be a prime number of the form $4k+1$. Show that*

$$\sum_{k=1}^{p-1}\left(\left[\frac{2k^2}{p}\right] - 2\left[\frac{k^2}{p}\right]\right) = \frac{p-1}{2}.$$

---

[14] The answer is $9 - 6\sqrt{n} \le \alpha \le 2$.

H 10. *Let p be a prime number of the form $4k+1$. Show that*

$$\sum_{i=1}^{k}\left[\sqrt{ip}\right]=\frac{p^2-1}{12}.$$

H 11. *Suppose that $n \geq 2$. Prove that*

$$\sum_{k=2}^{n}\left[\frac{n^2}{k}\right]=\sum_{k=n+1}^{n^2}\left[\frac{n^2}{k}\right]$$

H 12. *Let a, b, n be positive integers with $gcd(a,b)=1$. Prove that*

$$\sum_{k}\left\{\frac{ak+b}{n}\right\}=\frac{n-1}{2},$$

where k runs through a complete system of residues modulo m.

H 13. *Find the total number of different integer values the function*

$$f(x)=[x]+[2x]+\left[\frac{5x}{3}\right]+[3x]+[4x]$$

takes for real numbers x with $0 \leq x \leq 100$.

H 14. *Prove or disprove that there exists a positive real number u such that $[u^n]-n$ is an even integer for all positive integer n.*

H 15. *Determine all real numbers a such that*

$$4[an]=n+[a[an]] \; for \; all \, n \in \mathbf{N}$$

H 16. *Do there exist irrational numbers a and b such that $a>1$, $b>1$ and $[a^m]$ differs $[b^n]$ for any two positive integers m and n?*

10.2. **Euler phi Function.**

H 17. *Let n be an integer with $n \geq 2$. Show that $\phi(2^n-1)$ is divisible by n.*

H 18. *(Gauss) Show that for all $n \in \mathbf{N}$,*

$$n=\sum_{d|n}\phi(d).$$

H 19. *If p is a prime and n an integer such that $1<n \leq p$, then*

$$\phi\left(\sum_{k=0}^{p-1}n^k\right)\equiv 0 \; (mod \; p).$$

H 20. *Let m, n be positive integers. Prove that, for some positive integer a, each of $\phi(a)$, $\phi(a+1)$, $\cdots$, $\phi(a+n)$ is a multiple of m.*

H 21. *If n is composite, prove that $\phi(n) \leq n-\sqrt{n}$.*

H 22. *For a positive integer k, the number of positive integers less than k is denoted $\phi(k)$. Show that if m and n are relatively prime positive integers, then $\phi(5^m-1) \neq 5^n-1$.*

H 23. *Show that if the equation $\phi(x) = n$ has one solution it always has a second solution, n being given and x being the unknown.*

10.3. **Divisor Functions.**

H 24. *Let $d(n)$ denote the number of positive divisors of the natural number n. Prove that $d(n^2+1)^2$ does not become monotonic from any given point onwards.*

H 25. *For any $n \in \mathbf{N}$, let $d(n)$ denote the number of positive divisors of n. Determine all positive integers n such that $n = d(n)^2$.*

H 26. *For any $n \in \mathbf{N}$, let $d(n)$ denote the number of positive divisors of n. Determine all positive integers k such that*

$$\frac{d(n^2)}{d(n)} = k$$

for some $n \in \mathbf{N}$.

H 27. *For a positive integer n, let $d(n)$ be the number of all positive divisord of n. Find all positive integers n such that $d(n)^3 = 4n$.*

H 28. *For each positive integer n, let $d(n)$ be the number of distinct positive integers that divide n. Determine all positive integers for which $d(n) = \frac{n}{3}$ holds.*

H 29. *Let n be a positive integers. Let $\sigma(n)$ be the sum of the natural divisors d of n (including 1 and n). We say that an integer $m \geq 1$ is superabondant if*

$$\frac{\sigma(m)}{m} > \frac{\sigma(k)}{k},$$

for all $k \in \{1, 2, \cdots, m-1\}$. Prove that there exists an infinite number of superabondant numbers.

H 30. *Let $\sigma(n)$ denote the sum of the positive divisors of the positive integer n. and $\phi(n)$ the Euler phi-function. Show that $\phi(n) + \sigma(n) \geq 2n$ for all positive integers n.*

10.4. **More Functions.**

H 31. *Ramanujan's tau Function* [15] *$\tau : \mathbf{N} \to \mathbf{Z}$ has the generating function*

$$\sum_{n=1}^{\infty} \tau(n) x^n = x \prod_{n=1}^{\infty} (1 - x^n)^{24},$$

i.e. the co-efficients of x^n on the right hand side define $\tau(n)$.[16]
(1) Show that $\tau(mn) = \tau(m)\tau(n)$ for all $m, n \in \mathbf{N}$ with $gcd(m,n) = 1$. [17]
(2) Show that $\tau(n) \equiv \sum_{d|n} d^{11}$ (mod 691) for all $n \in \mathbf{N}$. [18]

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[15] In 1947, Lehmer conjectured that $\tau(n) \neq 0$ for all $n \in \mathbf{N}$.
[16] $\{\tau(n) | n \geq 1\} = \{1, -24, 252, -1472, \cdots\}$. For more terms, see the first page !
[17] This Ramanujan's conjecture was proved by Mordell.
[18] This Ramanujan's conjecture was proved by Watson.

H 32. *For every natural number n, $Q(n)$ denote the sum of the digits in the decimal representation of n. Prove that there are infinitely many natural number k with $Q(3^k) > Q(3^{k+1})$.*

H 33. *Let $S(n)$ be the sum of all different natural divisors of an odd natural number $n > 1$ (including 1 and n). Prove that $S(n)^3 < n^4$.*

H 34. *Let $((x)) = x - [x] - \frac{1}{2}$ if x is not an integer, and let $((x)) = 0$ otherwise. If n and k are integers, with $n > 0$, prove that*

$$\left(\left(\frac{k}{n}\right)\right) = -\frac{1}{2n}\sum_{m=1}^{n-1} \cot\frac{\pi m}{n} \sin\frac{2\pi k m}{n}.$$

H 35. *The function $\mu : \mathbf{N} \longrightarrow \mathbf{C}$ is defined by*

$$\mu(n) = \sum_{k \in R_n} \left(\cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n}\right),$$

where $R_n = \{k \in \mathbf{N} | 1 \le k \le n, gcd(k,n) = 1\}$. Show that for all positive integer n, $\mu(n)$ is an integer.

H 36. *Prove that there is a function f from the set of all natural numbers into itself such that for any natural number n, $f(f(n)) = n^2$.*

H 37. *Find all surjective function $f : \mathbf{N} \longrightarrow \mathbf{N}$ satisfying the condition $m|n \Longleftrightarrow f(m)|f(n)$ for all $m, n \in \mathbf{N}$.*

11. Polynomials

The only way to learn Mathematics is to do Mathematics. Paul Halmos

I 1. *Suppose* $p(x) \in \mathbf{Z}[x]$ *and* $P(a)P(b) = -(a-b)^2$ *for some distinct* $a, b \in \mathbf{Z}$. *Prove that* $P(a) + P(b) = 0$.

I 2. *Prove that there is no noncontant polynomial* $f(x)$ *with integral coefficients such that* $f(n)$ *is prime for all* $n \in \mathbf{N}$.

I 3. *Let* $n \geq 2$ *be an integer. Prove that if* k^2+k+n *is prime for all integers* k *such that* $0 \leq k \leq \sqrt{\frac{n}{3}}$, *then* k^2+k+n *is prime for all integers* k *such that* $0 \leq k \leq n-2$.

I 4. *A prime* p *has decimal digits* $p_n p_{n-1} \cdots p_0$ *with* $p_n > 1$. *Show that the polynomial* $p_n x^n + p_{n-1} x^{n-1} + \cdots + p_1 x + p_0$ *cannot be represented as a product of two nonconstant polynomials with integer coefficients*

I 5. *(Eisentein's Criterion) Let* $f(x) = a_n x^n + \cdots + a_1 x + a_0$ *be a nonconstant polynomial with integer coefficients. If there is a prime* p *such that* p *divides each of* a_0, a_1, $\cdots$,a_{n-1} *but* p *does not divide* a_n *and* p^2 *does not divide* a_0, *then* $f(x)$ *is irreducible in* $\mathbf{Q}[x]$.

I 6. *Prove that for a prime* p, $x^{p-1} + x^{p-2} + \cdots + x + 1$ *is irreducible in* $\mathbf{Q}[x]$.

I 7. *Let* $f(x) = x^n + 5x^{n-1} + 3$, *where* $n > 1$ *is an integer. Prove that* $f(x)$ *cannot be expressed as the product of two nonconstant polynomials with integer coefficients.*

I 8. *(Eugen Netto) Show that a polynomial of odd degree* $2m+1$ *over* $\mathbf{Z}$,

$$f(x) = c_{2m+1}x^{2m+1} + \cdots + c_1 x + c_0,$$

is irreducible if there exists a prime p *such that*

$$p \not| c_{2m+1}, p | c_{m+1}, c_{m+2}, \cdots, c_{2m}, p^2 | c_0, c_1, \cdots, c_m, \text{ and } p^3 \not| c_0.$$

I 9. *For non-negative integers* n *and* k, *let* $P_{n,k}(x)$ *denote the rational function*

$$\frac{(x^n-1)(x^n-x)\cdots(x^n-x^{k-1})}{(x^k-1)(x^k-x)\cdots(x^k-x^{k-1})}.$$

Show that $P_{n,k}(x)$ *is actually a polynomial for all* $n, k \in \mathbf{N}$.

I 10. *Suppose that the integers* a_1, a_2, $\cdots$, a_n *are distinct. Show that* $(x-a_1)(x-a_2)\cdots(x-a_n)-1$ *cannot be expressed as the product of two nonconstant polynomials with integer coefficients.*

I 11. *Show that the polynomail* $x^8 + 98x^4 + 1$ *can be expressed as the product of two nonconstant polynomials with integer coefficients.*

12. Sequences of Integers

A peculiarity of the higher arithmetic is the great difficulty which has often been experienced in proving simple general theorems which had been suggested quite naturally by numerical evidence. Harold Davenport

J 1. *If $a_1 < a_2 < \cdots < a_n$ are integers, show that*

$$\prod_{1\le i<j\le n} \frac{a_i - a_j}{i-j}$$

is an integer.[19]

J 2. *Show that the sequence $\{a_n\}_{n\ge 1}$ defined by $a_n = [n\sqrt{2}]$ contains an infinite number of integer powers of 2.*

J 3. *Let a_n be the last nonzero digit in the decimal representation of the number $n!$. Does the sequence $a_1,\ a_2,\ a_3,\ \cdots$ become periodic after a finite number of terms ?*

J 4. *Let $n > 6$ be an integer and $a_1, a_2, \dots, a_k$ be all the natural numbers less than n and relatively prime to n. If*

$$a_2 - a_1 = a_3 - a_2 = \cdots = a_k - a_{k-1} > 0,$$

prove that n must be either a prime number or a power of 2.

J 5. *Show that if an infinite arithmetic progression of positive integers contains a square and a cube, it must contain a sixth power.*

J 6. *Prove that there exists two strictly increasing sequences a_n and b_n such that $a_n(a_n+1)$ divides ${b_n}^2+1$ for every natural n.*

J 7. *Let $\{a_n\}$ be a strictly increasing positive integers sequece such that $gcd(a_i, a_j) = 1$ and $a_{i+2} - a_{i+1} > a_{i+1} - a_i$. Show that the infinite series*

$$\sum_{i=1}^{\infty} \frac{1}{a_i}$$

converges.

J 8. *Let $\{n_k\}_{k\ge 1}$ be a sequence of natural numbers such that for $i < j$, the decimal representation of n_i does not occur as the leftmost digits of the decimal representation of n_j. Prove that*

$$\sum_{k=1}^{\infty} \frac{1}{n_k} \le \frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{9}.$$

[19] The result follows immediately from the theory of Lie groups; the number turns out to be the dimension of an irreducible representation of $SU(n)$. [Rc]

J 9. *Let $P(x)$ be a nonzero polynomial with integral coefficients. Let $a_0 = 0$ and for $i \ge 0$ define $a_{i+1} = P(a_i)$. Show that $gcd(a_m, a_n) = a_{gcd(m,n)}$ for all $m, n \in \mathbf{N}$*

J 10. *An integer sequence $\{a_n\}_{n\ge1}$ is defined by*

$$a_0 = 0,\ a_1 = 1,\ a_{n+2} = 2a_{n+1} + a_n$$

Show that 2^k divides a_n if and only if 2^k divides n.

J 11. *An integer sequence $\{a_n\}_{n\ge1}$ is defined by*

$$a_1 = 1,\ a_{n+1} = a_n + [\sqrt{a_n}]$$

Show that a_n is a square if and only if $n = 2^k + k - 2$ for some $k \in \mathbf{N}$.

J 12. *Let $f(n) = n + [\sqrt{n}]$. Prove that, for every positive integer m, the sequence*

$$m, f(m), f(f(m)), f(f(f(m))), \cdots$$

contains at least one square of an integer.

J 13. *An integer sequence $\{a_n\}_{n\ge1}$ is given such that*

$$2^n = \sum_{d|n} a_d$$

for all $n \in \mathbf{N}$. Show that a_n is divisible by n.

J 14. *The sequence $\{a_n\}_{n\ge1}$ is defined by*

$$a_1 = 1,\ a_2 = 2,\ a_3 = 24,\ a_{n+2} = \frac{6{a_{n-1}}^2 a_{n-3} - 8a_{n-1}{a_{n-2}}^2}{a_{n-2}a_{n-3}} (n \ge 4)$$

Show that for all n, a_n is an integer.

J 15. *Show that there is a unique sequence of integers $\{a_n\}_{n\ge1}$ with*

$$a_1 = 1,\ a_2 = 2,\ a_4 = 12,\ a_{n+1}a_{n-1} = {a_n}^2 + 1\ \ (n \ge 2).$$

J 16. *The sequence $\{a_n\}_{n\ge1}$ is defined by*

$$a_1 = 1,\ a_{n+1} = 2a_n + \sqrt{3{a_n}^2 + 1}\ \ (n \ge 1)$$

Show that a_n is an integer for every n.

J 17. *Prove that the sequence $\{y_n\}_{n\ge1}$ defined by*

$$y_0 = 1,\ y_{n+1} = \frac{1}{2}\left(3y_n + \sqrt{5{a_n}^2 - 4}\right)\ \ (n \ge 0)$$

consists only of integers.

J 18. *(C. von Staudt) The Bernoulli sequence*[20] *$\{B_n\}_{n\ge0}$ is defined by*

$$B_0 = 1,\ B_n = -\frac{1}{n+1}\sum_{k=0}^{n}\binom{n+1}{k}B_k\ \ (n \ge 1)$$

[20] $B_0 = 1, B_1 = -\frac{1}{2}, B_2 = \frac{1}{6}, B_3 = 0, B_4 = -\frac{1}{30}, B_5 = 0, B_6 = \frac{1}{42}, \cdots$

Show that for all $n \in \mathbf{N}$,

$$(-1)^n B_n - \sum \frac{1}{p},$$

is an integer where the summation being extended over the primes p such that $p|2k-1$.

J 19. *An integer sequence $\{a_n\}_{n\geq 1}$ is defined by*

$$a_1 = 2,\ a_{n+1} = \left[\frac{3}{2}a_n\right]$$

Show that there are infinitely many even and infinitely many odd integers.

J 20. *An integer sequence satisfies $a_{n+1} = {a_n}^3 + 1999$. Show that it contains at most one square.*

J 21. *Let $a_1 = 11^{11}$, $a_2 = 12^{12}$, $a_3 = 13^{13}$, and*

$$a_n = |a_{n-1} - a_{n-2}| + |a_{n-2} - a_{n-3}|, n \geq 4.$$

Determine $a_{14^{14}}$.

J 22. *Let k be a fixed positive integer. The infinite sequence a_n is defined by the formulae*

$$a_1 = k+1, a_{n+1} = {a_n}^2 - ka_n + k \ \ (n \geq 1).$$

Show that if $m \neq n$, then the numbers a_m and a_n are relatively prime.

J 23. *The Fibonacci sequence $\{F_n\}$ is defined by*

$$F_1 = 1,\ F_2 = 1,\ F_{n+2} = F_{n+1} + F_n.$$

Show that $gcd(F_m, F_n) = F_{gcd(m,n)}$ for all $m, n \in \mathbf{N}$.

J 24. *The Fibonacci sequence $\{F_n\}$ is defined by*

$$F_1 = 1,\ F_2 = 1,\ F_{n+2} = F_{n+1} + F_n.$$

Show that $F_{mn-1} - {F_{n-1}}^m$ is divisible by ${F_n}^2$ for all $m \geq 1$ and $n > 1$.

J 25. *The Fibonacci sequence $\{F_n\}$ is defined by*

$$F_1 = 1,\ F_2 = 1,\ F_{n+2} = F_{n+1} + F_n.$$

Show that $F_{mn} - {F_{n+1}}^m + {F_{n-1}}^m$ is divisible by ${F_n}^3$ for all $m \geq 1$ and $n > 1$.

J 26. *The Fibonacci sequence $\{F_n\}$ is defined by*

$$F_1 = 1,\ F_2 = 1,\ F_{n+2} = F_{n+1} + F_n.$$

Show that ${F_{2n-1}}^2 + {F_{2n+1}}^2 + 1 = 3F_{2n-1}F_{2n+1}$ for all $n \geq 1$. [21]

J 27. *Prove that no Fibonacci number can be factored into a product of two smaller Fibonacci numbers, each greater than 1.*

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[21] See **A5**

J 28. *The sequence* $\{x_n\}$ *is defined by*

$$x_0 \in [0,1],\ x_{n+1} = 1 - |1 - 2x_n|.$$

Prove that the sequence is periodic if and only if x_0 *is irrational.*

J 29. *Let* x_1 *and* x_2 *be relatively prime positive integers. For* $n \geq 2$, *define* $x_{n+1} = x_n x_{n-1} + 1$.

(a) *Prove that for every* $i > 1$, *there exists* $j > i$ *such that* ${x_i}^i$ *divides* ${x_j}^j$.

(b) *Is it true that* x_1 *must divide* ${x_j}^j$ *for some* $j > 1$?

J 30. *For a given positive integer* k *denote the square of the sum of its digits by* $f_1(k)$ *and let* $f_{n+1}(k) = f_1(f_n(k))$. *Determine the value of* $f_{1991}(2^{1990})$.

J 31. *Let* $q_0, q_1, \cdots$ *be a sequence of integers such that*

(i) *for any* $m > n$, $m - n$ *is a factor of* $q_m - qn$, *and*
(ii) $|q_n| \leq n^{10}$ *for all integers* $n \geq 0$.

Show that there exists a polynomial $Q(x)$ *satisfying* $q_n = Q(n)$ *for all* n.

J 32. *Let* a, b *be integers greater than 2. Prove that there exists a positive integer* k *and a finite sequence* $n_1, n_2, \ldots, n_k$ *of positive integers such that* $n_1 = a$, $n_k = b$, *and* $n_i n_{i+1}$ *is divisible by* $n_i + n_{i+1}$ *for each* i $(1 \leq i < k)$.

J 33. *Define a sequence* $\{a_i\}$ *by* $a_1 = 3$ *and* $a_{i+1} = 3^{a_i}$ *for* $i \geq 1$. *Which integers between* 00 *and* 99 *inclusive occur as the last two digits in the decimal expansion of infinitely many* a_i?

J 34. *Let* m *be a positive integer. Define the sequence* $\{a_n\}_{n \geq 0}$ *by*

$$a_0 = 0,\ a_1 = m,\ a_{n+1} = m^2 a_n - a_{n-1}.$$

Prove that an ordered pair (a, b) *of non-negative integers, with* $a \leq b$, *gives a solution to the equation*

$$\frac{a^2 + b^2}{ab + 1} = m^2$$

if and only if (a, b) *is of the form* (a_n, a_{n+1}) *for some* $n \geq 0$. [22]

J 35. *Let* x_n, y_n *be two sequences defined recursively as follows*

$$x_0 = 1,\ x_1 = 4,\ x_{n+2} = 3x_{n+1} - x_n$$

$$y_0 = 1,\ y_1 = 2,\ y_{n+2} = 3y_{n+1} - y_n$$

for all $n = 0, 1, 2, \cdots$.
a) *Prove that* ${x_n}^2 - 5{y_n}^2 + 4 = 0$ *for all non-negative integers.*
b) *Suppose that* a, b *are two positive integers such that* $a^2 - b^2 + 4 = 0$. *Prove that there exists a non-negative integer* k *such that* $a = x_k$ *and* $b = y_k$.

[22]See **A3**

J 36. *The infinite sequence of 2's and 3's*

$$2, 3, 3, 2, 3, 3, 3, 2, 3, 3, 3, 2, 3, 3, 2, 3, 3,$$
$$3, 2, 3, 3, 3, 2, 3, 3, 3, 2, 3, 3, 2, 3, 3, 3, 2, \ldots$$

has the property that, if one forms a second sequence that records the number of 3's between successive 2's, the result is identical to the given sequence. Show that there exists a real number r such that, for any n, the nth term of the sequence is 2 if and only if $n = 1 + \lfloor rm \rfloor$ for some nonnegative integer m. (Note: $\lfloor x \rfloor$ denotes the largest integer less than or equal to x.)

J 37. *Let $\{u_n\}_{n\geq 0}$ be a sequence of positive integers defined by*

$$u_0 = 1,\ u_{n+1} = au_n + b,$$

where $a, b \in \mathbf{N}$. Prove that for any choice of a and b, the sequence $\{u_n\}_{n\geq 0}$ contains infinitely many composite numbers.

J 38. *A sequence of integers, $\{a_n\}_{n\geq 1}$ with $a_1 > 0$, is defined by*

$$a_{n+1} = \frac{a_n}{2} \quad \textit{if } n \equiv 0 \pmod 4,$$
$$a_{n+1} = 3a_n + 1 \quad \textit{if } n \equiv 1 \pmod 4,$$
$$a_{n+1} = 2a_n - 1 \quad \textit{if } n \equiv 2 \pmod 4,$$
$$a_{n+1} = \frac{a_n + 1}{4} \quad \textit{if } n \equiv 3 \pmod 4.$$

Prove that there is an integer m such that $a_m = 1$.

J 39. *Given is an integer sequence $\{a_n\}_{n\geq 0}$ such that $a_0 = 2$, $a_1 = 3$ and, for all positive integers $n \geq 1$, $a_{n+1} = 2a_{n-1}$ or $a_{n+1} = 3a_n - 2a_{n-1}$. Does there exist a positive integer k such that $1600 < a_k < 2000$?*

J 40. *A sequence with first two terms equal 1 and 24 respectively is defined by the following rule: each subsequent term is equal to the smallest positive integer which has not yet occurred in the sequence and is not coprime with the previous term. Prove that all positive integers occur in this sequence.*

J 41. *Each term of a sequence of natural numbers is obtained from the previous term by adding to it its largest digit. What is the maximal number of successive odd terms in such a sequence?*

J 42. *In the sequence $1, 0, 1, 0, 1, 0, 3, 5, \cdots$, each member after the sixth one is equal to the last digit of the sum of the six members just precedeing it. Prove that in this sequence one cannot find the following group of six consecutive members :*

$$0, 1, 0, 1, 0, 1$$

J 43. *Let a, b be odd positive integers. Define the sequence (f_n) by putting $f_1 = a$, $f_2 = b$, and by letting f_n for $n \geq 3$ be the greatest odd divisor of $f_{n-1} + f_{n-2}$. Show that f_n is constant for n sufficiently large and determine the eventual value as a function of a and b.*

J 44. *Numbers $d(n,m)$ with m,n integers, $0 \leq m \leq n$, are defined by $d(n,0) = d(n,n) = 1$ for all $n \geq 0$, and $md(n,m) = md(n-1,m) + (2n-m)d(n-1,m-1)$ for $0 < m < n$. Prove that all the $d(n,m)$ are integers.*

J 45. *Let k be a given positive integer. The sequence x_n is defined as follows : $x_1 = 1$ and x_{n+1} is the least positive integer which is not in $\{x_1, x_2, ..., x_n, x_1+k, x_2+2k, ..., x_n+nk\}$. Show that there exist real number a such that $x_n = [an]$ for all poisive integer n.*

J 46. *Let $\{a_n\}_{n\geq 1}$ be a sequence of positive integers such that*

$$0 < a_{n+1} - a_n \leq 2001 \; for \; all \; n \in \mathbf{N}$$

Show that there are infinitely many pairs (p,q) of positive integers such that $p > q$ and $a_q \mid a_p$.

J 47. *Let p be an odd prime p such that $2h \neq 1 \pmod p$ for all $h \in \mathbf{N}$ with $h < p-1$, and let a be an even integer with $a \in \left(\frac{p}{2}, p\right)$. The sequence $\{a_n\}_{n\geq 0}$ is defined by $a_0 = a$, $a_{n+1} = p - b_n \quad (n \in \{0,1,2,\cdots\})$, where b_n is the greatest odd divisor of a_n. Show that the sequence $\{a_n\}_{n\geq 0}$ is periodic and find its minimal (positive) period.*

J 48. *Let $p \geq 3$ be a prime number. The sequence $\{a_n\}_{n\geq 1}$ is defined by $a_n = n$ for all $0 \leq n \leq p-1$, and $a_n = a_{n-1} + a_{n-p}$, for all $n \geq p$. Compute $a_{p^3} \pmod p$.*

J 49. *Let $\{u_n\}_{n\geq 0}$ be a sequence of integers satisfying the recurrence relation $u_{n+2} = {u_{n+1}}^2 - u_n$ $(n \in \mathbf{N})$. Suppose that $u_0 = 39$ and $u_1 = 45$. Prove that 1986 divides infinitely many terms of this sequence.*

J 50. *The sequence $\{y_n\}_{n\geq 1}$ is defined by $y_1 = y_2 = 1$ and $y_{n+2} = (4k-5)y_{n+1} - y_n + 4 - 2k$ $(n \in \mathbf{N})$. Determine all integers k such that each term of this sequence is a perfect square.*

J 51. *The sequence $\{a_n\}_{n\geq 1}$ is defined by $a_1 = 1$ and*

$$a_{n+1} = \frac{a_n}{2} + \frac{1}{4a_n} \; (n \in \mathbf{N}).$$

Prove that $\sqrt{\frac{2}{2{a_n}^2-1}}$ is a positive integer for $n > 1$.

J 52. *Let k be a positive integer. Prove that there exists an infinite, monotone increasing sequence of integers $\{a_n\}_{n\geq 1}$ such that*

$$a_n \; divides \; {a_{n+1}}^2 + k \; and \; a_{n+1} \; divides \; {a_n}^2 + k$$

for all $n \in \mathbf{N}$.

J 53. *Let the sequence $\{K_n\}_{n\geq 1}$ be defined by $K_1 = 2$, $K_2 = 8$, $K_{n+2} = 3K_{n+1} - K_n + 5(-1)^n$. Prove that K_n is prime, then n must be a power of 3.*

J 54. *The sequence* $\{a_n\}_{n\ge1}$ *is defined by*

$$a_n = 1 + 2^2 + 3^3 + \cdots + n^n.$$

Prove that there are infinitely many n *such that* a_n *is composite.*

J 55. *One member of an infinite arithmetic sequence in the set of natural numbers is a perfect square. Show that there are infinitely many members of this sequence having this property.*

J 56. *In the sequence* 00, 01, 02, 03, $\cdots$, 99 *the terms are rearranged so that each term is obtained from the previous one by increasing or decreasing one of its digits by* 1 *(for example,* 29 *can be followed by* 19, 39, *or* 28, *but not by* 30 *or* 20*). What is the maximal number of terms that could remain on their places?*

J 57. *Each term of an infinite sequence of natural numbers is obtained from the previous term by adding to it one of its nonzero digits. Prove that this sequence contains an even number.*

J 58. *In an increasing infinite sequence of positive integers, every term starting from the* 2002*-th term divides the sum of all preceding terms. Prove that every term starting from some term is equal to the sum of all preceding terms.*

J 59. *Does there exist positive integers* $a_1 < a_2 < \cdots < a_{100}$ *such that for* $2 \le k \le 100$, *the least common multiple of* a_{k-1} *and* a_k *is greater than the least common multiple of* a_k *and* a_{k+1} *?*

J 60. *Does there exist positive integers* $a_1 < a_2 < \cdots < a_{100}$ *such that for* $2 \le k \le 100$, *the greatest common divisor of* a_{k-1} *and* a_k *is greater than the greatest common divisor of* a_k *and* a_{k+1} *?*

J 61. *The sequence* $\{x_n\}_{n\ge1}$ *is defined by* $x_1 = 2$ *and* $x_{n+1} = \frac{2+x_n}{1-2x_n}$ $(n \in \mathbf{N})$. *Prove that (a)* $x_n \neq$ *for all* $n \in \mathbf{N}$ *and (b)* $\{x_n\}_{n\ge1}$ *is not periodic.*

J 62. *The sequence* $\{a_n\}_{n\ge1}$ *is defined by* $a_1 = 1$, $a_2 = 12$, $a_3 = 20$ *and* $a_{n+3} = 2a_{n+2} + 2a_{n+1} - a_n$ $(n \in \mathbf{N})$. *Prove that* $1 + 4a_na_{n+1} \neq$ *is a square for all* $n \in \mathbf{N}$.

13. Combinatorial Number Theory

In great mathematics there is a very high degree of unexpectedness, combined with inevitability and economy. Godfrey Harold Hardy

K 1. *Let p be a prime. Find all k such that the set $\{1, 2, \cdots, k\}$ can be partitioned into p subsets with equal sum of elements.*

K 2. *Prove that the set of integers of the form $2^k - 3 (k = 2, 3, ...)$ contains an infinite subset in which every two members are relatively prime.*

K 3. *The set of positive integers is parititioned into finitely many subsets. Show that some subset S has the following property : for every positive integer n, S contains infinitely many multiples of n.*

K 4. *Let M be a positive integer and consider the set*

$$S = \{n \in \mathbf{N} | M^2 \le n < (M + 1)^2\}.$$

Prove that the products of the form ab with $a, b \in S$ are distinct.

K 5. *Let S be a set of integers such that*

- *there exist $a, b \in S$ with $gcd(a, b) = gcd(a - 2, b - 2) = 1$.*
- *if x and y are elements of S, then $x^2 - y$ also belongs to S. Prove that S is the set of all integers.*

K 6. *Show that for each $n \ge 2$, there is a set S of n integers such that $(a - b)^2$ divides ab for every distinct $a, b \in S$*

K 7. *Let a and b be positive integers greater than 2. Prove that there exists a positive integer k and a finite sequence $n_1, \cdots, n_k$ of positive integers such that $n_1 = a$, $n_k = b$, and $n_i n_{i+1}$ is divisible by $n_i + n_{i+1}$ for each i $(1 \le i \le k)$.*

K 8. *Let n be an integer, and let X be a set of $n+2$ integers each of absolute value at most n. Show that there exist three distinct numbers $a, b, c \in X$ such that $c = a + b$.*

K 9. *Let $m \ge 2$ be an integer. Find the smallest integer $n > m$ such that for any partition of the set $\{m, m + 1, \cdots, n\}$ into two subsets, at least one subset contains three numbers a, b, c such that $c = a^b$.*

K 10. *Let $S = \{1, 2, 3, \ldots, 280\}$. Find the smallest integer n such that each n-element subset of S contains five numbers which are pairwise relatively prime.*

K 11. *Let m and n be positive integers. If $x_1, x_2, \cdots, x_m$ are postive integers whose average is less than $n + 1$ and if $y_1, y_2, \cdots, y_n$ are postive integers whose average is less than $m + 1$, prove that some sum of one or more x's equals some sum of one or more y's.*

K 12. *Let n and k be given relatively prime natural numbers, $k < n$. Each number in the set $M = \{1, 2, ..., n-1\}$ is colored either blue or white. It is given that*

- *for each $i \in M$, both i and $n-i$ have the same color;*
- *for each $i \in M, i \neq k$, both i and $|i-k|$ have the same color.*

Prove that all numbers in M must have the same color.

K 13. *Let p be a prime number, $p \geq 5$, and k be a digit in the p-adic representation of positive integers. Find the maximal length of a non constant arithmetic progression whose terms do not contain the digit k in their p-adic representation.*

K 14. *Is it possible to choose 1983 distinct positive integers, all less than or equal to 10^5, no three of which are consecutive terms of an arithmetic progression?*

K 15. *Is it possible to find 100 positive integers not exceeding 25000 such that all pairwise sums of them are different ?*

K 16. *Find the maximum number of pairwise disjoint sets of the form*

$$S_{a,b} = \{n^2 + an + b | n \in \mathbf{Z}\},$$

with $a, b \in \mathbf{Z}$.

K 17. *Let p be an odd prime number. How many p-element subsets A of $\{1, 2, \ldots\ 2p\}$ are there, the sum of whose elements is divisible by p?*

K 18. *Let $m, n \geq 2$ be positive integers, and let $a_1, a_2, \cdots, a_n$ be integers, none of which is a multiple of m^{n-1}. Show that there exist integers $e_1, e_2, \cdots, e_n$, not all zero, with $|e_i| < m$ for all i, such that $e_1a_1 + e_2a_2 + \cdots + e_na_n$ is a multiple of m^n.*

K 19. *Determine the smallest integer $n \geq 4$ for which one can choose four different numbers a, b, c, and d from any n distinct integers such that $a + b - c - d$ is divisible by 20*

K 20. *A sequence of integers $a_1, a_2, a_3, \cdots$ is defined as follows : $a_1 = 1$, and for $n \geq 1$, a_{n+1} is the smallest integer greater than a_n such that $a_i + a_j \neq 3a_k$ for any i, j, and k in $\{1, 2, 3, \cdots, n+1\}$, not necessarily distinct. Determine a_{1998}.*

K 21. *Prove that for each positive integer n, there exists a positive integer with the following properties :*

- *It has exactly n digits.*
- *None of the digits is 0.*
- *It is divisible by the sum of its digits.*

K 22. *Let k, m, n be integers such that $1 < n \leq m - 1 \leq k$. Determine the maximum size of a subset S of the set $\{1, 2, \cdots, k\}$ such that no n distinct elements of S add up to m.*

K 23. *Find the number of subsets of* $\{1, 2, \cdots, 2000\}$*, the sum of whose elements is divisible by* 5.

K 24. *Let* A *be a non-empty set of positive integers. Suppose that there are positive integers* $b_1, \cdots, b_n$ *and* $c_1, \cdots, c_n$ *such that*

(i) *for each* i *the set* $b_iA + c_i = \{b_ia + c_i | a \in A\}$ *is a subset of* A*, and*

(ii) *the sets* b_iA+c_i *and* b_jA+c_j *are disjoint whenever* $i \neq j$.

Prove that

$$\frac{1}{b_1} + \cdots + \frac{1}{b_n} \leq 1.$$

K 25. *A set of three nonnegative integers* $\{x, y, z\}$ *with* $x < y < z$ *is called historic if* $\{z-y, y-x\} = \{1776, 2001\}$. *Show that the set of all nonnegative integers can be written as the unions of pairwise disjoint historic sets.*

K 26. *Let* p *and* q *be relatively prime positive integers. A subset* S *of* $\{0, 1, 2, \cdots\}$ *is called ideal if* $0 \in S$ *and, for each element* $n \in S$*, the integers* $n + p$ *and* $n + q$ *belong to* S. *Determine the number of ideal subsets of* $\{0, 1, 2, \cdots\}$.

K 27. *Prove that the set of positive integers cannot be partitioned into three nonempty subsets such that, for any two integers* x, y *taken from two different subsets, the number* $x^2 - xy + y^2$ *belongs to the third subset.*

K 28. *Let* A *be a set of* N *residues* (mod N^2). *Prove that there exists a set* B *of* N *residues* (mod N^2) *such that the set* $A + B = \{a + b | a \in A, b \in B\}$ *contains at least half of all the residues* (mod N^2).

K 29. *Determine the largest positive integer* n *for which there exists a set* S *with exactly* n *numbers such that*

(i) *each member in* S *is a positive integer not exceeding* 2002,

(ii) *if* a *and* b *are two (not necessarily different) numbers in* S*, then there product* ab *does not belong to* S.

K 30. *Prove that, for any integer* $a_1 > 1$ *there exist an increasing sequence of positive integers* $a_1, a_2, a_3, \cdots$ *such that*

$$a_1 + a_2 + \cdots + a_n \mid {a_1}^2 + {a_2}^2 + \cdots + {a_n}^2$$

for all $k \in \mathbf{N}$.

K 31. *An odd integer* $n \geq 3$ *is said to be "nice" if and only if there is at least one permution* $a_1, \cdots, a_n$ *of* $1, \cdots, n$ *such that the* n *sums* $a_1 - a_2 + a_3 - \cdots - a_{n-1} + a_n$, $a_2 - a_3 + a_3 - \cdots - a_n + a_1$, $a_3 - a_4 + a_5 - \cdots - a_1 + a_2$, $\cdots$, $a_n - a_1 + a_2 - \cdots - a_{n-2} + a_{n-1}$ *are all positive. Determine the set of all "nice" integers.*

K 32. *Assume that the set of all positive integers is decomposed into* r *distinct subsets* $A_1 \cup A_2 \cup \cdots \cup A_r = \mathbf{N}$. *Prove that one of them, say* A_i*, has the following property : There exist a positive integer* m *such that for any*

k one can find numbers $a_1, \cdots, a_k$ in A_i with $0 < a_{j+1} - a_j \le m$ $(1 \le j \le k-1)$.

K 33. *Determine for which positive integers k, the set*

$$X = \{1990, 1990+1, 1990+2, \cdots, 1990+k\}$$

can be partitioned into two disjoint subsets A and B such that the sum of the elements of A is equal to the sum of the elements of B.

K 34. *Prove that $n \ge 3$ be a prime number and $a_1 < a_2 < \cdots < a_n$ be integers. Prove that $a_1, \cdots, a_n$ is an arithmetic progression if and only if there exists a partition of $\{0, 1, 2, \cdots\}$ into classes $A_1, A_2, \cdots, A_n$ such that*

$$a_1 + A_1 = a_2 + A_2 = \cdots = a_n + A_n,$$

where $x + A$ denotes the set $\{x + a | a \in A\}$.

K 35. *Let a and b be non-negative integers such that $ab \ge c^2$ where c is an integer. Prove that there is a positive integer n and integers $x_1, x_2, \cdots, x_n, y_1, y_2, \cdots, y_n$ such that*

$${x_1}^2 + \cdots + {x_n}^2 = a, {y_1}^2 + \cdots + {y_n}^2 = b, x_1y_1 + \cdots + x_ny_n = c$$

K 36. *Let n, k be positive integers such that n is not divisible by 3 and k is greater or equal to n. Prove that there exists a positive integer m which is divisible by n and the sum of its digits in the decimal representation is k.*

K 37. *Prove that for every real number M there exists an infinite arithmetical progression such that*

- *each term is a positive integer and the common difference is not divisible by 10.*
- *the sum of digits of each term exceeds M.*

K 38. *Find the smallest positive integer n, for which there exist n different positive integers $a_1, a_2, \cdots, a_n$ satifying the conditions :*

a) the smallest common multiple of $a_1, a_2, \cdots, a_n$ is 1985;
b) for each $i, j \in \{1, 2, \cdots, n\}$, the numbers a_i and a_j have a common divisor;
c) the product $a_1a_2 \cdots a_n$ is a perfect square and is divisible by 243.

Find all n-tuples $(a_1, \cdots, a_n)$, satisfying a), b), and c).

K 39. *Let X be a non-empty set of positive integers which satisfies the following :*

(a) If $x \in X$, then $4x \in X$.
(b) If $x \in X$, then $[\sqrt{x}] \in X$.

Prove that $X = \mathbf{N}$.

K 40. *Prove that for every positive integer n there exists an n-digit number divisible by 5^n all of whose digits are odd.*

K 41. *Let N_n denote the number of ordered n-tuples of positive integers $(a_1, a_2, \ldots, a_n)$ such that $1/a_1 + 1/a_2 + \ldots + 1/a_n = 1$. Determine whether N_{10} is even or odd.*

K 42. *Is it possible to find a set A of eleven positive integers such that no six elements of A have a sum which is divisible by 6 ?*

K 43. *A set C of positive integers is called good if for every integer j there exist $a, b \in C (a \neq b)$ such that the numbers $a+k$ and $b+k$ are not relatively prime. Prove that if the sum of the elements of a good set C equals 2003, then there exists $c \in C$ such that the set $C - \{c\}$ is good.*

K 44. *Find the set of all positive integers n with the property that the set $\{n, n+1, n+2, n+3, n+4, n+5\}$ can be partitioned into two sets such that the product of the numbers in one set equals the product of the numbers in the other set.*

K 45. *Suppose p is a prime with $p \equiv 3 \pmod 4$. Show that for any set of $p-1$ consecutive integers, the set cannot be divided two subsets so that the the product of the members of the one set is equal to the product of the members of the other set.*

14. Additive Number Theory

On Ramanujan, G. H. Hardy Said : I remember once going to see him when he was lying ill at Putney. I had ridden in taxi cab number 1729 *and remarked that the number seemed to me rather a dull one, and that I hoped it was not an unfavorable omen. "No," he replied,*

"it is a very interesting number; it is the smallest number expressible as the sum of two cubes in two different ways."

L 1. *Show that any integer can be expressed as a sum of two squares and a cube.*

L 2. *Show that each integer* n *can be written as the sum of five perfect cubes (not necessarily positive).*

L 3. *Prove that infinitely many positive integers cannot be written in the form* ${x_1}^3 + {x_2}^5 + {x_3}^7 + {x_4}^9 + {x_5}^11$, *where* $x_1, x_2, x_3, x_4, x_5 \in \mathbf{N}$.

L 4. *Determine all positive integers that are expressible in the form* $a^2 + b^2 + c^2 + c$, *where* a, b, c *are integers.*

L 5. *Show that any positive rational number can be represented as the sum of three positive rational cubes.*

L 6. *Show that every integer greater than* 1 *can be written as a sum of two square-free integers.*

L 7. *Prove that every integer* $n \geq 12$ *is the sum of two composite numbers.*

L 8. *Prove that any positive integer can be represented as an aggregate of different powers of* 3, *the terms in the aggregate being combined by the signs* $+$ *and* $-$ *appropriately chosen.*

L 9. *The integer* 9 *can be written as a sum of two consecutive integers : 9=4+5 ; moreover it can be written as a sum of (more than one) consecutive positive integers in exactly two ways, namely 9=4+5= 2+3+4. Is there an integer which can be written as a sum of* 1990 *consecutive integers and which can be written as a sum of (more than one) consecutive integers in exactly* 1990 *ways ?*

L 10. *For each positive integer* n, $S(n)$ *is defined to be the greatest integer such that, for every positive integer* $k \leq S(n)$, n^2 *can be written as the sum of* k *positive squares.*

(a) Prove that $S(n) \leq n^2 - 14$ *for each* $n \geq 4$.
(b) Find an integer n *such that* $S(n) = n^2 - 14$.
(c) Prove that there are infintely many integers n *such that* $S(n) = n^2 - 14$.

L 11. *For each positive integer n , let $f(n)$ denote the number of ways of representing n as a sum of powers of 2 with nonnegative integer exponents. Representations which differ only in the ordering of their summands are considered to be the same. For instance, $f(4) = 4$, because the number 4 can be represented in the following four ways:*

$$4; 2+2; 2+1+1; 1+1+1+1.$$

Prove that, for any integer $n \geq 3$,

$$2^{n^2/4} < f(2^n) < 2^{n^2/2}.$$

L 12. *The positive function $p(n)$ is defined as the number of ways that the positive integer n can be written as a sum of positive integers.* [23] *Show that, for all $n > 1$,*

$$2^{[\sqrt{n}]} < p(n) < n^{3[\sqrt{n}]}.$$

L 13. *Let $a_1 = 1$, $a_2 = 2$, be the sequence of positive integers of the form $2^\alpha 3^\beta$, where α and β are nonnegative integers. Prove that every positive integer is exprssible in the form*

$$a_{i_1} + a_{i_2} + \cdots + a_{i_n},$$

where no summand is a multiple of any other.

L 14. *Let n be a non-negative integer. Find the non-negative integers a, b, c, d such that*

$$a^2 + b^2 + c^2 + d^2 = 7 \cdot 4^n.$$

L 15. *Find all integers $m > 1$ such that m^3 is a sum of m squares of consecutive integers.*

L 16. *A positive integer n is a square-free integer if there is no prime p such that $p^2 | n$. Show that every integer greater than 1 can be written as a sum of two square-free integers.*

L 17. *Prove that there exist infinitely many integers n such that $n, n+1, n+2$ are each the sum of the squares of two integers.*

L 18. *(Jacobsthal) Let p be a prime number of the form $4k+1$. Suppose that r is a quadratic residue of p and that s is a quadratic nonresidue of p. Show that $p = a^2 + b^2$, where*

$$a = \frac{1}{2} \sum_{i=1}^{p-1} \left(\frac{i(i^2 - r)}{p} \right), b = \frac{1}{2} \sum_{i=1}^{p-1} \left(\frac{i(i^2 - s)}{p} \right).$$

Here, $\left(\frac{k}{p}\right)$ denotes the Legendre Symbol.

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[23] For example, $5 = 4+1 = 3+2 = 3+1+1 = 2+2+1 = 2+1+1+1 = 1+1+1+1+1$, and so $p(5) = 7$.

L 19. *Let p be a prime with $p \equiv 1 (mod\, 4)$. Let a be the unique integer such that*

$$p = a^2 + b^2, a \equiv -1 (mod\, 4), b \equiv 0 (mod\, 2)$$

Prove that

$$\sum_{i=0}^{p-1} \left(\frac{i^3 + 6i^2 + i}{p} \right) = 2 \left(\frac{2}{p} \right) a.$$

L 20. *Let n be an integer of the form $a^2 + b^2$, where a and b are relatively prime integers and such that if p is a prime, $p \leq \sqrt{n}$, then p divides ab. Determine all such n.*

L 21. *If an integer n is such that $7n$ is the form $a^2 + 3b^2$, prove that n is also of that form.*

L 22. *Let A be the set of positive integers represented by the form $a^2 + 2b^2$, where a, b are integers and $b \neq 0$. Show that p is a prime number and $p^2 \in A$, then $p \in A$.*

L 23. *Show that an integer can be expressed as the difference of two squares if and only if it is not of the form $4k + 2 (k \in \mathbf{Z})$.*

L 24. *Show that there are infinitely many positive integers which cannot be expressed as the sum of squares.*

L 25. *Show that any integer can be expressed as the form $a^2 + b^2 - c^2$, where $a, b, c \in \mathbf{Z}$.*

L 26. *Let a and b be positive integers with $gcd(a, b) = 1$. Show that every integer greater than $ab - a - b$ can be expressed in the form $ax + by$, where $x, y \in \mathbf{N}_0$.*

L 27. *Let a, b and c be positive integers, no two of which have a common divisor greater than 1. Show that $2abc - ab - bc - ca$ is the largest integer which cannot be expressed in the form $xbc + yca + zab$, where $x, y, z \in \mathbf{N}_0$*

L 28. *Determine, with proof, the largest number which is the product of positive integers whose sum is* 1976.

L 29. *(Zeckendorf) Any positive integer can be represented as a sum of Fibonacci numbers, no two of which are consecutive.*

L 30. *Show that the set of positive integers which cannot be represented as a sum of distinct perfect squares is finite.*

L 31. *Let $a_1, a_2, a_3, \cdots$ be an increasing sequence of nonnegative integers such that every nonnegative integer can be expressed uniquely in the form $a_i + 2a_j + 4a_l$, where i, j, and k are not necessarily distinct. Determine a_{1998}.*

L 32. *A finite sequence of integers $a_0, a_1, \cdots, a_n$ is called quadratic if for each $i \in \{1, 2, \cdots, n\}$ we have the equality $|a_i - a_{i-1}| = i^2$.*

(a) Prove that for any two integers b and c, there exists a natural number n and a quadratic sequence with $a_0 = b$ and $a_n = c$.
(b) Find the smallest natural number n for which there exists a quadratic sequence with $a_0 = 0$ and $a_n = 1996$.

L 33. *A composite positive integer is a product ab with a and b not necessarily distinct integers in $\{2, 3, 4, \dots\}$. Show that every composite is expressible as $xy + xz + yz + 1$, with x, y, z positive integers.*

L 34. *Let $a_1, a_2, \cdots, a_k$ be relatively prime positive integers. Determine the largest integer not expressible in the form*

$$x_1 a_2 a_3 \cdots a_k + x_2 a_1 a_3 \cdots a_k + \cdots + x_k a_1 a_2 \cdots a_{k-1}$$

for some nonnegative integers $x_1, x_2, \cdots, x_k$.

15. The Geometry of Numbers

Srinivasa Aiyangar Ramanujan said *"An equation means nothing to me unless it expresses a thought of God."*

M 1. *Does there exist a convex pentagon, all of whose vertices are lattice points in the plane, with no lattice point in the interior?*

M 2. *Show there do not exist four points in the Euclidean plane such that the pairwise distances between the points are all odd integers.*

M 3. *Prove no three lattice points in the plane form an equilateral triangle.*

M 4. *The sides of a polygon with* 1994 *sides are* $a_i = \sqrt{i^2+4} \quad (i = 1, 2, \cdots, 1994)$. *Prove that its vertices are not all on lattice points.*

M 5. *A triangle has lattice points as vertices and contains no other lattice points. Prove that its area is* $\frac{1}{2}$.

M 6. *Let* R *be a convex region*[24] *symmetrical about the origin with area greater than* 4. *Then* R *must contain a lattice point*[25] *different form the origin.*

M 7. *Show that the number* $r(n)$ *of representations of* n *as a sum of two squares has average value* π, *that is*

$$\frac{1}{n}\sum_{m=1}^{n} r(m) \to \pi \text{ as } n \to \infty.$$

M 8. *Prove that on a coordinate plane it is impossible to draw a closed broken line such that (i) coordinates of each vertex are ratoinal, (ii) the length of its every edge is equal to* 1, *(iii) the line has an odd number of vertices.*

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[24]For any two points of R, their midpoint also lies in R.

[25]A point with integral coordinates

16. Miscellaneous Problems

Mathematics is not yet ready for such problems. Paul Erdös

N 1. *It is given that* 2^{333} *is a* 101*-digit number whose first digit is* 1*. How many of the numbers* 2^k, $1 \le k \le 332$, *have first digit* 4*?*

N 2. *Is there a power of* 2 *such that it is possible to rearange the digits giving another power of* 2 *?*

N 3. *If* x *is a real number such that* $x^2 - x$ *is an integer, and for some* $n \ge 3$, $x^n - x$ *is also an integer, prove that* x *is an integer.*

N 4. *(Tran Nam Dung) Suppose that both* $x^3 - x$ *and* $x^4 - x$ *are integers for some real number* x*. Show that* x *is an integer.*

N 5. *Suppose that* x *and* y *are complex numbers such that*

$$\frac{x^n - y^n}{x - y}$$

are integers for some four consecutive positive integers n*. Prove that it is an integer for all positive integers* n*.*

N 6. *Let* n *be a positive integer. Show that*

$$\sum_{i=1}^{n} \tan^2 \frac{i\pi}{2n+1}$$

is an odd integer.

N 7. *The set* $S = \{\frac{1}{n} \mid n \in \mathbf{N}\}$ *contains arithmetic progressions of various lengths. For instance,* $\frac{1}{20}$, $\frac{1}{8}$, $\frac{1}{5}$ *is such a progression of length* 3 *and common difference* $\frac{3}{40}$*. Moreover, this is a maximal progression in* S *since it cannot be extended to the left or the right within* S *(*$\frac{11}{40}$ *and* $\frac{-1}{40}$ *not being members of* S*). Prove that for all* $n \in \mathbf{N}$*, there exists a maximal arithmetic progression of length* n *in* S*.*

N 8. *Suppose that*

$$\prod_{n=1}^{1996} (1 + nx^{3^n}) = 1 + a_1 x^{k_1} + a_2 x^{k_2} + \cdots + a_m x^{k_m}$$

where a_1, a_2, ..., a_m *are nonzero and* $k_1 < k_2 < \cdots < k_m$*. Find* a_{1996}*.*

N 9. *Let* p *be an odd prime. Show that there is at most one non-degenerate integer triangle with perimeter* $4p$ *and integer area. Characterize those primes for which such triangle exist.*

N 10. *For each positive integer* n*, prove that there are two consecutive positive integers each of which is the product of* n *positive integers* > 1*.*

N 11. *Let*

$$\begin{array}{cccc} a_{1,1} & a_{1,2} & a_{1,3} & \cdots \\ a_{2,1} & a_{2,2} & a_{2,3} & \cdots \\ a_{3,1} & a_{3,2} & a_{3,3} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{array}$$

be a doubly infinite array of positive integers, and suppose each positive integer appears exactly eight times in the array. Prove that $a_{m,n} > mn$ *for some pair of positive integers* (m,n).

N 12. *The digital sum of a natural number* n *is denoted by* $S(n)$. *Prove that* $S(8n) \geq \frac{1}{8}S(n)$ *for each* n.

N 13. *Let* p *be an odd prime. Determine positive integers* x *and* y *for which* $x \leq y$ *and* $\sqrt{2p} - \sqrt{x} - \sqrt{y}$ *is nonnegative and as small as possible.*

N 14. *Let* $\alpha(n)$ *be the number of digits equal to one in the dyadic representation of a positive integer* n. *Prove that*

(a) the inequality $\alpha(n^2) \leq \frac{1}{2}\alpha(n)(1+\alpha(n))$ *holds,*
(b) the above inequality is equality for infinitely many positive integers, and
(c) there exists a sequence $\{n_i\}$ *such that* $\frac{\alpha({n_i}^2)}{\alpha(n_i)} \to 0$ *as* $i \to \infty$.

N 15. *Show that if* a *and* b *are positive integers, then*

$$\left(a+\frac{1}{2}\right)^n + \left(b+\frac{1}{2}\right)^n$$

is an integer for only finitely many positive integer n.

N 16. *Determine the maximum value of* m^2+n^2,*where* m *and* n *are integers satisfying* $m,n \in \{1,2,...,1981\}$ *and* $(n^2-mn-m^2)^2 = 1$.

N 17. *Denote by* S *the set of all primes* p *such that the decimal representation of* $\frac{1}{p}$ *has the fundamental period of divisible by* 3. *For every* $p \in S$ *such that* $\frac{1}{p}$ *has the fundamental period* $3r$ *one may write*

$$\frac{1}{p} = 0.a_1a_2\cdots a_{3r}a_1a_2\cdots a_{3r}\cdots,$$

where $r = r(p)$ *; for every* $p \in S$ *and every integer* $k \geq 1$ *define* $f(k,p)$ *by*

$$f(k,p) = a_k + a_{k+r(p)} + a_{k+2r(p)}.$$

a) Prove that S *is finite.*
b) Find the hightst value of $f(k,p)$ *for* $k \geq 1$ *and* $p \in S$.

N 18. *Determine all pairs* (a,b) *of real numbers such that* $a[bn] = b[an]$ *for all positive integer* n. *(Note that* $[x]$ *denotes the greatest integer less than or equal to* x.*)*

N 19. *Let n be a positive integer that is not a perfect cube. Define real numbers a, b, c by*

$$a = n^{\frac{1}{3}}, b = \frac{1}{a - [a]}, c = \frac{1}{b - [b]},$$

where $[x]$ denotes the integer part of x. Prove that there are infinitely many such integers n with the property that there exist integers r, s, t, not all zero, such that $ra + sb + tc = 0$.

N 20. *Find, with proof, the number of positive integers whose base-n representation consists of distinct digits with the property that, except for the leftmost digit, every digit differs by ± 1 from some digit further to the left.*

N 21. *The decimal expression of the natural number a consists of n digits, while that of a^3 consists of m digits. Can $n + m$ be equal to 2001?*

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H 28. *Canada 1992*

H 29. *IMO Short List 1983 (Belgium)*

H 30 (Rh pp.104)**.** *Quantum, Problem M59, Contributed by B. Martynov*

4. More Functions

H 31 (Tau)**.** *and [Tau2]*

H 32. *Germany 1996*

H 33. *Belarus 1999, Proposed by D. Bazylev*

H 34 (Tma, pp.175)**.**

H 35.

H 36. *Singapore 1996*

H 37. *Turkey 1995*

Polynomials

I 1. *Math. Magazine, Problem Q800, Proposed by Bjorn Poonen*

I 2.

I 3. *IMO 1987/6*

I 4. *Balkan Mathematical Olympiad 1989*

I 5 (Twh, pp. 111)**.**

I 6 (Twh, pp. 114)**.**

I 7. *IMO 1993/1*

I 8 (Ac, pp. 87)**.** *For a proof, see [En].*

I 9. *Crux Mathematicorum with Mathematical Mayhem, Problem A230, Proposed by Naoki Sato*

I 10 (Ae, pp. 257).

I 11 (Ae, pp. 258).

Sequences of Integers

J 1.

J 2. *IMO Long List 1985 (RO3)*

J 3. *IMO Short List 1991 P14 (USS 2)*

J 4. *IMO 1991/2*

J 5. *IMO Short List 1993*

J 6. *IMO Short List 1999 N3*

J 7. *Pi Mu Epsilon Journal, Problem 339, Proposed by Paul Erdös*

J 8. *Iran 1998*

J 9.

J 10. *IMO Short List 1988*

J 11. *Amer. Math. Monthly, Problem E2619, Proposed by Thomas C. Brown*

J 12. *Putnam 1983*

J 13. *IMO Short List 1989*

J 14. *Putnam 1999*

J 15. *United Kingdom 1998*

J 16. *Serbia 1998*

J 17. *United Kingdom 2002*

J 18 (KiMr pp. 233).

J 19. *Putnam 1983*

J 20. *APMC 1999*

J 21. *IMO Short List 2001 N3*

J 22. *Poland 2002*

J 23 (Nv pp.58).

J 24 (Nv pp.74).

J 25 (Nv pp.75).

J 26 (Eb1 pp.21).

J 27. *Math. Magazine, Problem 1390, Proposed by J. F. Stephany*

J 28 (Ae pp.228).

J 29. *IMO Short List 1994 N6*

J 30. *IMO Short List 1990 HUN1*

J 31. *Taiwan 1996*

J 32. *USA 2002*

J 33. *Putnam 1985/A4*

J 34. *Canada 1998*

J 35. *Vietnam 1999*

J 36. *Putnam 1993/A6*

J 37. *Germany 1995 - Arne Smeets : 2003/11/24* [32]

J 38. *Crux Mathematicorum with Mathematical Mayhem, Problem 2446, Proposed by Carherine Shevlin*

J 39. *Netherlands 1994 - Arne Smeets : 2003/12/12*

J 40 (Tt). *Tournament of the Towns 2002 Fall/A-Level- Arne Smeets : 2003/12/12*

J 41 (Tt). *Tournament of the Towns 2003 Spring/O-Level*

J 42 (JtPt, pp. 93). *Russia 1984 - Arne Smeets : 2003/12/12*

J 43. *USA 1993*

J 44. *IMO Long List 1987 (GB)*

J 45. *Vietnam 2000 - Tran Nam Dung : 2003/12/13*

J 46. *Vietnam 2001 - Tran Nam Dung : 2003/12/13*

J 47. *Vietnam 1999 - Tran Nam Dung : 2003/12/13*

J 48. *Poland 1995 - Arne Smeets : 2003/12/13*

J 49. *Canada 1986 - Arne Smeets : 2003/12/13*

J 50. *Bulgaria 2003 - Arne Smeets : 2003/12/13*

J 51. *China 1991*

J 52. *Math. Magazine, Problem 1545, Proposed by Erwin Just*

J 53. *Math. Magazine, Problem 1558, Proposed by Mansur Boase*

J 54 (Rh, pp. 276).

J 55. *Croatia 1994*

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J 56 (Tt)**.** *Tournament of the Towns 2003 Spring/O-Level*

J 57 (Tt)**.** *Tournament of the Towns 2002 Fall/O-Level*

J 58 (Tt)**.** *Tournament of the Towns 2002 Spring/A-Level*

J 59 (Tt)**.** *Tournament of the Towns 2001 Fall/A-Level*

J 60 (Tt)**.** *Tournament of the Towns 2001 Fall/A-Level*

J 61 (Ae, pp. 227)**.**

J 62 (Ae, pp. 226)**.**

Combinatorial Number Theory

K 1. *IMO Long List 1985 (PL2)*

K 2. *IMO 1971/3*

K 3. *Berkeley Math Circle Monthly Contest 1999-2000*

K 4. *India 1998*

K 5. *USA 2001*

K 6. *USA 1998*

K 7. *Romania 1998*

K 8. *India 1998*

K 9. *Romania 1998*

K 10. *IMO 1991/3*

K 11. *Math. Magazine, Problem 1466, Proposed by David M. Bloom*

K 12. *IMO 1985/2*

K 13. *Romania 1997, Proposed by Marian Andronache and Ion Savu*

K 14. *IMO 1983/5*

K 15. *IMO Short List 2001*

K 16. *Turkey 1996*

K 17. *IMO 1995/6*

K 18. *IMO Short List 2002 N5*

K 19. *IMO Short List 1998 P16*

K 20. *IMO Short List 1998 P17*

K 21. *IMO ShortList 1998 P20*

K 22. *IMO Short List 1996*

K 23 (TaZf pp.10)**.** *High-School Mathematics (China) 1994/1*

K 24. *IMO Short List 2002 A6*

K 25. *IMO Short List 2001 C4*

K 26. *IMO Short List 2000 C6*

K 27. *IMO Short List 1999 A4*

K 28. *IMO Short List 1999 C4*

K 29. *Australia 2002*

K 30 (Ae pp.228)**.**

K 31. *IMO ShortList 1991 P24 (IND 2)*

K 32. *IMO Short List 1990 CZE3*

K 33. *IMO Short List 1990 MEX2*

K 34. *USA 2002*

K 35. *IMO Short List 1995*

K 36. *IMO Short List 1999*

K 37. *IMO Short List 1999*

K 38. *Romania 1995*

K 39. *Japan 1990*

K 40. *USA 2003*

K 41. *Putnam 1997/A5*

K 42. *British Mathematical Olympiad 2000 - Arne Smeets : 2003/12/13*

K 43. *Bulgaria 2003 - Arne Smeets : 2003/12/13*

K 44. *IMO 1970/4*

K 45. *Crux Mathematicorum with Mathematical Mayhem, Problem A233, Proposed by Mohammed Aassila*

Additive Number Theory

L 1. *Amer. Math. Monthly, Problem 10426, Proposed by Noam Elkies and Irving Kaplanky*

L 2. *Netherlands 1994 - Arne Smeets : 2003/12/12* [33]

L 3. *Belarus 2002 Proposed by V. Bernik - Arne Smeets : 2003/12/13*

L 4. *Math. Magazine, Problem Q817, Proposed by Robert B. McNeill*

L 5.

[33] Contributor, Date

L 6 (IHH, pp. 474).

L 7 (Tma, pp. 22).

L 8 (Rdc pp.24).

L 9. *IMO Short List 1990 AUS3*

L 10. *IMO 1992/6*

L 11. *IMO 1997/6*

L 12 (Hua pp.199).

L 13. *Math. Magazine, Problem Q814, Proposed by Paul Erdös*

L 14. *Romania 2001, Proposed by Laurentiu Panaitopol*

L 15. *Amer. Math. Monthly, Problem E3064, Proposed by Ion Cucurezeanu*

L 16.

L 17. *Putnam 2000*

L 18.

L 19. *Amer. Math. Monthly, Problem 2760, Proposed by Kenneth S. Williams*

L 20. *APMO 1994/3*

L 21. *India 1998*

L 22. *Romania 1997, Proposed by Marcel Tena*

L 23.

L 24.

L 25.

L 26.

L 27. *IMO 1983/3*

L 28. *IMO 1976/4*

L 29.

L 30. *IMO Short List 2000 N6*

L 31. *IMO Short List 1998 P21*

L 32. *IMO Short List 1996 N3*

L 33. *Putnam 1988/B1*

L 34. *Math. Magazine, Problem 1561, Proposed by Emre Alkan*

The Geometry of Numbers

M 1. *Math. Magazine, Problem 1409, Proposed by Gerald A. Heur*

M 2. *Putnam 1993/B5*

M 3.

M 4. *Israel 1994*

M 5.

M 6 (Hua pp.535)**.**

M 7 (GjJj pp.215)**.**

M 8. *IMO Short List 1990 USS3*

Miscellaneous Problems

N 1 (Tt)**.** *Tournament of the Towns 2001 Fall/A-Level*

N 2 (Pt)**.** *Tournament of the Towns*

N 3. *Ireland 1998*

N 4. *Vietnam 2003 - Tran Nam Dung : 2003/12/13* [34]

N 5. *Amer. Math. Monthly, Problem E2998, Proposed by Clark Kimberling*

N 6.

N 7. *British Mathematical Olympiad 1997 - Arne Smeets : 2003/12/13*

N 8. *Turkey 1996 - Arne Smeets : 2003/12/12*

N 9. *Crux Mathematicorum with Mathematical Mayhem, Problem 2331, Proposed by Paul Yiu*

N 10 (Rh, pp. 165)**.** *Unused problems for 1985 Canadian Mathematical Olympiad*

N 11. *Putnam 1985/B3*

N 12. *Latvia 1995*

N 13. *IMO Short List 1992 P17*

N 14.

N 15 (Ns pp.4)**.**

N 16. *IMO 1981/3*

N 17. *IMO Short List 1999 N4*

N 18. *IMO Short List 1998 P15*

N 19. *IMO Short List 2002 A5*

N 20. *USA 1990*

N 21 (Tt)**.** *Tournament of the Towns 2001 Spring/O-Level*

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Appendix

How Many Problems Are In This Book ?

`Divisibility Theory I` : 67 problems
`Divisibility Theory II` : 45 problems
`Arithmetic in` Z_n : 34 problems
`Primes and Composite Numbers` : 34 problems
`Rational and Irrational Numbers` : 35 problems
`Diophantine Equations I` : 37 problems
`Diophantine Equations II` : 32 problems
`Functions in Number Theory` : 37 problems
`Polynomials` : 11 problems
`Sequences of Integers` : 62 problems
`Combinatorial Number Theory` : 45 problems
`Additive Number Theory` : 34 problems
`The Geometry of Numbers` : 8 problems
`Miscellaneous Problems` : 21 problems

502 PROBLEMS