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THE COEFFICIENT PROBLEM FOR BAZILEVIC FUNCTIONS

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Let S denote the usual class of normalized functions analytic and univalent in the unit disc U and let S* and C denote the subclasses of starlike and close-to-convex functions respectively. Let P denote the class of all functions $h(z) = 1 + b_1 z + \cdots$ analytic in U with $\mathrm{Re}\, h(z) > 0$.

A function f is called a Bazilević function of type α, $\alpha > 0$ and is said to belong to the class $B(\alpha)$ if there are functions σ and h in S* and P respectively such that

$$(1)\quad f(z) = [\alpha \int_0^z \sigma^\alpha(\zeta) h(\zeta) \zeta^{-1} d\zeta]^{1/\alpha},$$

all powers being principal powers. I. Bazilević [1] showed that such functions are univalent in U.

There has been considerable interest in the coefficients of a function $f(z) = z + a_2 z^2 + \cdots$ in $B(\alpha)$. Since such functions are univalent, the natural conjecture is that $|a_n| \leq n$. This has been verified by D. K. Thomas [7] for $\alpha = 1/2N$, N an integer and by J. Zamorski [9] for $\alpha = 1/N$, N an integer. In 1971, F. R. Keough and S. S. Miller [3] obtained Zamorski's result by a different method, also obtaining sharp estimates on the coefficients of a function in the class $B_m(\alpha)$ of m-fold symmetric functions in $B(\alpha)$. Keough and Miller also gave an example of a function in $B(1/2)$ which is not close-to-convex and hence these classes are the largest ones for which the Bieberbach conjecture is known to be correct. (It is well-known that $S^* \in B(\alpha)$ for all α and that $B(1) = C$.) J. Plaster [5] has given a simple geometric criterion to determine if a function is not a Bazilević function.

In this note we solve the coefficient problem for $B_m(\alpha)$ for all $\alpha > 0$, the extremal function being $z(1 - xz^m)^{-2/m}$, $|x| = 1$ in each case. The paper closes with an open problem.

THEOREM 1. $\cap_{\alpha>0} B(\alpha) = S^*$.

PROOF. T. Shiel-Small [6] has shown that $f \in B(\alpha)$ if and only if for each fixed r, $0 < r < 1$, and $0 < \theta_1 < \theta_2 < 2\pi$,

$$(2)\quad \mathrm{Re} \int_{\theta_1}^{\theta_2} \left(1 + \frac{re^{i\theta} f''(re^{i\theta})}{f'(re^{i\theta})} + (\alpha - 1)\frac{re^{i\theta} f' re^{i\theta})}{f(re^{i\theta})}\right) d\theta > -\pi.$$

If f belongs to each $B(\alpha)$, then (2) holds for each α. After dividing both sides of (2) by α and letting $\alpha \to \infty$, we obtain

$$(3)\quad \int_{\theta_1}^{\theta_2} \mathrm{Re} \frac{re^{i\theta} f'(re^{i\theta})}{f(re^{i\theta})} d\theta \geqslant 0.$$

Inequality (3) and the maximum principle show that $\mathrm{Re} zf'(z)/f(z) > 0$ in U and hence f is starlike.

As was remarked earlier, the reverse inclusion is well-known.

THEOREM 2. *Let* $\sigma_\alpha \in S^*$, $h_\alpha \in P$ *and let* $f_\alpha \in B(\alpha)$ *be defined by*

$$f_\alpha(z) = [\alpha \int_0^z \sigma_\alpha^\alpha(\zeta) h_\alpha(\zeta) \zeta^{-1} d\zeta]^{1/\alpha}$$

and let $\lim_{\alpha \to \infty} \sigma_\alpha$ *and* $\lim_{\alpha \to \infty} h_\alpha$ *exist.*

PROOF. After a short calculation using (1), we obtain

$$(4)\quad 1 + \frac{zf_\alpha''(z)}{f_\alpha'(z)} + (\alpha - 1\frac{zf_\alpha'(z)}{f_\alpha(z)} = \alpha\frac{z\sigma_\alpha'(z)}{\sigma_\alpha(z)} + \frac{zh_\alpha'(z)}{h_\alpha(z)}.$$

The statement follows if we divide by α and let $\alpha \to \infty$.

Before stating the next theorem, we recall a result proved in [3].

LEMMA. $\phi(z) \in B_m(\alpha)$ *if and only if*

$$\phi(z) = [f(z^m)]^{1/m},$$

where $f(z) \in B_1(\alpha/m)$.

Here $B_m(\alpha)$ denotes the class of m-fold symmetric functions in $B(\alpha)$. We will also use the notation $g(z) << f(z)$ to mean that if $g(z) = \Sigma a_n z^n$ and $f(z) = \Sigma b_n z^n$, then $|a_n| \leqslant b_n$, $n = 0,1,2,\ldots$ and $g(z) <<_p f(z)$ to mean $|a_n| \leqslant b_n$, $n = 0,1,\ldots,p$.

THEOREM. *Let* $g \in B_m(\alpha)$, $\alpha > 0$, m *a positive integer. Then* $g(z) << z(1 - z^m)^{-2/m}$.

PROOF. By (1) and the Lemma, there are m-fold symmetric functions σ and h in S^* and P respectively so that

$$(5)\quad g(z) = [\alpha \int_0^z \sigma^\alpha(\zeta) h(\zeta) \zeta^{-1} d\zeta]^{1/\alpha}.$$

Define $g_M(z) = [g(z^M)]^{1/M}$. A trivial extension of the Lemma together with (1) yields

(6) $g_M(z) = [\alpha M \int_0^z \sigma_M^{\alpha M}(\zeta) h_M(\zeta)\zeta^{-1} d\zeta]^{1/\alpha M}$

where $\sigma_M^M(\zeta) = \sigma(\zeta^M)$ and $h_M(\zeta) = h(\zeta^M)$. Let p be a fixed positive integer. We will show that if $\sigma(z) \neq z(1 - xz^m)^{-2/m}$, $|x| = 1$, then for M sufficiently large, g_M is so close to σ_M that

$$g_M(z) \ll_{mMp+1} z(1 - z^{mM})^{-2/mM}.$$

The result will then follow easily.

Suppose first that g, g_M, σ_M, h_M are analytic for $|z| \leqslant 1$. It follows from (6) that

(7) $g_M(z) = \sigma_M(z)[(h_M(z) g_M(z)/z g_M'(z)]^{1/\alpha M}$.

The nth coefficient of a function in $B_m(\alpha)$ is an analytic function of the first n coefficients of the starlike function σ and the first n-1 coefficients of the function h of positive real part. As is well-known, a function maximizing this coefficient must have

$$\sigma_M(z) = z\, \Pi_{j=1}^n (1 - x_j z^M)^{-2\beta_j/M}$$

and

$$h_M(z) = \Sigma_{j=1}^n \lambda_j \frac{1 + y_j z^M}{1 - y_j z^M},$$

where $|x_j| = |y_j| = 1$, $\lambda_j > 0$, $\beta_j > 0$, $\Sigma\lambda_j = \Sigma\beta_j = 1$.

It follows from (6) that

(7) $g_M(z) = \sigma_M(z)[h_M(z) g_M(z)/z g_M'(z)]^{1/\alpha M}$.

We need only consider functions σ_M and h_M of the type indicated above. The distortion theorem implies that every zero of h_M on $|z| = 1$ must occur at either a pole of σ_M or a zero of $z g_M'(z)/g_M(z)$ and that similarly each pole of $z g_M'(z)/g_M(z)$ must occur at a pole of $h_M(z)$ or $\sigma_M(z)$. It is clear that the term in brackets in (7) approaches 1 as $M \to \infty$; in fact, it follows from (4) that

$$(z g_M'(z)/g_M(z))^{1/M} = (z\sigma_M'(z)/\sigma_M(z))^{1/M} + 0(1/M^2).$$

We now suppose that g_M is analytic for $|z|$ by considering $r^{-1} g_M(rz)$.

Let

$$g_M(z) = \Sigma_{k=0}^\infty b_{Nk+1} z^{Nk+1}$$

where N = mM and let

(8) $P_M(z)^{1/\alpha M} = (h_M(z)g_M(z)/zg'_M(z))^{1/\alpha M} = 1 + \Sigma\, c_{Nk}z^{Nk}$.

The argument given above shows that $P_M(z) \to 1$ as $M \to \infty$ and consequently if $\epsilon > 0$ is given,

$$|\Sigma_{k=1}^{\infty} c_{Nk}z^{Nk}| < \epsilon/\alpha M$$

for $|z| \leqslant 1$ and N sufficiently large. It follows that $|c_{Nk}| < \epsilon/\alpha M$ for N sufficiently large, k = 1,2,... . Let

$$\sigma_M(z) = \Sigma_{k=0}^{\infty} d_{Nk+1}z^{Nk+1}$$

and recall that

$$z(1 - z^N)^{-2/N} = \Sigma_{k=0}^{\infty} \binom{-2/N}{k}(-1)^{Nk}z^{Nk+1}$$

and that [8]

(9) $|d_{Nk+1}| \leqslant |\binom{-2/N}{k}|$.

It follows from (7), (8), and (9) that

(10) $|b_{Np+1}| = |\Sigma_{k=0}^{p} d_{Nk+1}c_{N(p-k)}|$

$$\leqslant |d_{Np+1}| + |\Sigma_{k=0}^{p-1} d_{Nk+1}c_{N(p-k)}|$$

$$\leqslant |d_{Np+1}| + (p - 1)\epsilon/\alpha M$$

since $|\binom{-2/N}{k}| \leqslant 1$ for $N \geqslant 2$.

In fact, since by assumption σ_M is analytic for $|z| \leqslant 1$, $\sigma_M(z) = r^{-1}\alpha_M(rz)$ where α_M is an N = mM-fold symmetric function starlike in $|z| < 1$,

$$|d_{Np+1}| \leqslant |\binom{-2/N}{p}|r^{Np}$$

and hence (10) implies

$$|b_{Np+1}| < |\binom{-2/N}{p}|$$

for N sufficiently large and fixed p = 1,2,... . Then

$$g_M(z) << z(1 - z^N)^{-2/N} = z\, \Sigma_{k=0}^{\infty} \binom{-2/N}{k}(-z)^{Nk}.$$

Since $g(z) = [g_M(z^{1/M})]^M$, the coefficients of g are polynomials with non-negative coefficients in the coefficients of $g_M(z)$ and the coefficients of $g_M(z^{1/M})$ are positive, the result follows if g is analytic for $|z| \leqslant 1$.

COROLLARY. *If* $f(z) = z + a_2z^2 + \cdots \in B(\alpha)$, *then* $|a_n| \leqslant n$.

REMARKS. 1. It is clear from the proof that the only extremal functions occur when $\sigma(z) = z(1 - xz)^{-2}$. It is quite likely that $h(z) = (1 + xz)/(1 - xz)$ is necessary for f to be an extremal function, but we are unable to prove this.

2. The solution to the coefficient problem would follow trivially from known results if we could show that $B(\alpha) \subset B(\beta)$ if $\alpha > \beta$. Theorem 1, the example in [3] and the inclusion $C(1/\alpha) \subset C(1/\beta)$ if $\alpha > \beta$ for $1/\alpha$-convex functions [4] give some support to this conjecture.

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REFERENCES

1. I. Bazilević, *On a case of integrability in quadratures of the Loewner-Kufarev equation,* Mat. Sborn. N. S., 37(1955), 471-476.
2. L. Brickman, D. J. Hallenbeck, T. H. MacGregor and D. R. Wilken, *Convex hulls and extreme points of families of starlike and convex mappings,* Trans. Amer. Math. Soc., 185(1973), 413-428.
3. F. R. Keough and S. S. Miller, *On the coefficients of Bazilević functions,* Proc. Amer. Math. Soc., 30(1971), 492-496.
4. P. T. Mocanu, *Une propriété de convexité généralisée dans la theorie de la représentation conforme,* Mathematica (Cluj), 11(1969), 127-133.
5. J. Plaster, *A simple geometric criterion for non-Bazilević-ness,* to appear.
6. T. Shiel-Small, *On Bazilević functions,* Quart. J. Math., Oxford, 23(1972), 135-142.
7. D. K. Thomas, *On an extremal problem in univalent functions,* Math. Z., 109(1969), 344-348.
8. H. Waadeland, *Uber k-fact symmetrische, sternformige schlichte Abbildungen des Einheitkreises,* Math. Scand., 3(1955), 150-154.
9. J. Zamorski, *On Bazilevic schlicht functions,* Ann. Polon. Math., 12(1962), 83-90.

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