FOURIER-STIELTJES TRANSFORMS TENDING TO ZERO

LAWRENCE J. WALLEN

ABSTRACT. Let μ be a Borel measure on the circle, $\hat{\mu}$ its Fourier transform. It is shown that a certain thinness condition on the positive part of the support of $\hat{\mu}$ forces a power of μ (in the sense of convolution) to be absolutely continuous.

Let μ be a Borel measure on $[0, 2\pi]$, $\hat{\mu}$ its Fourier transform (i.e. the sequence of Fourier coefficients $\hat{\mu}(n) = \int_0^{2\pi} e^{-inx} d\mu(x)$), and let S^+ be the set of positive integers n such that $\hat{\mu}(n) \neq 0$. There are many theorems which permit one to conclude that if S^+ is sparse, then μ is absolutely continuous (this class will henceforth be denoted by A.C.). The classical F. and M. Riesz theorem is, of course, the prototype. It asserts that if S^+ is finite, $\mu \in$ A.C. Newer and more recondite results permit the same conclusion if S^+ has Hadamard gaps, or S^+ is the set of perfect squares or S^+ is the set of primes (see [1]).

In this note, we prove an elementary theorem that allows us to infer from a certain thinness condition on S^+ that $\hat{\mu}$ tends to zero. We prove this, not by showing that $\mu \in$ A.C., but by showing that some power of μ in the sense of convolution is in A.C. The thinness condition is similar to one used by Glicksberg to obtain the same sort of conclusion (see [2]). However, Glicksberg's theorem seems to be a harmonic analysis theorem, while our much shallower result seems to be function theoretic.

One last bit of notation is this: $f(\theta)d\theta$ is the measure ν defined by $d\nu/d\mu = f(\theta)$ for $f \in L^\infty$.

THEOREM. *Let* $S^+ = \{n_i\}$ *be such that* $\lim_i(n_{i+p} - n_i) = \infty$ *for some positive integer* p. *Then*

$$\underbrace{\mu \star \mu \star \cdots \star \mu}_{(p+1) \text{ times}} \in \text{A.C.}$$

PROOF. The condition on $\{n_i\}$ is clearly equivalent to the following assertion: given $0 < k_1 < k_2 < \cdots < k_p$, the set of n such that $n \in S^+$, $n + k_1 \in S^+, \cdots, n + k_p \in S^+$ is finite. Fixing $k_1, \cdots, k_p$ for the

Received by the editors July 18, 1969.

AMS Subject Classifications. Primary 4210; Secondary 4225.

Key Words and Phrases. Fourier-Stieltjes coefficients, convolution, absolutely continuous measure, F. and M. Riesz Theorem.

[1] This work was supported by NSF Grant GP 9523.

moment, we have then that

$$\hat{\mu}(n)\hat{\mu}(n+k_1)\cdots\hat{\mu}(n+k_p) = (\mu \star \exp[ik_1\theta]\mu \star \cdots \star \exp[ik_p\theta]\mu)\hat{\ } (n)$$

vanishes for all but a finite set of positive integers n. The theorem of F. and M. Riesz implies that $\mu \star \exp[ik_1\theta]\mu \star \cdots \star \exp[ik_p\theta]\mu \in$ A.C. Let $\mu = \alpha + \sigma$ where $\alpha \in$ A.C. and σ is singular. Since A.C. is an ideal, we have the fact that $\sigma \star \exp[ik_1\theta]\sigma \star \cdots \star \exp[ik_p\theta]\sigma \in$ A.C. Hence if $P(\theta)$ is a trigonometric polynomial of the form $P(\theta) = \sum \alpha_j \exp[iq_j\theta]$ where $q_j > k_{p-1}$, then $\sigma \star \exp[ik_1\theta]\sigma \star \cdots \star \exp[ik_{p-1}\theta]\sigma \star P(\theta)\sigma \in$ A.C. According to a famous Theorem of Szegö (see [3]), there exists a sequence P_n of such trigonometric polynomials such that $\int |1 - P_n(\theta)| d|\sigma| \to 0$. Thus $p_n(\theta)\sigma \to \sigma$ in the variation norm and so $\sigma \star \exp[ik_1\theta]\sigma \star \cdots \star \exp[ik_{p-1}\theta]\sigma \star \sigma \in$ A.C. This procedure can now be iterated to finally get $\sigma \star \cdots \star \sigma \in$ A.C., concluding the proof.

References

1. Y. Meyer, *Spectres des mesures et mesures absolument continues*, Studia Math. **30** (1968), 87–99. MR **37** #3281.

2. I. Glicksberg, *Fourier-Stieltjes transforms with small supports*, Illinois J. Math. **9** (1965), 418–427. MR **33** #506.

3. N. I. Ahieser, *Lectures on the theory of approximation*, OGIZ, Moscow, 1947; English transl., Ungar, New York, 1956. MR **10**, 33; MR **20** #1872.

University of Hawaii