Monotone Sequences & Cauchy Sequences

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1 Monotone Sequences and Cauchy Sequences

1.1 Monotone Sequences

The techniques we have studied so far require we know the limit of a sequence in order to prove the sequence converges. However, it is not always possible to find the limit of a sequence by using the definition, or the limit rules. This happens when the formula defining the sequence is too complex to work with. It also happens with sequences defined recursively. Furthermore, it is often the case that it is more important to know if a sequence converges than what it converges to. In this section, we look at two ways to prove a sequence converges without knowing its limit.

We begin by looking at sequences which are monotone and bounded. These terms were defined at the beginning of this chapter.

You will recall that in order to show that a sequence is increasing, several methods can be used.

(1) Direct approach,simply show that $a_{n+1} \geq a_n$ for every n.

(2) Equivalently, show that $a_{n+1} - a_n \geq 0$ for every n.

(3) Equivalently, show that $\dfrac{a_{n+1}}{a_n} \geq 1$ for every n if both a_n and a_{n+1} are positive.

(4) If $a_n = f(n)$, one can show a sequence (a_n) is increasing by showing that f is increasing that is by showing that $f'(x) \geq 0$.

(5) By induction.

We now state and prove an important theorem about the convergence of increasing sequences.

Theorem 1 *An increasing sequence (a_n) which is bounded above converges. Furthermore,* $\lim_{n\to\infty} a_n = \sup\{a_n\}$.

Corollary 2 *A decreasing sequence (a_n) which is bounded below converges. Furthermore,* $\lim_{n\to\infty} a_n = \inf\{a_n\}$.

Theorem 3 *A monotone sequence converges if and only if it is bounded.*

Example 1 *Prove that the sequence whose general term is* $a_n = \sum_{k=0}^{n} \frac{1}{k!}$ *converges.*

Example 2 *Find* $\lim_{n\to\infty} a_n$ *where* (a_n) *is defined by:*

$$a_1 = 2$$
$$a_{n+1} = \frac{1}{2}(a_n + 6)$$

Example 3 *Let* $a_n = \left(1 + \frac{1}{n}\right)^n$. *Show* $\lim a_n$ *exists. This limit is in fact the number* e*, but we won't show that. Again, to show that* (a_n) *converges, we show that it is increasing and bounded above.*

1.2 Cauchy Sequences

Definition 1 (Cauchy Sequence) *A sequence* (x_n) *is said to be a Cauchy sequence if for each* $\epsilon > 0$ *there exists a positive integer* N *such that* $m, n > N \Longrightarrow |x_m - x_n| < \epsilon$.

We begin with some remarks.

Remark 1 *These series are named after the French mathematician Augustin Louis Cauchy (1789-1857).*

Remark 2 *It is important to note that the inequality* $|x_m - x_n| < \epsilon$ *must be valid for all integers* m, n *that satisfy* $m, n > N$. *In particular, am sequence* (x_n) *satisfying* $|x_{n+1} - x_n| < \epsilon$ *for all* $n > N$ *may not be a Cauchy sequence.*

Remark 3 *A Cauchy sequence is a sequence for which the terms are eventually close to each other.*

Remark 4 *In theorem* **??**, *we proved that if a sequence converged then it had to be a Cauchy sequence. In fact, as the next theorem will show, there is a stronger result for sequences of real numbers.*

We now look at some examples.

Example 4 *Consider* (x_n) *where* $x_n = \dfrac{1}{n}$. *Prove that this is a Cauchy sequence.*

Example 5 *Consider* (x_n) *where* $x_n = \displaystyle\sum_{k=1}^{n} \frac{1}{k^2}$

We now look at important properties of Cauchy sequences.

Theorem 4 *Every Cauchy sequence is bounded.*

Theorem 5 *A sequence of real numbers converges if and only if it is a Cauchy sequence.*

Remark 5 *The key to this theorem is that we are dealing with a sequence of real numbers. The fact that a Cauchy sequence of* *real number converges is linked to the fact that $\mathbb{R}$ is complete. In fact, this is sometimes used as a definition of completeness. Some texts say that a set is complete if every Cauchy sequence converges in that set. It is possible to find a Cauchy sequence of rational numbers which does not converge in $\mathbb{Q}$.*

We finish this section with an important theorem.

Definition 2 *A* ***nested sequence of intervals*** *is a sequence $\{I_n\}$ of intervals with the property that $I_{n+1} \subset I_n$ for all n.*

Theorem 6 (Nested Intervals) *If $\{[a_n, b_n]\}$ is a nested sequence of closed intervals then there exists a point z that belongs to all the intervals. Furthermore, if $\lim a_n = \lim b_n$ then the point z is unique.*

1.3 Exercises

(1) Prove that $n! \geq 2^{n-1}$ for every $n \geq 1$

(2) Prove that if k is an integer such that $1 < k \leq n$, then

$$\frac{n(n-1)(n-2)\dots(n-k+1)}{n^k} = \left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\dots\left(1 - \frac{k-}{n}\right.$$

(3) Prove that the sequence given by

$$a_1 = 2$$
$$a_{n+1} = \frac{1}{2}(a_n + 6)$$

is increasing and bounded above by 6. (hint: use induction for both).

(4) Show that the sequence defined by

$$a_1 = 1$$
$$a_{n+1} = 3 - \frac{1}{a_n}$$

is increasing and satisfies $a_n < 3$ for all n. Then, find its limit.

(5) Let a and b be two positive numbers such that $a > b$. Let a_1 be their arithmetic mean, that is $a_1 = \frac{a+b}{2}$. Let b_1 be their geometric mean, that is $b_1 = \sqrt{ab}$. Define $a_{n+1} = \frac{a_n + b_n}{2}$ and $b_{n+1} = \sqrt{a_n b_n}$.
   (a) Use mathematical induction to show that $a_n > a_{n+1} > b_{n+1} > b_n$.
   (b) Deduce that both (a_n) and (b_n) converge.
   (c) Show that $\lim a_n = \lim b_n$. Gauss called the common value of these limits the **arithmetic-geometric mean**.

(6) Prove theorem 4.

(7) Answer the why? parts in the proof of theorem 5.

(8) Finish proving theorem 6.