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Certain Class of Analytic and Univalent Functions Involving the Ruscheweyh Derivative Operator

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Abstract. A class of univalent functions which provides an interesting transition from starlike functions to convex functions is defined by means of the Ruscheweyh derivative. Some interesting sufficient conditions involving coefficient inequalities for functions in these classes are discussed which generalize the results derived by Hayami, Owa and Srivastava.

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1. **Introduction, Definition and Preliminaries**

Let $\mathcal{A}_0$ denote the class of functions f of the form,

$$f(z) = a_0 + a_1 z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the open unit disc $\mathcal{U} = \{z : z \in \mathbb{C} \quad \text{and} \quad |z| < 1\}$.

If $f \in \mathcal{A}_0$ is given by (1.1), together with the normalizations $a_0 = 0$ and

$a_1 = 1$, then we say that $f \in \mathcal{A}$.

If $f \in \mathcal{A}$ satisfies,

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha \quad (z \in \mathcal{U}; \quad 0 \leq \alpha < 1) \tag{1.2}$$

then f is said to be starlike of order α in $\mathcal{U}$. We denote by $\mathcal{S}^*(\alpha)$ the subclass of $\mathcal{A}$ consisting of functions f which are starlike of order α in $\mathcal{U}$. Similarly, we say that f is in the class $\mathcal{K}(\alpha)$ of convex functins of order α in $\mathcal{U}$ if $f \in \mathcal{A}$ satisfies,

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha \quad (z \in \mathcal{U}; \quad 0 \leq \alpha < 1). \tag{1.3}$$

It can be easily observed from (1.2) and (1.3) that,

$$f \in \mathcal{K}(\alpha) \Leftrightarrow zf'(z) \in \mathcal{S}^*(\alpha) \quad (0 \leq \alpha < 1).$$

Let $\mathcal{SP}(\lambda, \alpha)$ denote the subclass of functions $f \in \mathcal{A}$ which satisfy the condition,

$$\Re\left\{e^{i\lambda}\left(\frac{zf'(z)}{f(z)} - \alpha\right)\right\} > 0 \quad (z \in \mathcal{U}; \quad 0 \leq \alpha < 1; \quad -\pi/2 < \lambda < \pi/2). \tag{1.4}$$

Definition 1.1. *Let $\mathcal{V}(\alpha, b, \delta)$ denote the subclass of $\mathcal{A}$ consisting of functions f satisfying the condition,*

$$\Re\left\{1 - \frac{2}{b} + \frac{2}{b}.\frac{D^{\delta+1}f(z)}{D^{\delta}f(z)}\right\} > \alpha \tag{1.5}$$

where $b \neq 0$, $\delta > -1$, $0 \leq \alpha < 1$ and $D^{\delta}f$ is the Ruscheweyh derivative of f [3] given by,

$$D^{\delta}f(z) = \frac{z}{(1-z)^{1+\delta}} * f(z) = z + \sum_{n=2}^{\infty} \tau_n(\delta) a_n z^n \tag{1.6}$$

where $*$ stands for the convolution or Hadamard product of two power series and

$$\tau_n(\delta) = \frac{\Gamma(\delta + n)}{(n-1)!\Gamma(\delta + 1)}. \tag{1.7}$$

This class is obtained by putting $k = 2$ and $\lambda = 0$ in the class $\mathcal{V}_k^{\lambda}(\alpha, b, \delta)$ introduced by Latha and Nanjunda Rao [2]. The class $\mathcal{V}_k^{\lambda}(\alpha, b, \delta)$ is of special interest for it contains many well known as well as new classes of analytic univalent functions studied in literature. It provides a transition from starlike functions to convex functions. More specifically $\mathcal{V}_2^0(\alpha, 2, 0)$ is the family of starlike functions of order α and $\mathcal{V}_2^0(\alpha, 1, 1)$ is the class of convex functions of order α.

Observe that, for the parametric values $b = 2$, $\delta = 0$ and $b = \delta = 1$ in $\mathcal{V}(\alpha, b, \delta)$ we obtain the classes $\mathcal{S}^*(\alpha)$ and $\mathcal{K}(\alpha)$ respectively.

Definition 1.2. *Let $\mathcal{V}(\alpha, b, \delta; \lambda)$ denote the subclass of $\mathcal{A}$, which consists of functions $f \in \mathcal{A}$ if and only if,*

$$\Re\left\{e^{i\lambda}\left(1 - \frac{2}{b} + \frac{2}{b}.\frac{D^{\delta+1}f(z)}{D^{\delta}f(z)} - \alpha\right)\right\} > 0 \tag{1.8}$$

where $0 \leq \alpha < 1$; $-\pi/2 < \lambda < \pi/2$ and $z \in \mathcal{U}$.

We observe that $\mathcal{V}(\alpha, b, \delta; 0) = \mathcal{V}(\alpha, b, \delta)$ and $\mathcal{V}(\alpha, 2, 0; \lambda) = \mathcal{SP}(\lambda, \alpha)$.

Also, let $\mathcal{B}$ denote the class of functions $p(z)$ of the form,

$$p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n \tag{1.9}$$

which are analytic in $\mathcal{U}$.

We require the following lemmas in our present investigation.

Lemma 1.3. [1] *A function $p(z) \in \mathcal{B}$ satisfies the condition,*

$$\Re\{p(z)\} > 0 \quad (z \in \mathcal{U})$$

if and only if

$$p(z) \neq \frac{\xi - 1}{\xi + 1} \quad (z \in \mathcal{U}; \quad \xi \in \mathbb{C}; \quad |\xi| = 1).$$

Lemma 1.4. *A function $f \in \mathcal{A}$ is in the class $\mathcal{V}(\alpha, b, \delta)$ if and only if,*

$$1 + \sum_{n=2}^{\infty} A_n z^{n-1} \neq 0, \tag{1.10}$$

where

$$A_n = \frac{\Gamma(\delta + n)}{(n-1)!\Gamma(\delta + 2)}\left\{\frac{[(n-1) + b(1-\alpha)(\delta+1)] + (n-1)\xi}{b(1-\alpha)}\right\} a_n$$

Proof. Let us define $p(z)$ by,

$$p(z) = \frac{\left(1 - \frac{2}{b} + \frac{2}{b}.\frac{D^{\delta+1}f(z)}{D^{\delta}f(z)}\right) - \alpha}{1 - \alpha} \quad (f \in \mathcal{V}(\alpha, b, \delta)) \tag{1.11}$$

Then $p(z) \in \mathcal{B}$ and $\Re\{p(z)\} > 0 \quad (z \in \mathcal{U})$.
Using Lemma 1.3, we have

$$\frac{\left(1 - \frac{2}{b} + \frac{2}{b}.\frac{D^{\delta+1}f(z)}{D^{\delta}f(z)}\right) - \alpha}{1 - \alpha} \neq \frac{\xi - 1}{\xi + 1} \quad (z \in \mathcal{U}; \quad \xi \in \mathbb{C}; \quad |\xi| = 1)$$

which yields,

$$(\xi+1)D^{\delta+1}f(z)+[b(1-\alpha)-(\xi+1)]D^{\delta}f(z)\neq 0$$
$$(f\in\mathcal{V}(\alpha,b,\delta);\quad z\in\mathcal{U};\quad \xi\in\mathbb{C};\quad |\xi|=1)$$

Hence we have,

$$(\xi+1)\left[z+\sum_{n=2}^{\infty}\tau_n(\delta+1)a_nz^n\right]+[b(1-\alpha)+(1+\xi)]\left[z+\sum_{n=2}^{\infty}\tau_n(\delta)a_nz^n\right]\neq 0$$

ie.,

(1.12)

$$b(1-\alpha)z\left(1+\sum_{n=2}^{\infty}\frac{\Gamma(\delta+n)}{(n-1)!\Gamma(\delta+2)}\times\left[\frac{(n-1)+b(1-\alpha)(\delta+1)+(n-1)\xi}{b(1-\alpha)}\right]a_nz^{n-1}\right)\neq 0$$
$$(z\in\mathcal{U};\quad \xi\in\mathbb{C};\quad |\xi|=1)$$

Dividing both sides of by (1.12) by $2(1-\alpha)z \quad (z\neq 0)$ we obtain,

$$1+\sum_{n=2}^{\infty}\frac{\Gamma(\delta+n)}{(n-1)!\Gamma(\delta+2)}\left[\frac{(n-1)+b(1-\alpha)(\delta+1)+(n-1)\xi}{b(1-\alpha)}\right]a_nz^{n-1}\neq 0$$
$$(z\in\mathcal{U};\quad \xi\in\mathbb{C};\quad |\xi|=1)$$

This completes the proof. □

Remark 1.5. *It follows from the normalization conditions $a_0=0$ and $a_1=1$ that,*

$$A_0=\frac{\Gamma(\delta)}{\Gamma(\delta+2)}\left\{\frac{b(1-\alpha)(\delta+1)-1-\xi}{b(1-\alpha)}\right\}a_0=0$$

and $A_1=a_1=1$.

Remark 1.6. *The parametric substitutions $\delta=0$, $b=2$ yield Lemma 2 in* [1].

Remark 1.7. *The assertion (1.10) of Lemma 1.4 is equivalent to,*

$$\frac{1}{z}\left\{f(z)*\frac{\dfrac{z+[(\xi+1)-b(1-\alpha)]}{b(1-\alpha)}}{(1-z)^{\delta+3}}\right\}\neq 0 \tag{1.13}$$

It can be observed that for appropriate parametric substitutions the results due to Silverman, Silvia and Telage [4] *are obtained. ie., Theorem* 1 *and Theorem* 2 *of* [4] *can be obtained by putting $\delta=b=1$ and $\delta=0$, $b=2$ in (1.13) respectively.*

2. Coefficient Conditions For Functions In The Class $\mathcal{V}(\alpha, b, \delta)$

Our results for functions f to be in the class $\mathcal{V}(\alpha, b, \delta)$ is contained in the following.

Theorem 2.1. *If $f \in \mathcal{A}$ satisfies the condition,*

$$\begin{aligned}\sum_{n=2}^{\infty}\Bigg(\Bigg|\sum_{k=1}^{n}\Bigg[\sum_{j=1}^{k}&\frac{\Gamma(\delta+j)}{(j-1)!\Gamma(\delta+2)}(-1)^{k-j}\\ &\times[(j-1)+b(1-\alpha)(\delta+1)]\binom{\beta}{k-j}a_j\Bigg]\binom{\gamma}{n-k}\Bigg|\\ +\Bigg|\sum_{k=1}^{\infty}\Bigg[\sum_{j=1}^{k}&\frac{\Gamma(\delta+j)}{(j-1)!\Gamma(\delta+2)}(-1)^{k-j}(j-1)\binom{\beta}{k-j}a_j\Bigg]\binom{\gamma}{n-k}\Bigg|\Bigg)\\ \leq b(1-\alpha) \quad &(0 \leq \alpha < 1; \quad \beta \in \mathbb{R}; \quad \gamma \in \mathbb{R})\end{aligned} \tag{2.1}$$

then $f \in \mathcal{V}(\alpha, b, \delta)$.

Proof. We note that,

$$(1-z)^{\beta} \neq 0 \quad \text{and} \quad (1+z)^{\gamma} \neq 0 \quad (z \in \mathcal{U}; \quad \beta \in \mathbb{R}; \quad \gamma \in \mathbb{R}).$$

Hence, if the inequality,

$$\left(1+\sum_{n=2}^{\infty}A_n z^{n-1}\right)(1-z)^{\beta}(1+z)^{\gamma} \neq 0 \quad (z \in \mathcal{U}; \quad \beta \in \mathbb{R}; \quad \gamma \in \mathbb{R}) \tag{2.2}$$

holds true, then we have

$$1+\sum_{n=2}^{\infty}A_n z^{n-1} \neq 0$$

which is the relation (1.10) of Lemma 1.4. It can be observed that,

$$\left(1+\sum_{n=2}^{\infty}A_n z^{n-1}\right)\left(\sum_{n=0}^{\infty}(-1)^n b_n z^n\right)\left(\sum_{n=0}^{\infty}c_n z^n\right) \neq 0 \tag{2.3}$$

where, for convinience,

$$b_n = \binom{\beta}{n} \quad \text{and} \quad c_n = \binom{\gamma}{n}$$

Considering the Cauchy product of the first two factors, (2.3) can be rewritten as,

$$\left(1+\sum_{n=2}^{\infty}B_n z^{n-1}\right)\left(\sum_{n=0}^{\infty}c_n z^n\right) \neq 0 \tag{2.4}$$

where

$$B_n = \sum_{j=1}^{n}(-1)^{n-j}A_j b_{n-j}.$$

Furthermore, by applying the same method for the Cauchy product in (2.4), we find that,

$$1+\sum_{n=2}^{\infty}\left(\sum_{k=1}^{\infty}B_k c_{n-k}\right)z^{n-1}\neq 0 \quad (z\in\mathcal{U})$$

or equivalently, that

$$1+\sum_{n=2}^{\infty}\left[\sum_{k=1}^{n}\left(\sum_{j=1}^{k}(-1)^{k-j}A_j b_{k-j}\right)c_{n-k}\right]z^{n-1}\neq 0 \quad (z\in\mathcal{U}).$$

Thus, if $f\in\mathcal{A}$ satisfies,

$$\sum_{n=2}^{\infty}\left|\sum_{k=1}^{n}\left(\sum_{j=1}^{k}(-1)^{k-j}A_j b_{k-j}\right)c_{n-k}\right|\leq 1$$

that is if,

$$\begin{aligned}
&\frac{1}{b(1-\alpha)}\sum_{n=2}^{\infty}\left|\sum_{k=1}^{n}\left(\sum_{j=1}^{k}\frac{\Gamma(\delta+j)}{(j-1)!\Gamma(\delta+2)}(-1)^{k-j}\right.\right.\\
&\qquad\left.\left.\times[(j-1)+b(1-\alpha)(\delta+1)+(j-1)\xi]a_j b_{k-j}\right)c_{n-k}\right|\\
&\leq\frac{1}{b(1-\alpha)}\sum_{n=2}^{\infty}\left\{\left|\sum_{k=1}^{n}\left(\sum_{j=1}^{k}\frac{\Gamma(\delta+j)}{(j-1)!\Gamma(\delta+2)}(-1)^{k-j}\right.\right.\right.\\
&\qquad\left.\left.\times[(j-1)+b(1-\alpha)(\delta+1)]a_j b_{k-j}\right)c_{n-k}\right|\\
&\qquad\left.+|\xi|\left|\sum_{k=1}^{n}\left(\sum_{j=1}^{k}\frac{\Gamma(\delta+j)}{(j-1)!\Gamma(\delta+2)}(j-1)b_{k-j}a_j\right)c_{n-k}\right|\right\}\\
&\leq 1 \qquad (0\leq\alpha<1;\quad \xi\in\mathbb{C};\quad |\xi|=1)
\end{aligned}$$

then $f\in\mathcal{V}(\alpha,b,\delta)$. □

Remark 2.2. *In the hypothesis (2.1) of Theorem 2.1, for the parametric values $b=2$, $\delta=0$ we obtain Theorem 1 in* [1] *and for $b=\delta=1$ we obtain Theorem 2 in* [1].

By specializing on the parameters α, β, γ, b and δ in Theorem 2.1, we can deduce the following interesting corollaries.

For $\alpha=\delta=0$ and $b=2$ we have,

Corollary 2.3. *If $f\in\mathcal{A}$ satisfies,*

$$\begin{aligned}
&\sum_{n=2}^{\infty}\left(\left|\sum_{k=1}^{n}\left[\sum_{j=1}^{k}(-1)^{k-j}(j+1)\binom{\beta}{k-j}a_j\right]\binom{\gamma}{n-k}\right|\right.\\
&\qquad\left.+\left|\sum_{k=1}^{\infty}\left[\sum_{j=1}^{k}(-1)^{k-j}(j-1)\binom{\beta}{k-j}a_j\right]\binom{\gamma}{n-k}\right|\right)\leq 2 \quad (\beta\in\mathbb{R};\quad \gamma\in\mathbb{R})
\end{aligned}$$

then $f\in\mathcal{S}^*$.

For $\alpha = 0$ and $\delta = b = 1$ we have,

Corollary 2.4. *If $f \in \mathcal{A}$ satisfies,*

$$\sum_{n=2}^{\infty}\left(\left|\sum_{k=1}^{n}\left[\sum_{j=1}^{k}(-1)^{k-j}j(j+1)\binom{\beta}{k-j}a_j\right]\binom{\gamma}{n-k}\right| + \left|\sum_{k=1}^{\infty}\left[\sum_{j=1}^{k}(-1)^{k-j}j(j-1)\binom{\beta}{k-j}a_j\right]\binom{\gamma}{n-k}\right|\right) \leq 2 \quad (\beta \in \mathbb{R}; \quad \gamma \in \mathbb{R})$$

then $f \in \mathcal{K}$.

For $\beta = \gamma = \delta = 0$ and $b = 2$ we have,

Corollary 2.5. *If $f \in \mathcal{A}$ satisfies,*

$$\sum_{n=2}^{\infty}(n-\alpha)|a_n| \leq 1-\alpha \qquad (0 \leq \alpha < 1)$$

then $f \in S^*(\alpha)$.

For $\beta = \gamma = 0$ and $\delta = b = 1$ we have,

Corollary 2.6. *If $f \in \mathcal{A}$ satisfies,*

$$\sum_{n=2}^{\infty}n(n-\alpha)|a_n| \leq 1-\alpha \qquad (0 \leq \alpha < 1)$$

then $f \in \mathcal{K}(\alpha)$.

For $\alpha = \beta = \gamma = \delta = 0$ and $b = 2$ we derive,

Corollary 2.7. *If $f \in \mathcal{A}$ satisfies,*

$$\sum_{n=2}^{\infty}n|a_n| \leq 1$$

then $f \in \mathcal{S}^*$.

For $\alpha = \beta = \gamma = 0$ and $\delta = b = 1$ we have,

Corollary 2.8. *If $f \in \mathcal{A}$ satisfies,*

$$\sum_{n=2}^{\infty}n^2|a_n| \leq 1$$

then $f \in \mathcal{K}$.

3. Coefficient Conditions For Functions In The Class $\mathcal{V}(\alpha, b, \delta; \lambda)$

Lemma 3.1. *A function $f \in \mathcal{A}$ is in the class $\mathcal{V}(\alpha, b, \delta; \lambda)$ if and only if,*

$$1 + \sum_{n=2}^{\infty} C_n z^{n-1} \neq 0 \tag{3.1}$$

where

$$C_n = \frac{\Gamma(\delta+n)}{(n-1)!\Gamma(\delta+2)} \left\{ \frac{n-1+b(1-\alpha)(\delta+1)e^{-i\lambda}\cos\lambda + (n-1)\xi}{b(1-\alpha)e^{-i\lambda}\cos\lambda} \right\} a_n$$

Proof. Defining $p(z)$ by,

$$p(z) = \frac{e^{i\lambda}\left(1 - \frac{2}{b} + \frac{2}{b}.\frac{D^{\delta+1}f(z)}{D^{\delta}f(z)} - \alpha\right) - i(1-\alpha)\sin\lambda}{(1-\alpha)\cos\lambda} \tag{3.2}$$

$$(f \in \mathcal{V}(\alpha, b, \delta; \lambda))$$

we see that, $p(z) \in \mathcal{B}$ and $\Re\{p(z)\} > 0 \quad (z \in \mathcal{U})$.
From Lemma 1.3, it follows that,

$$\frac{e^{i\lambda}\left(1 - \frac{2}{b} + \frac{2}{b}.\frac{D^{\delta+1}f(z)}{D^{\delta}f(z)} - \alpha\right) - i(1-\alpha)\sin\lambda}{(1-\alpha)\cos\lambda} \neq \frac{\xi-1}{\xi+1} \tag{3.3}$$

$$(z \in \mathcal{U}; \quad \xi \in \mathbb{C}; \quad |\xi| = 1)$$

From (3.3) we observe that,

$$p(0) \neq \frac{\xi-1}{\xi+1} \quad (\xi \in \mathbb{C}; \quad |\xi| = 1).$$

Also from (3.3) it follows that,

$$\frac{e^{i\lambda}[bD^{\delta}f(z) - 2D^{\delta}f(z) + 2D^{\delta+1}f(z) - \alpha b D^{\delta}f(z)] - i(1-\alpha)bD^{\delta}f(z)\sin\lambda}{(1-\alpha)\cos\lambda}$$
$$\neq \left(\frac{\xi-1}{\xi+1}\right) bD^{\delta}f(z) \quad (z \in \mathcal{U}; \quad \xi \in \mathbb{C}; \quad |\xi| = 1)$$

which readily yields,

$$(\xi+1)\{e^{i\lambda}[2D^{\delta+1}f(z) - (b\alpha+2-b)D^{\delta}f(z)] - i(1-\alpha)bD^{\delta}f(z)\sin\lambda\}$$
$$\neq (\xi-1)(1-\alpha)bD^{\delta}f(z)\cos\lambda \quad (z \in \mathcal{U}; \quad \xi \in \mathbb{C}; \quad |\xi| = 1)$$

or equivalently,

(3.4)

$$2(\xi+1)e^{i\lambda}D^{\delta+1}f(z) - (b\alpha+2-b)e^{i\lambda}D^{\delta}f(z) - \xi(b\alpha+2-b)e^{i\lambda}D^{\delta}f(z)$$
$$- i(1-\alpha)bD^{\delta}f(z)\sin\lambda - i\xi(1-\alpha)bD^{\delta}f(z)\sin\lambda$$
$$\neq \xi(1-\alpha)bD^{\delta}f(z)\cos\lambda - (1-\alpha)bD^{\delta}f(z)\cos\lambda$$
$$(z \in \mathcal{U}; \quad \xi \in \mathbb{C}; \quad |\xi| = 1)$$

Further simplification yields,

$$2(\xi+1)e^{i\lambda}D^{\delta+1}f(z)-(b\alpha+2-b)e^{i\lambda}D^{\delta}f(z)-\xi(b\alpha+2-b)e^{i\lambda}D^{\delta}f(z) - \xi(1-\alpha)be^{i\lambda}D^{\delta}f(z)+(1-\alpha)be^{-i\lambda}D^{\delta}f(z)\neq 0$$

$$(z\in\mathcal{U};\quad \xi\in\mathbb{C};\quad |\xi|=1)$$

ie.,

$$2(\xi+1)e^{i\lambda}D^{\delta+1}f(z)+[be^{-i\lambda}-2\alpha b\cos\lambda-(2\xi-b+2)e^{i\lambda}]D^{\delta}f(z)\neq 0$$

$$(z\in\mathcal{U};\quad \xi\in\mathbb{C};\quad |\xi|=1)$$

Since $a_0=a_1-1=0$ we have,

$$2(\xi+1)e^{i\lambda}\left(z+\sum_{n=2}^{\infty}\frac{\Gamma(\delta+n+1)}{(n-1)!\Gamma(\delta+2)}a_nz^n\right) + [be^{-i\lambda}-2\alpha b\cos\lambda-(2\xi-b+2)e^{i\lambda}]\left(z+\sum_{n=2}^{\infty}\frac{\Gamma(\delta+n)}{(n-1)!\Gamma(\delta+1)}a_nz^n\right)\neq 0$$

$$(z\in\mathcal{U};\quad \xi\in\mathbb{C};\quad |\xi|=1)$$

Equivalently,

(3.5)

$$2b(1-\alpha)z\cos\lambda\left(1+\sum_{n=2}^{\infty}\frac{\Gamma(\delta+n)}{(n-1)!\Gamma(\delta+2)} \times\left[\frac{2(n-1)(\xi+1)+(\delta+1)(be^{-2i\lambda}-2\alpha be^{-i\lambda}\cos\lambda+b)}{2b(1-\alpha)\cos\lambda e^{-i\lambda}}\right]a_nz^{n-1}\right)\neq 0$$

$$(z\in\mathcal{U};\quad \xi\in\mathbb{C};\quad |\xi|=1)$$

Finally, upon dividing both sides of by (3.5) by

$$2b(1-\alpha)z\cos\lambda\neq 0$$

and noting that

$$e^{-2i\lambda}=-1+2e^{-i\lambda}\cos\lambda$$

we obtain,

$$1+\sum_{n=2}^{\infty}\frac{\Gamma(\delta+n)}{(n-1)!\Gamma(\delta+2)}\left[\frac{(n-1)+b(\delta+1)(1-\alpha)e^{-i\lambda}\cos\lambda+(n-1)\xi}{b(1-\alpha)e^{-i\lambda}\cos\lambda}\right]a_n\neq 0$$

$$(0\leq\alpha<1;\quad -\pi/2<\lambda<\pi/2;\quad \xi\in\mathbb{C};\quad |\xi|=1)$$

which completes the proof of Lemma 3.1. □

Remark 3.2. *For the parametric substitutions $\lambda=\delta=0$ and $b=2$ we obtain Lemma 3 of* [1].

Remark 3.3. *The assertion (3.1) of Lemma 3.1 is equivalent to,*

$$\frac{1}{z}\left\{ f(z) * \frac{\dfrac{2\xi + (2-b) - be^{-2i\lambda} + 2\alpha b \cos\lambda e^{-i\lambda}}{b(1+e^{-2i\lambda} - 2\alpha\cos\lambda e^{-i\lambda})}}{(1-z)^{\delta+2}} \right\} \neq 0. \tag{3.6}$$

Note that for the parametric substitutions $\delta = \alpha = \lambda = 0$ and $b = 2$ in (3.6) we obtain a result due to Silverman, Silvia and Telage [3] *(see Theorem 4,* [4]*).*

Theorem 3.4. *If $f \in \mathcal{A}$ satisfies the condition,*

(3.7)

$$\begin{aligned} &\sum_{n=2}^{\infty}\left(\left|\sum_{k=1}^{n}\left[\sum_{j=1}^{k}\frac{\Gamma(\delta+j)}{(j-1)!\Gamma(\delta+2)}(-1)^{k-j}\right.\right.\right. \\ &\qquad \left.\times[2(j-1)+b(\delta+1)(1-\alpha)(1+e^{-2i\lambda})]\binom{\beta}{k-j}a_j\right]\binom{\gamma}{n-k}\Bigg| \\ &+\left.\left|\sum_{k=1}^{\infty}\left[\sum_{j=1}^{k}\frac{\Gamma(\delta+j)}{(j-1)!\Gamma(\delta+2)}(-1)^{k-j}2(j-1)\binom{\beta}{k-j}a_j\right]\binom{\gamma}{n-k}\right|\right) \\ &\leq 2b(1-\alpha)\cos\lambda \quad (0 \leq \alpha < 1; \quad -\pi/2 < \lambda < \pi/2; \quad \beta \in \mathbb{R}; \quad \gamma \in \mathbb{R}) \end{aligned}$$

then $f \in \mathcal{V}(\alpha, b, \delta; \lambda)$.

Proof. Applying the same method as in the proof of Theorem 2.1, we observe that f is in the class $\mathcal{V}(\alpha, b, \delta; \lambda)$ if,

$$\sum_{n=2}^{\infty}\left[\sum_{k=1}^{n}\left(\sum_{j=1}^{k}(-1)^{k-j}C_j b_{k-j}\right)c_{n-k}\right] \leq 1 \tag{3.8}$$

where

$$b_n = \binom{\beta}{n} \quad \text{and} \quad c_n = \binom{\gamma}{n},$$

the coefficients C_n being given as in Lemma 3.1.
From (3.8) it follows that,

$$\frac{1}{|b(1-\alpha)e^{-i\lambda}\cos\lambda|}\sum_{n=2}^{\infty}\left|\sum_{k=1}^{n}\left(\sum_{j=1}^{k}\frac{\Gamma(\delta+j)}{(j-1)!\Gamma(\delta+2)}(-1)^{k-j}\right.\right.$$
$$\left.\left.\times[(j-1)+b(\delta+1)(1-\alpha)e^{-i\lambda}\cos\lambda+(j-1)\xi]a_jb_{k-j}\right)c_{n-k}\right|$$
$$\leq\frac{1}{2b(1-\alpha)\cos\lambda}\sum_{n=2}^{\infty}\left\{\left|\sum_{k=1}^{n}\left(\sum_{j=1}^{k}\frac{\Gamma(\delta+j)}{(j-1)!\Gamma(\delta+2)}(-1)^{k-j}\right.\right.\right.$$
$$\left.\left.\times[2(j-1)+b(\delta+1)(1-\alpha)(1+e^{-2i\lambda})]a_jb_{k-j}\right)c_{n-k}\right|$$
$$\left.+|\xi|\left|\sum_{k=1}^{n}\left(\sum_{j=1}^{k}\frac{\Gamma(\delta+j)}{(j-1)!\Gamma(\delta+2)}2(j-1)b_{k-j}a_j\right)c_{n-k}\right|\right\}$$
$$\leq 1\qquad(0\leq\alpha<1;\quad -\pi/2<\lambda<\pi/2;\quad \xi\in C;\quad |\xi|=1)$$

which implies that if f satisfies (3.7) then $f\in\mathcal{V}(\alpha,b,\delta;\lambda)$. This completes the proof. □

Remark 3.5. *For $\lambda=0$, Theorem 3.4 implies Theorem 2.1. Also for the parametric substitutions $b=2$ and $\delta=0$, Theorem 3.4 yields Theorem 3 of* [1].

For $\alpha=\delta=0$ and $b=2$ we have,

Corollary 3.6. *If $f\in\mathcal{A}$ satisfies,*

$$\sum_{n=2}^{\infty}\left(\left|\sum_{k=1}^{n}\left[\sum_{j=1}^{k}(-1)^{k-j}(j+e^{-2i\lambda})\binom{\beta}{k-j}a_j\right]\binom{\gamma}{n-k}\right|\right.$$
$$\left.+\left|\sum_{k=1}^{\infty}\left[\sum_{j=1}^{k}(-1)^{k-j}(j-1)\binom{\beta}{k-j}a_j\right]\binom{\gamma}{n-k}\right|\right)\leq 2\cos\lambda$$
$$(0\leq\alpha<1;\quad \beta\in\mathbb{R};\quad \gamma\in\mathbb{R};\quad -\pi/2<\lambda<\pi/2)$$

then $f\in\mathcal{SP}(\lambda)=\mathcal{SP}(\lambda,0)$.

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