

New Proof of the Equation

$$\sum_{k=1}^{\infty} \frac{\mu(k)}{k} = 0.$$

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New Proof of the Equation

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by

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Berlin 1899.

For my dear parents.

The function $\mu(k)$ is normally defined as the number-theoretic function for which

1. $\mu(1) = 1$,
2. $\mu(k) = 0$ when $k > 1$ is divisible by a square,
3. $\mu(k) = (-1)^r$ when k is the product of r distinct primes.

This statement was first expressed by Euler, that

$$\sum_{k=1}^{\infty} \frac{\mu(k)}{k} = 0$$

holds; that is, $\lim_{x \rightarrow \infty} \sum_{k=1}^x \frac{\mu(k)}{k}$ exists and equals 0, the recent proof of which is due to von Mangoldt¹). The same goes for the investigations of Hadamard and de la Vallée Poussin over the Riemann ζ -function, and it seems also, that without the use of these works, the present means of analysis is not enough to give a proof of Euler's statement. However, if one expects the results of those investigations to be in agreement with those of von Mangoldt, then one, as will be executed in the following, can arrive at the target along a quite short path.

The proof, which forms the contents of this dissertation, uses first the theorem²) of Hadamard³) and de la Vallée Poussin⁴):

“If $\vartheta(x) := \sum_{p \leq x} \log p$, then $\lim_{x \rightarrow \infty} \frac{\vartheta(x)}{x} = 1$.” However, apart from the use of this theorem, it is as elementary as can be for such a transcendent statement.

¹) “Beweis der Gleichung $\sum_{k=1}^{\infty} \frac{\mu(k)}{k} = 0$ ”; Proceedings of the Royal Prussian Academy of Science of Berlin, 1897, pp. 835–852.

²) This theorem is proven without the use of von Mangoldt's proof.

³) Bulletin de la société mathématique de France, Volume 24, 1896, p. 217.

⁴) Annales de la société scientifique de Bruxelles, Volume 20, Part 2, p. 251.

1

Denote by $g(x)$ the sum $\sum_{k=1}^{[x]} \frac{\mu(k)}{k}$, where $[x]$ denotes the greatest integer less than or equal to x ; more simply we write

$$g(x) = \sum_{k=1}^x \frac{\mu(k)}{k}, \quad (1)$$

where k ranges over all positive integers less than or equal to x . The sum has meaning only for $x \geq 1$; thus, for $x < 1$, set $g(x) = 0$.

With the above notation, we read the two lemmas, which von Mangoldt proves in a simple way at the start of his paper⁵⁾ and which are also applied in the following one, as:

For all x

$$|g(x)| \leq 1 \quad (2)$$

and for all $x \geq 1$

$$\left| \log x \cdot g(x) - \sum_{k=1}^x \frac{\mu(k) \log k}{k} \right| \leq 3 + \gamma \quad (3)$$

where γ denotes Euler's constant.

The inequality (3), which von Mangoldt only derived in order to apply it in a certain place in his proof⁷⁾, serves in the following one as the basis of the whole investigation.

Concerning the sum $\sum_{k=1}^x \frac{\mu(k) \log k}{k}$, Möbuis⁸⁾ believed he had proved that for sufficiently large x , its difference from -1 is arbitrarily small: however, his proof is not sound. Although new writers consider it probable⁹⁾

⁵⁾ 1. c., pp. 837–839.

⁶⁾ This lemma was, as indicated there, proved in the writings of Gram: "Undersøgelser abgaaende Maengden af Primtal under en given Graense," K. Danske Videnskabernes Selskabs Skrifter, 6te Raekke, naturvidenskabelig og matematisk Afdeling, II, 1884, pp. 197–198.

⁷⁾ 1. c., p. 843.

⁸⁾ "Über eine besondere Art von Umkehrung Reihen," Journal für die reine und angewandte Mathematik, Volume 9, p. 122.

⁹⁾ E.g., Mertens proved, which the general validity of the inequality condition assumes:

$$\left| \sum_{k=1}^x \mu(k) \right| \leq \sqrt{x},$$

that this theorem is correct, if this relation is generally fulfilled (Proceedings of the Vienna Academy, math.-nat. Kl., Volume 106, Dept. 2a, p. 774.)

that

$$\lim_{x \rightarrow \infty} \sum_{k=1}^x \frac{\mu(k) \log k}{k}$$

exists and equals -1 , it has yet to be proven that for all x , $\sum_{k=1}^x \frac{\mu(k) \log k}{k}$ is contained between two finite boundaries. Now since (3) yields

$$\left| g(x) - \frac{1}{\log x} \sum_{k=1}^x \frac{\mu(k) \log k}{k} \right| \leq \frac{3 + \gamma}{\log x},$$

it follows, with use of the Euler–v. Mangoldt Theorem, that

$$\lim_{x \rightarrow \infty} g(x) = 0$$

so that

$$\frac{1}{\log x} \sum_{k=1}^x \frac{\mu(k) \log k}{k}$$

approaches 0 as $x \rightarrow \infty$.

If, in reverse, it was successfully proven that

$$\lim_{x \rightarrow \infty} \frac{1}{\log x} \sum_{k=1}^x \frac{\mu(k) \log k}{k}$$

exists and equals 0, then one would trivially have that

$$\sum_{k=1}^x \frac{\mu(k)}{k} = 0,$$

since for any δ there is a G such that for all $x \geq G$

$$\left| \frac{1}{\log x} \sum_{k=1}^x \frac{\mu(k) \log k}{k} \right| \leq \frac{\delta}{2}$$

and

$$0 < \frac{3 + \gamma}{\log x} \leq \frac{\delta}{2},$$

thus it follows for $x \geq G$:

$$\begin{aligned}
|g(x)| &= \left| \left(g(x) - \frac{1}{\log x} \sum_{k=1}^x \frac{\mu(k) \log k}{k} \right) + \frac{1}{\log x} \sum_{k=1}^x \frac{\mu(k) \log k}{k} \right| \\
&\leq \left| \left(g(x) - \frac{1}{\log x} \sum_{k=1}^x \frac{\mu(k) \log k}{k} \right) \right| + \left| \frac{1}{\log x} \sum_{k=1}^x \frac{\mu(k) \log k}{k} \right| \\
&\leq \frac{3 + \gamma}{\log x} + \left| \frac{1}{\log x} \sum_{k=1}^x \frac{\mu(k) \log k}{k} \right| \\
&\leq \frac{\delta}{2} + \frac{\delta}{2} = \delta,
\end{aligned}$$

also

$$\lim_{x \rightarrow \infty} g(x) = 0.$$

The proof, that for

$$f(x) = \sum_{k=1}^x \frac{\mu(k) \log k}{k}, \quad (4)$$

one has

$$\lim_{x \rightarrow \infty} \frac{f(x)}{\log x} = 0 \quad (5)$$

will be supplied in what follows.

In order not to have to interrupt the course of the investigation, we note the following simple lemma, which was already known to Gram¹⁰): it is

$$\sum_{\nu=1}^x \frac{1}{\nu} g\left(\frac{x}{\nu}\right) = g(x) + \frac{1}{2}g\left(\frac{x}{2}\right) + \frac{1}{3}g\left(\frac{x}{3}\right) + \cdots + \frac{1}{[x]}g\left(\frac{x}{[x]}\right) = 1. \quad (6)$$

The ν -th summand $\frac{1}{\nu}g\left(\frac{x}{\nu}\right)$ contains the sum of the terms

$$\frac{1}{\nu} \frac{\mu(1)}{1} = \frac{\mu(1)}{\nu}, \quad \frac{1}{\nu} \frac{\mu(2)}{2} = \frac{\mu(2)}{2\nu}, \dots, \quad \frac{1}{\nu} \frac{\mu(n)}{n} = \frac{\mu(n)}{n\nu}, \dots, \quad \frac{1}{\nu} \frac{\mu\left[\frac{x}{\nu}\right]}{\left[\frac{x}{\nu}\right]} = \frac{\mu\left[\frac{x}{\nu}\right]}{\left[\frac{x}{\nu}\right]\nu},$$

the sum $\sum_{\nu=1}^x \frac{1}{\nu}g\left(\frac{x}{\nu}\right)$ consists of terms of the form $\frac{\mu(n)}{t}$, where n is a divisor of t , and t runs through the integers from 1 to $[x]$; that is,

$$\sum_{\nu=1}^x \frac{1}{\nu}g\left(\frac{x}{\nu}\right) = \sum_{t=1}^x \frac{1}{t} \sum_{n|t} \mu(n);$$

¹⁰) 1. c., p. 197, where separately for all valid r in equation (43) set $r = 1$.

now since $\sum_{n|t} \mu(n)$ is 1 for $t = 1$ and 0 otherwise, we have

$$\sum_{\nu=1}^x \frac{1}{\nu} g\left(\frac{x}{\nu}\right) = 1.$$

2

If one lets $x = p_1 p_2 \cdots p_r$ in the defining equation (4) of $f(x)$, replaces¹¹⁾ $\log(x)$ by $\log p_1 + \cdots + \log p_r$ and gathers like terms in which the logarithm is applied to the same prime number, then (4) becomes an equation of the form

$$f(x) = \sum F(p, x) \log p$$

where the sum extends over all prime numbers $p \leq x$. As was easily given by von Mangoldt¹²⁾ for another purpose,

$$F(p, x) = - \left(\frac{1}{p} \sum_{k=1}^{\frac{x}{p}} \frac{\mu(k)}{k} + \frac{1}{p^2} \sum_{k=1}^{\frac{x}{p^2}} \frac{\mu(k)}{k} + \frac{1}{p^3} \sum_{k=1}^{\frac{x}{p^3}} \frac{\mu(k)}{k} + \cdots \right),$$

a series which has only a finite number of non-zero summands, since the summation index of the i -th sum runs from 1 to $\left[\frac{x}{p^i}\right]$, so that $p^i \leq x$ gives $i \leq \frac{\log x}{\log p}$. Therefore

$$f(x) = - \sum_{p \leq x} \log p \left(\frac{1}{p} g\left(\frac{x}{p}\right) + \frac{1}{p^2} g\left(\frac{x}{p^2}\right) + \frac{1}{p^3} g\left(\frac{x}{p^3}\right) + \cdots \right),$$

$$f(x) = - \sum_{p \leq x} \frac{\log p}{p} g\left(\frac{x}{p}\right) - \sum_{p \leq x} \log p \left(\frac{1}{p^2} g\left(\frac{x}{p^2}\right) + \frac{1}{p^3} g\left(\frac{x}{p^3}\right) + \cdots \right). \quad (7)$$

According to (2), for all y

$$|g(y)| \leq 1,$$

¹¹⁾ The $\log k$ is multiplied by the factor $\frac{\mu(k)}{k}$, which occurs only for k that are the product of distinct primes.

¹²⁾ 1. c., p. 840.

so that the absolute value of the second sum in (7) is

$$\begin{aligned}
& \left| \sum_{p \leq x} \log p \left(\frac{1}{p^2} g \left(\frac{x}{p^2} \right) + \frac{1}{p^3} g \left(\frac{x}{p^3} \right) + \cdots \right) \right| \\
& \leq \sum_{p \leq x} \log p \left(\frac{1}{p^2} \left| g \left(\frac{x}{p^2} \right) \right| + \frac{1}{p^3} \left| g \left(\frac{x}{p^3} \right) \right| + \cdots \right) \\
& \leq \sum_{p \leq x} \log p \left(\frac{1}{p^2} + \frac{1}{p^3} + \frac{1}{p^4} + \cdots + \frac{1}{p^n} + \cdots \right) \\
& \leq \sum_{p \leq x} \log p \left(\frac{1}{p^2} + \frac{1}{2p^2} + \frac{1}{2^2 p^2} + \cdots + \frac{1}{2^{n-2} p^2} + \cdots \right) \\
& \leq \sum_{p \leq x} \frac{\log p}{p^2} \left(1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots \right) \\
& \leq 2 \sum_{p \leq x} \frac{\log p}{p^2} \\
& < 2 \sum_{\nu=1}^{\infty} \frac{\log \nu}{\nu^2}.
\end{aligned}$$

It is well known that $\sum_{\nu=1}^{\infty} \frac{\log \nu}{\nu^2}$ is convergent; thus as $x \rightarrow \infty$ the sum on the right-hand side of (7) approaches either a certain bound, or its value oscillates between two finite uncertain bounds. In either case, the quotient with $\log x$ approaches 0 as $x \rightarrow \infty$.

Denote by $h(x)$, the function defined by

$$h(x) = \sum_{p \leq x} \frac{\log p}{p} g \left(\frac{x}{p} \right). \quad (8)$$

If $\lim_{x \rightarrow \infty} \frac{h(x)}{\log x}$ exists and equals 0, then according to (7)

$$\frac{f(x) + h(x)}{\log x} = \frac{-1}{\log x} \sum_{p \leq x} \log p \left(\frac{1}{p^2} g \left(\frac{x}{p^2} \right) + \frac{1}{p^3} g \left(\frac{x}{p^3} \right) + \cdots \right),$$

and as we just saw, as x gets large the right-hand side approaches 0, from which the correctness of statement (5) follows.

The proof of Euler's statement, that

$$\sum_{k=1}^x \frac{\mu(k)}{k} = 0,$$

thus depends on the proof of the statement

$$\lim_{x \rightarrow \infty} \frac{h(x)}{\log x} = 0,$$

which will be furnished in the following section.

3

Recall that the function $\vartheta(x)$ ¹³⁾ is defined for all positive ν by

$$\vartheta(\nu) - \vartheta(\nu - 1) = \begin{cases} \log \nu & \text{if } \nu \text{ is prime,} \\ 0 & \text{if } \nu \text{ is composite or } 1, \\ \log \nu = 0 & \text{if } \nu = 1. \end{cases}$$

And

$$h(x) = \sum_{\nu=1}^x \frac{\vartheta(\nu) - \vartheta(\nu - 1)}{\nu} g\left(\frac{x}{\nu}\right),$$

where the sum ranges over the integers between 1 and x .

In the place of the function $\vartheta(x)$, use

$$\vartheta(x) = x\{1 + \varepsilon(x)\}, \tag{9}$$

where the function $\varepsilon(x)$ takes only non-negative values of x , and $\varepsilon(0) = 0$. We note the following properties of $\varepsilon(x)$:

1. Since by definition, $\vartheta(x)$ is never negative, then always

$$\varepsilon(x) \geq -1.$$

2. As shown by Mertens¹⁴⁾, for all x

$$\vartheta(x) < 2x,$$

so that always

$$\varepsilon(x) < 1;$$

therefore, we gain the inequality

$$|\varepsilon(x)| \leq 1. \tag{10}$$

¹³⁾ See the theorem of Hadamard and de la Vallée Poussin in the introduction.

¹⁴⁾ "Ein Beitrag zur analytischen Zahlentheorie," Journal für die reine und angewandte Mathematik, Volume 78, p. 48.

3. The theorem cited in the introduction¹⁵), that

$$\lim_{x=\infty} \frac{\vartheta(x)}{x} = 1,$$

gives

$$\lim_{x=\infty} \varepsilon(x) = 0. \quad (11)$$

The introduction of the function $\varepsilon(x)$ yields, for $h(x)$,

$$\begin{aligned} h(x) &= \sum_{\nu=1}^x \frac{\nu + \nu\varepsilon(\nu) - (\nu-1) - (\nu-1)\varepsilon(\nu-1)}{\nu} g\left(\frac{x}{\nu}\right) \\ &= \sum_{\nu=1}^x \left\{ \frac{1}{\nu} g\left(\frac{x}{\nu}\right) + \left(\varepsilon(\nu) - \frac{\nu-1}{\nu} \varepsilon(\nu-1) \right) g\left(\frac{x}{\nu}\right) \right\} \\ &= \sum_{\nu=1}^x \frac{1}{\nu} g\left(\frac{x}{\nu}\right) + \sum_{\nu=1}^x \left(\varepsilon(\nu) - \varepsilon(\nu-1) + \frac{1}{\nu} \varepsilon(\nu-1) \right) g\left(\frac{x}{\nu}\right). \end{aligned}$$

Using Eqs. (8) and (9),

$$\sum_{\nu=1}^x \frac{1}{\nu} g\left(\frac{x}{\nu}\right) = 1;$$

yielding

$$h(x) - 1 = \sum_{\nu=1}^x (\varepsilon(\nu) - \varepsilon(\nu-1)) g\left(\frac{x}{\nu}\right) + \sum_{\nu=1}^x \frac{1}{\nu} \varepsilon(\nu-1) g\left(\frac{x}{\nu}\right). \quad (12)$$

For the first of the two sums in (12) we get

$$\begin{aligned} &\sum_{\nu=1}^x (\varepsilon(\nu) - \varepsilon(\nu-1)) g\left(\frac{x}{\nu}\right) \\ &= \sum_{\nu=1}^x \varepsilon(x) \left(g\left(\frac{x}{\nu}\right) - g\left(\frac{x}{\nu+1}\right) \right) + \varepsilon([x]) g\left(\frac{x}{[x]+1}\right) \\ &= \sum_{\nu=1}^x \varepsilon(\nu) \left(g\left(\frac{x}{\nu}\right) - g\left(\frac{x}{\nu+1}\right) \right), \end{aligned}$$

where $x < [x] + 1$ so that $g\left(\frac{x}{[x]+1}\right) = 0$.

¹⁵) See the theorem of Hadamard and de la Vallée Poussin in the introduction.

If in the second sum in (12) we write $\nu + 1$ in place of ν , we have

$$\sum_{\nu=0}^{x-1} \frac{1}{\nu+1} \varepsilon(\nu) g\left(\frac{x}{\nu+1}\right) = \sum_{\nu=1}^{x-1} \frac{1}{\nu+1} \varepsilon(\nu) g\left(\frac{x}{\nu+1}\right)$$

and so

$$h(x) - 1 = \sum_{\nu=1}^x \varepsilon(\nu) \left(g\left(\frac{x}{\nu}\right) - g\left(\frac{x}{\nu+1}\right) \right) + \sum_{\nu=1}^{x-1} \frac{1}{\nu+1} \varepsilon(\nu) g\left(\frac{x}{\nu+1}\right). \quad (13)$$

Let δ be an arbitrary small positive quantity. Then by (11), there is a G such that for all $\nu \geq G$

$$|\varepsilon(\nu)| \leq \frac{\delta}{3}. \quad (14)$$

This yields

$$\begin{aligned} & \left| \sum_{\nu=1}^x \varepsilon(\nu) \left(g\left(\frac{x}{\nu}\right) - g\left(\frac{x}{\nu+1}\right) \right) \right| \leq \left| \sum_{\nu=1}^{G-1} \varepsilon(\nu) \left(g\left(\frac{x}{\nu}\right) - g\left(\frac{x}{\nu+1}\right) \right) \right| \\ & \quad + \left| \sum_{\nu=G}^x \varepsilon(\nu) \left(g\left(\frac{x}{\nu}\right) - g\left(\frac{x}{\nu+1}\right) \right) \right| \\ & \leq \sum_{\nu=1}^{G-1} |\varepsilon(\nu)| \left| \left(g\left(\frac{x}{\nu}\right) - g\left(\frac{x}{\nu+1}\right) \right) \right| + \sum_{\nu=G}^x |\varepsilon(\nu)| \left| \left(g\left(\frac{x}{\nu}\right) - g\left(\frac{x}{\nu+1}\right) \right) \right|. \end{aligned}$$

As the right-hand side is increased, $|\varepsilon(\nu)|$ goes to 1 in the first sum (by (10)), and goes to $\delta/3$ in the second sum (by (14)), yielding

$$\begin{aligned} & \left| \sum_{\nu=1}^x \varepsilon(\nu) \left(g\left(\frac{x}{\nu}\right) - g\left(\frac{x}{\nu+1}\right) \right) \right| \leq \sum_{\nu=1}^{G-1} \left| \left(g\left(\frac{x}{\nu}\right) - g\left(\frac{x}{\nu+1}\right) \right) \right| \quad (15) \\ & \quad + \frac{\delta}{3} \sum_{\nu=G}^x \left| \left(g\left(\frac{x}{\nu}\right) - g\left(\frac{x}{\nu+1}\right) \right) \right|. \end{aligned}$$

Now

$$\begin{aligned} \sum_{\nu=1}^{G-1} \left| \left(g\left(\frac{x}{\nu}\right) - g\left(\frac{x}{\nu+1}\right) \right) \right| & \leq \sum_{\nu=1}^{G-1} \left\{ \left| g\left(\frac{x}{\nu}\right) \right| - \left| g\left(\frac{x}{\nu+1}\right) \right| \right\} \\ & = \sum_{\nu=1}^{G-1} (1 + 1) \quad (\text{by (2)}), \end{aligned}$$

so that

$$\sum_{\nu=1}^{G-1} \left| \left(g\left(\frac{x}{\nu}\right) - g\left(\frac{x}{\nu+1}\right) \right) \right| \leq 2(G-1) \quad (16)$$

and

$$\begin{aligned} \sum_{\nu=G}^x \left| \left(g\left(\frac{x}{\nu}\right) - g\left(\frac{x}{\nu+1}\right) \right) \right| &\leq \sum_{\nu=1}^x \left| \left(g\left(\frac{x}{\nu}\right) - g\left(\frac{x}{\nu+1}\right) \right) \right| \\ &= \sum_{\nu=1}^x \left| \sum_{k=1}^{\frac{x}{\nu}} \frac{\mu(k)}{k} - \sum_{k=1}^{\frac{x}{\nu+1}} \frac{\mu(k)}{k} \right| \\ &= \sum_{\nu=1}^x \left| \sum \frac{\mu(k)}{k} \right|, \end{aligned}$$

where k ranges over all integers in the interval $\left(\frac{x}{\nu+1}, \frac{x}{\nu}\right]$. Therefore

$$\begin{aligned} \sum_{\nu=G}^x \left| \left(g\left(\frac{x}{\nu}\right) - g\left(\frac{x}{\nu+1}\right) \right) \right| &\leq \sum_{\nu=1}^x \sum_{\frac{x}{\nu} \geq k > \frac{x}{\nu+1}} \frac{|\mu(k)|}{k} \leq \sum_{\nu=1}^x \sum_{\frac{x}{\nu} \geq k > \frac{x}{\nu+1}} \frac{1}{k} \\ &= \sum_{x \geq k > \frac{x}{2}} \frac{1}{k} + \sum_{\frac{x}{2} \geq k > \frac{x}{3}} \frac{1}{k} + \sum_{\frac{x}{3} \geq k > \frac{x}{4}} \frac{1}{k} + \cdots + \sum_{\frac{x}{[x]-1} \geq k > \frac{x}{[x]}} \frac{1}{k} \\ &\quad + \sum_{\frac{x}{[x]} \geq k > 1} \frac{1}{k} = \sum_{k=1}^x \frac{1}{k}, \end{aligned}$$

and since always

$$\sum_{k=1}^x \frac{1}{k} \leq \log x + 1,$$

we have

$$\sum_{\nu=G}^x \left| \left(g\left(\frac{x}{\nu}\right) - g\left(\frac{x}{\nu+1}\right) \right) \right| \leq \log x + 1. \quad (17)$$

Replacing both sums of the right-hand side of inequality (15) by the results gained in (16) and (17) yields

$$\left| \sum_{\nu=1}^x \varepsilon(x) \left(g\left(\frac{x}{\nu}\right) - g\left(\frac{x}{\nu+1}\right) \right) \right| \leq 2(G-1) + \frac{\delta}{3}(\log x + 1). \quad (18)$$

The handling of the second sum in (13) is somewhat simpler. We have

$$\begin{aligned}
\left| \sum_{\nu=1}^{x-1} \frac{1}{\nu+1} \varepsilon(\nu) g\left(\frac{x}{\nu+1}\right) \right| &\leq \sum_{\nu=1}^{x-1} \frac{1}{\nu+1} |\varepsilon(\nu)| \left| g\left(\frac{x}{\nu+1}\right) \right| \\
&\leq \sum_{\nu=1}^{x-1} \frac{|\varepsilon(\nu)|}{\nu+1} \\
&= \sum_{\nu=1}^{G-1} \frac{|\varepsilon(\nu)|}{\nu+1} + \sum_{\nu=G}^{x-1} \frac{|\varepsilon(\nu)|}{\nu+1} \\
&\leq \sum_{\nu=1}^{G-1} \frac{1}{\nu+1} + \sum_{\nu=G}^{x-1} \frac{1}{\nu+1} \frac{\delta}{3} \\
&\leq \sum_{\nu=1}^{G-1} 1 + \frac{\delta}{3} \sum_{\nu=1}^{x-1} \frac{1}{\nu},
\end{aligned}$$

so that

$$\left| \sum_{\nu=1}^{x-1} \frac{1}{\nu+1} \varepsilon(\nu) g\left(\frac{x}{\nu+1}\right) \right| \leq G - 1 + \frac{\delta}{3} (\log x + 1). \quad (19)$$

With help from the inequalities (18) and (19), (13) becomes

$$\begin{aligned}
|h(x)| &\leq 1 + 2G - 2 + \frac{\delta}{3} \log x + \frac{\delta}{3} + G - 1 + \frac{\delta}{3} \log x + \frac{\delta}{3} \\
&= 3G - 2 + \frac{2}{3} \delta + \frac{2}{3} \delta \log x,
\end{aligned}$$

thus for all

$$x \geq e^{\frac{3G-2+\frac{2}{3}\delta}{\frac{1}{2}\delta}},$$

we have

$$3G - 2 + \frac{2}{3} \delta \leq \frac{1}{3} \delta \log x,$$

so that

$$|h(x)| \leq \frac{1}{3} \delta \log x + \frac{2}{3} \delta \log x = \delta \log x,$$

which yields

$$\left| \frac{h(x)}{\log x} \right| \leq \delta. \quad (20)$$

For such a δ there is a ξ assignable, such that for all $x \geq \xi$, (20) is fulfilled; therefore the $\lim_{x \rightarrow \infty} \frac{h(x)}{\log x}$ exists and equals 0. Thus all the results shown in

the first two paragraphs of this work are valid; that is, the $\lim_{x=\infty} \sum_{k=1}^x \frac{\mu(k)}{k}$ exists and equals 0, and thus the correctness of the equation named in the title, briefly

$$\sum_{k=1}^x \frac{\mu(k)}{k} = 0.$$

4

If we define¹⁶⁾

$$M(x) = \sum_{k=1}^x \mu(k),$$

then with help of the proven result,

$$\lim_{x=\infty} g(x) = 0,$$

we have

$$\lim_{x=\infty} \frac{M(x)}{x} = 0.$$

Von Mangoldt¹⁷⁾ proved this indirectly by use of the identity

$$g(x) = \sum_{k=1}^x \frac{\mu(k)}{k} = \sum_{k=1}^x (M(k) - M(k-1)) \frac{1}{k}.$$

It can be furnished as follows directly. From the equation

$$M(x) = \sum_{k=1}^x \mu(k) = \sum_{k=1}^x \frac{\mu(k)}{k} \cdot k = \sum_{k=1}^x (g(k) - g(k-1)) \cdot k,$$

it follows that

$$\begin{aligned} M(x) &= \sum_{k=1}^{x-1} g(k)(k - (k+1)) + g(x)[x] \\ &= - \sum_{k=1}^{x-1} g(k) + g(x)[x], \end{aligned}$$

¹⁶⁾ von Mangoldt, 1. c., p. 850.

¹⁷⁾ 1. c., pp. 849–851.

so that since for $\delta > 0$ there is a G such that for all $k \geq G$

$$|g(k)| \leq \frac{\delta}{3}$$

for all $x \geq G$

$$\begin{aligned} |M(x)| &\leq \sum_{k=1}^{G-1} |g(k)| + \sum_{k=G}^{x-1} |g(k)| + |g(x)| \cdot x \\ &\leq G - 1 + \frac{\delta}{3}([x] - G) + \frac{\delta}{3}x, \end{aligned}$$

so that

$$\left| \frac{M(x)}{x} \right| \leq \frac{G - 1 - \frac{\delta}{3}G}{x} + \frac{2}{3}\delta,$$

then for

$$x \geq \frac{G - 1 - \frac{\delta}{3}G}{\frac{\delta}{3}}$$

and at the same time greater than or equal to G ,

$$\left| \frac{M(x)}{x} \right| \leq \frac{1}{3}\delta + \frac{2}{3}\delta = \delta,$$

with which the statement

$$\lim_{x \rightarrow \infty} \frac{M(x)}{x} = 0$$

is proved.

Theses

1. It is desirable during every existence proof of a mathematical quantity to be led, at the same time on the way to the result, to the actual existing quantity.
2. A boundary between arithmetic and analytic areas of mathematics cannot be drawn.
3. The concept of the semiconvergent series is a relative concept.
4. Out of the impossibility of perpetual motion of second kind comes the proof of the second law of thermodynamics.
5. It did not succeed, the justifying of psychology on an exactly mathematical basis.