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Starlike Functions in the Hornich Space

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Abstract. We will show that the set of starlike univalent functions in $\mathbb{D}$ is starlike in the Hornich space, i.e. for starlike functions f and $0 \le \alpha \le 1$ the function $\int_0^z (f'(\zeta))^\alpha \, d\zeta$ is also starlike. This solves a problem given by Kim, Ponnusamy and Sugawa in [6]. An important step in proving this result will be to show that for starlike functions f and $z \in \mathbb{D}$ we have $|\int_0^1 \arg(z/\gamma'(t))\, dt| < \pi/2$, where $\gamma(t) := f^{-1}(tf(z))$, $0 \le t \le 1$.

Keywords. Starlike functions; Hornich operations; Integral transform.

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1. Introduction

Let $\mathcal{A}$ denote the class of functions f analytic in $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ normalized by $f(0) = f'(0) - 1 = 0$ and $\mathcal{H}$ the class of functions f in $\mathcal{A}$ that are locally univalent, i.e. that satisfy $f' \neq 0$ in $\mathbb{D}$. In [4] H. Hornich introduced an addition and a scalar multiplication for functions in the class $\mathcal{H}$. For f, $g \in \mathcal{H}$ and $\alpha \in \mathbb{C}$ the *Hornich operations* are defined by

$$(f \oplus g)(z) := \int_0^z f'(\zeta)g'(\zeta)\, d\zeta \qquad \text{and} \qquad (\alpha \odot f)(z) := \int_0^z [f'(\zeta)]^\alpha \, d\zeta,$$

where the branch of $(f')^\alpha$ is taken so that $(f')^\alpha(0) = 1$. It is clear that with these operations the set $\mathcal{H}$ becomes a complex vector space — the so called *Hornich space* — with zero element id: $z \mapsto z$. Obviously the set $\mathcal{S}$ of functions in $\mathcal{A}$ that are univalent in $\mathbb{D}$ is a subset of $\mathcal{H}$. Even though in general the Hornich sum of two functions in $\mathcal{S}$ is not univalent, it will at least be locally univalent. This, of course, does not have to be true for the usual sum of two functions in $\mathcal{S}$.

Therefore it seems a logical step to examine the structure of $\mathcal{S}$ in the Hornich space more closely. For instance, Pfaltzgraff [9] showed that $\alpha \odot \mathcal{S} \subset \mathcal{S}$ if $|\alpha| < 1/4$. On the other hand, for each $|\alpha| > 1/3$, $\alpha \neq 1$, Royster [10] was able

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to find a function $f \in \mathcal{S}$ such that $\alpha \odot f \notin \mathcal{S}$. These, however, are the only results obtained so far in this direction.

Regarding the more special class $\mathcal{K}$ of functions in $\mathcal{S}$ that map the unit disk onto a convex set, it is shown in [2] that $\mathcal{K}$ is convex in $\mathcal{H}$, i.e. that for f, $g \in \mathcal{K}$ and $0 \le t \le 1$ also $[t \odot f] \oplus [(1-t) \odot g] \in \mathcal{K}$. Y. J. Kim and Merkes [7] proved that the same is true for the set $\mathcal{C} \subset \mathcal{S}$ of *close-to-convex functions* (cf. [11, p. 46] for the definition of close-to-convex functions).

For α, $\beta \ge 0$ the *Kaplan class* $K(\alpha, \beta)$ is defined to be the set of functions f which are analytic and non-vanishing in $\mathbb{D}$ and satisfy $f(0) = 1$ as well as

$$-\alpha\pi - \frac{1}{2}(\alpha - \beta)(\theta_1 - \theta_2) \le \arg\, f(re^{i\theta_2}) - \arg\, f(re^{i\theta_1})$$

for $0 < r < 1$ and $\theta_1 < \theta_2 < \theta_1 + 2\pi$ [11, p. 33]. Both $\mathcal{K}$ and $\mathcal{C}$ can be characterized in terms of the Kaplan classes: $f \in \mathcal{A}$ is in $\mathcal{K}$ if and only if $f' \in K(0, 2)$ and in $\mathcal{C}$ if and only if $f' \in K(1, 3)$ [11, p. 46]. Therefore the results mentioned above are only special cases of the following theorem.

Theorem A. *For α, $\beta \ge 0$ let $C(\alpha, \beta)$ be the set of functions $f \in \mathcal{A}$ such that $f' \in K(\alpha, \beta)$. Then for $f \in C(\alpha, \beta)$, $g \in C(\mu, \lambda)$ and all t, $s \ge 0$ we have*

$$(t \odot f) \oplus (s \odot g) \in C(t\alpha + s\mu, t\beta + s\lambda).$$

In particular, for all α, $\beta \ge 0$ the set $C(\alpha, \beta)$ is convex in the Hornich space.

From the definition of the Kaplan classes it can be easily seen that for $f \in K(\alpha, \beta)$, $g \in K(\mu, \lambda)$ and $t \ge 0$ we have $fg \in K(\alpha + \mu, \beta + \lambda)$ and $f^t \in K(t\alpha, t\beta)$ and so the proof of Theorem A becomes almost trivial.

Things become more difficult if we consider the class $\mathcal{S}^* \subset \mathcal{S}$ of *starlike functions*, i.e. functions that map the unit disk univalently onto a starlike set. In contrast to the classes $\mathcal{K}$ and $\mathcal{C}$, a characterization of $\mathcal{S}^*$ in terms of the $C(\alpha, \beta)$ is not known. In fact, only little is known about the derivatives of starlike functions besides the fact that $f \in \mathcal{A}$ is starlike if and only if $\operatorname{Re} zf'(z)/f(z) > 0$ for $z \in \mathbb{D}$.

In [6] Kim, Ponnusamy and Sugawa showed that $\mathcal{S}^*$ is not convex in the Hornich space. Having also proved that the straight line between the *Koebe function* $k(z) = z/(1-z)^2$ and the zero element id lies in $\mathcal{S}^*$, they raised the question if the class $\mathcal{S}^*$ was at least starlike in $\mathcal{H}$, i.e. if for all $f \in \mathcal{S}^*$ and all $0 \le \alpha \le 1$ the relation $\alpha \odot f \in \mathcal{S}^*$ would hold.

In this paper we will show that the answer to this question is positive.

Theorem 1. *For all $f \in \mathcal{S}^*$ the straight line $\alpha \odot f$, $0 \le \alpha \le 1$, between f and* id *is contained in $\mathcal{S}^*$.*

As we will see, this theorem is a consequence of the following statement concerning the mapping behaviour of the integral transforms

$$I_f(z) := \int_0^z \frac{f'(\zeta)}{f(z)} \log \frac{zf'(\zeta)}{f(z)}\, d\zeta, \qquad z \in \mathbb{D},$$

of starlike functions f. Obviously I_f is analytic in $\mathbb{D}$ for $f \in \mathcal{S}^*$ (even for $f \in \mathcal{S}$).

Theorem 2. *Let $f \in \mathcal{S}^*$. Then $|\operatorname{Im} I_f(z)| < \pi/2$ for $z \in \mathbb{D}$.*

If we choose the special integration path $\gamma(t) := f^{-1}(tf(z))$, $0 \le t \le 1$, then $\gamma'(t)f'(\gamma(t)) = f(z)$ for $0 \le t \le 1$ and so $I_f(z) = \int_0^1 \log(z/\gamma'(t))\,dt$. In particular, Theorem 2 has the following equivalent formulation.

Theorem 2′. *For $f \in \mathcal{S}^*$ and $z \in \mathbb{D}$ we have*

$$\left| \int_0^1 \arg \frac{z}{\gamma'(t)}\, dt \right| < \frac{\pi}{2},$$

where $\gamma(t) := f^{-1}(tf(z))$, $0 \le t \le 1$.

It seems this property of starlike functions has been unknown until now.

The proof of Theorem 1 will be presented in Section 3 and Theorem 2 will be shown in Section 4. Before this we will point out some preliminary facts in Section 2.

For the proof of Theorem 2 we will need (a slightly extended version of) the Clunie-Jack Lemma [5], a consequence of the maximum modulus principle that has already found many other applications. We give it here as a reference.

Clunie-Jack Lemma. *Let F be an analytic function in $\mathbb{D}$ with $F(0) = 0$. Suppose that F has an analytic extension to a neighborhood N of some point $w \in \mathbb{T} := \{z \in \mathbb{C} \colon |z| = 1\}$ and that $|F(z)| \le |F(w)|$ for all $z \in \mathbb{D} \cup (\mathbb{T} \cap N)$. Then $wF'(w)/F(w) \ge 1$.*

2. Preliminaries

The following statement will be used in the proof of Theorem 2.

Lemma 3. *For all $a \in \mathbb{R}$ and $b \ge \pi/2$ we have*

$$\pi\left(\log\left(b + \frac{\pi}{2}\right) + 1 + a\right) - 2b\left(b + \frac{\pi}{2}\right)\cosh\frac{\pi a}{2b} < 0. \tag{1}$$

Proof. Using the estimate $\log(1+x) < x$, $x > 0$, we find that the left-hand side of (1) is smaller than $\pi^2/2 + \pi a + \pi b - 2b\pi\cosh(\pi a/2b)$ and therefore it will be enough to show that

$$\frac{a}{2b} - \cosh\frac{\pi a}{2b} < \frac{\pi}{4b} - \frac{1}{2}. \tag{2}$$

It is easy to see that $(\operatorname{arsinh}(1/\pi) - \sqrt{1+\pi^2})/\pi = -.94\ldots < -1/2$ is the maximum value of $x - \cosh(\pi x)$ in $\mathbb{R}$. Since $\pi/4b > 0$ when $b \ge \pi/2$, (2) is proven. ■

We will now present some properties of the integral transform I_f.

Lemma 4. *Let $f \in \mathcal{S}^*$. Then* $\operatorname{Im} I_f$ *is in* h^∞*, i.e.* $\operatorname{Im} I_f$ *is a bounded harmonic function in* $\mathbb{D}$*. Further, if f has an analytic extension to a neighborhood N of a point $w = e^{it_0} \in \mathbb{T}$, then $I_f(z)$ is continuous in $\mathbb{D} \cup (\mathbb{T} \cap N)$ and $t \mapsto I_f(e^{it})$ is differentiable in t_0. If $f'(w) = 0$, then*

$$\lim_{z \to w, z \in \overline{\mathbb{D}}} f'(z) \log f'(z) = 0. \tag{3}$$

Proof. Let $z \in \mathbb{D}$ and set $\gamma(t) := f^{-1}(tf(z))$, $t \in [0,1]$. Then $\gamma'(t)f'(\gamma(t)) = f(z)$ for $t \in [0,1]$ and thus

$$\operatorname{Im} I_f(z) = \int_0^1 \arg f'(\gamma(t))dt - \arg \frac{f(z)}{z}.$$

As $|\arg f'(z)| < 3\pi/2$ and $|\arg f(z)/z| < \pi$ in $\mathbb{D}$ for all functions $f \in \mathcal{S}^*$ (cf. [3]) we find that $|\operatorname{Im} I_f(z)| < 5\pi/2$ for all $z \in \mathbb{D}$. Since I_f is analytic in $\mathbb{D}$, it follows that $\operatorname{Im} I_f \in h^\infty$.

Now suppose that f has an analytic extension to a neighborhood N of a point $w = e^{it_0} \in \mathbb{T}$. Since by Koebe's 1/4-Theorem $f(w) \neq 0$, it is clear that if f' does not vanish in w, then $I_f(z)$ is continuous in $\mathbb{D} \cup (\mathbb{T} \cap N)$ and $t \mapsto I_f(e^{it})$ is differentiable in t_0. As

$$I_f(z) = \frac{1}{f(z)} \int_0^z f'(\zeta) \log f'(\zeta)\, d\zeta - \log \frac{f(z)}{z}, \tag{4}$$

it is also clear that the same will hold in the case $f'(w) = 0$ once we have shown (3). Thus, suppose that $f'(w) = 0$. Since $|\arg f'(z)| < 3\pi/2$ in $\mathbb{D}$ for all functions $f \in \mathcal{S}^*$, we have

$$\lim_{z \to w, z \in \overline{\mathbb{D}}} |f'(z) \log f'(z)| \leq \lim_{z \to w, z \in \overline{\mathbb{D}}} |f'(z)| \left(\log |f'(z)| + \frac{3\pi}{2} \right) = 0,$$

as required. ■

Our examination of the transform I_f will be based on the fact that it satisfies the following differential equation.

Lemma 5. *Let $f \in \mathcal{S}^*$ and suppose that f has an analytic extension to a neighborhood of a point $w = re^{it_0} \in \overline{\mathbb{D}}$, $0 \leq r \leq 1$, $t_0 \in \mathbb{R}$. Then, if $f'(w) \neq 0$,*

$$\left. \frac{d}{dt} I_f(re^{it}) \right|_{t=t_0} = iwI_f'(w) = i + \frac{iwf'(w)}{f(w)} \left(\log \frac{wf'(w)}{f(w)} - 1 - I_f(w) \right), \tag{5}$$

while

$$\left. \frac{d}{dt} I_f(e^{it}) \right|_{t=t_0} = \lim_{z \to w, z \in \overline{\mathbb{D}}} izI_f'(z) = i, \tag{6}$$

in the case that $f'(w) = 0$.

Proof. Using the representation (4) of I_f, a simple calculation shows that if $f'(w) \neq 0$

$$\begin{aligned}\frac{d}{dt} I_f(re^{it})\Big|_{t=t_0} &= i + \frac{iwf'(w)}{f(w)} \left(\log f'(w) - 1 - \int_0^w \frac{f'(\zeta)}{f(w)} \log f'(\zeta)\, d\zeta \right) \\ &= i + \frac{iwf'(w)}{f(w)} \left(\log \frac{wf'(w)}{f(w)} - 1 - I_f(w) \right).\end{aligned}$$

If $f'(w) = 0$, then, because of (3),

$$\frac{d}{dt} I_f(e^{it})\Big|_{t=t_0} = i - \frac{iwf'(w)}{f^2(w)} \int_0^w f'(\zeta) \log f'(\zeta)\, d\zeta - \frac{iwf'(w)}{f(w)} = i.$$

■

Lemma 6. *Let $f \in \mathcal{S}^*$ and suppose that there is a $w = e^{is} \in \mathbb{T}$ such that f has an analytic extension to a neighborhood N of w and satisfies $\operatorname{Re} wf'(w)/f(w) = 0$ and $|\operatorname{Im} I_f(z)| < |\operatorname{Im} I_f(w)|$ in $\mathbb{D}$. Then $f'(w) \neq 0$ and $\operatorname{Im} I_f(w) < \pi/2$ if $\arg wf'(w)/f(w) = \pi/2$ and $\operatorname{Im} I_f(w) > -\pi/2$ if $\arg wf'(w)/f(w) = -\pi/2$.*

Proof. Since $|\operatorname{Im} I_f(z)| < |\operatorname{Im} I_f(w)|$ in $\mathbb{D}$, we have $[d/dt \operatorname{Im} I_f(e^{it})]_{t=s} = 0$ and therefore $f'(w) \neq 0$ by Lemma 5. Taking the imaginary part of (5) and setting $r := iwf'(w)/f(w)$ and $\varphi := \arg(-ir)$, we obtain $0 = 1 + r(\varphi - \operatorname{Im} I_f(w))$. Since $\varphi = \pm\pi/2$ and $r\varphi < 0$, the lemma follows. ■

3. Proof of Theorem 1

For $f \in \mathcal{S}^*$ and $0 \le \alpha \le 1$ set $f_\alpha := \alpha \odot f$. Theorem 1 will be proved if we can show that for all $f \in \mathcal{S}^*$, $0 \le \alpha \le 1$ and $z \in \mathbb{D}$ we have $\operatorname{Re} f_\alpha(z)/(zf'_\alpha(z)) > 0$.

Let us denote by $\mathcal{S}^*_c$ the set of functions $f \in \mathcal{S}^*$ that are analytic in a neighborhood of the closed unit disk and by $\mathcal{S}^*_{c,+}$ the set of functions $f \in \mathcal{S}^*_c$ that satisfy $\operatorname{Re} zf'(z)/f(z) > 0$ in $\overline{\mathbb{D}}$. If f is any function in $\mathcal{S}^*$, then, for $0 < r < 1$, $g(z) := f(rz)/r$ belongs to $\mathcal{S}^*_{c,+}$ and we have

$$\operatorname{Re} \frac{zg'_\alpha(z)}{g_\alpha(z)} = \operatorname{Re} \frac{z\,(g'(z))^\alpha}{\int_0^z (g'(\zeta))^\alpha\, d\zeta} = \operatorname{Re} \frac{rz\,(f'(rz))^\alpha}{\int_0^{rz} (f'(\zeta))^\alpha\, d\zeta} = \operatorname{Re} \frac{rzf'_\alpha(rz)}{f_\alpha(rz)}$$

for $z \in \mathbb{D}$, $0 < r < 1$ and $0 \le \alpha \le 1$. Therefore we only have to show that Theorem 1 is true for functions in the class $\mathcal{S}^*_{c,+}$.

Thus, let $f \in \mathcal{S}^*_{c,+}$. Then there is a $c > 0$ such that $\operatorname{Re} zf'(z)/f(z) > c$ in $\overline{\mathbb{D}}$ and so there must be a $0 \le \alpha^* < 1$ such that $f_\alpha \in \mathcal{S}^*_{c,+}$ for $\alpha^* < \alpha \le 1$. Suppose that Theorem 1 is wrong and that the f we have chosen is a counterexample to it. This means we can assume that $\alpha^* > 0$ is such that $f_\alpha \in \mathcal{S}^*_{c,+}$ for $\alpha^* < \alpha \le 1$, but that for each $\epsilon > 0$ there is an $\alpha \in (\alpha^* - \epsilon, \alpha^*)$ with $f_\alpha \notin \mathcal{S}^*_{c,+}$. Since $\mathcal{S}^*$ is compact, f_{α^*} must then be a member of $\mathcal{S}^*_c$ and satisfy $\operatorname{Re} wf'_{\alpha^*}(w)/f_{\alpha^*}(w) = 0$ and $f'_{\alpha^*}(w) \neq 0$ for a w on the unit circle ($f'(w) \neq 0$ obviously implies $f'_\alpha(w) \neq 0$

for $0 \le \alpha \le 1$). This, however, is a contradiction to the next lemma since $(f_{\alpha^*})_\alpha = f_{\alpha\alpha^*} \in \mathcal{S}^*$ for $1 < \alpha \le 1/\alpha^*$. Theorem 1 must therefore be true.

Lemma 7. *Let $f \in \mathcal{S}_c^*$ with $\operatorname{Re} wf'(w)/f(w) = 0$ and $f'(w) \neq 0$ for some $w \in \mathbb{T}$. Then there is a number $\alpha^* > 1$ such that $f_\alpha \notin \mathcal{S}^*$ for $1 < \alpha < \alpha^*$.*

Proof. Since $f'(w) \neq 0$, also $wf'(w)/f(w) \neq 0$ and so $wf'(w)/f(w) = i\phi$ for a $\phi \neq 0$. Assume that $\phi > 0$.

To prove the lemma it will clearly be enough to show that

$$\frac{d}{d\alpha} \operatorname{Re} \frac{f_\alpha(w)}{wf'_\alpha(w)}\bigg|_{\alpha=1} < 0.$$

Now,

$$\begin{aligned}
\frac{d}{d\alpha} \operatorname{Re} \frac{f_\alpha(w)}{wf'_\alpha(w)}\bigg|_{\alpha=1} &= \frac{d}{d\alpha} \operatorname{Re} \int_0^w \frac{1}{w}\left(\frac{f'(\zeta)}{f'(w)}\right)^\alpha d\zeta\bigg|_{\alpha=1} \\
&= \operatorname{Re} \int_0^w \frac{f'(\zeta)}{wf'(w)} \log\frac{f'(\zeta)}{f'(w)}\, d\zeta \\
&= \operatorname{Re} \frac{f(w)}{iwf'(w)} i\left(\int_0^w \frac{f'(\zeta)}{f(w)} \log\frac{wf'(\zeta)}{f(w)}\, d\zeta - \log\frac{wf'(w)}{f(w)}\right) \\
&= \frac{1}{\phi}\left(\operatorname{Im} I_f(w) - \frac{\pi}{2}\right).
\end{aligned}$$

Since it follows readily from Theorem 2 and Lemma 6 that $\operatorname{Im} I_f(w) < \pi/2$, the lemma is proven in the case $\phi > 0$.

In the case $\phi < 0$ one can proceed similarly to complete the proof. ∎

4. Proof of Theorem 2

For $n \in \mathbb{N}$ let $\mathcal{D}_n$ be the set of functions f of the form

$$f(z) = \frac{z}{\prod_{j=1}^n (1 - ze^{-i\theta_j})^{\alpha_j}}, \tag{7}$$

where $\theta_j < \theta_{j+1}$ for $1 \le j \le n$ ($\theta_{n+1} := \theta_1 + 2\pi$) as well as $\alpha_j > 0$ and $\sum_{j=1}^n \alpha_j = 2$; in addition, set $\mathcal{D} := \bigcup_{n\in\mathbb{N}} \mathcal{D}_n$. A function $f \in \mathcal{D}_n$ as above maps the unit disk onto the complement of n rays lying on straight lines through the origin: for $\theta_j < t < \theta_{j+1}$, $j = 1, \ldots, n$, the argument of $f(e^{it})$ is constant and to each $1 \le j \le n$ there is a $\phi_j \in (\theta_j, \theta_{j+1})$ such that $f'(e^{i\phi_j}) = 0$ and such that $t \mapsto |f(e^{it})|$ decreases from ∞ to $|f(e^{i\phi_j})|$ in (θ_j, ϕ_j) and increases from $|f(e^{i\phi_j})|$ to infinity in (ϕ_j, θ_{j+1}). In particular, we have

$$\arg\frac{e^{it}f'(e^{it})}{f(e^{it})} = \frac{\pi}{2} \qquad \text{and} \qquad \arg\frac{e^{it}f'(e^{it})}{f(e^{it})} = -\frac{\pi}{2} \tag{8}$$

for $t \in (\theta_j, \phi_j)$ and $t \in (\phi_j, \theta_{j+1})$, respectively. All this is not hard to see if we observe that for $f \in \mathcal{D}_n$ and $r(t) := ie^{it}f'(e^{it})/f(e^{it})$ we have

$$r(t) = -\sum_{j=1}^{n} \frac{\alpha_j}{2} \cot \frac{t-\theta_j}{2}, \qquad \text{and hence} \qquad r'(t) = \sum_{j=1}^{n} \frac{\alpha_j}{4\sin^2 \frac{t-\theta_j}{2}} > 0$$

and

$$f(e^{it}) = -e^{i\left(\sum_{j=1}^{n} \theta_j \alpha_j\right)/2} \prod_{j=1}^{n} \left(2 \sin \frac{\theta_j - t}{2}\right)^{-\alpha_j}$$

in $\theta_1 < t < \theta_{n+1}$, $t \neq \theta_j$.

The set $\mathcal{D}$ is dense in $\mathcal{S}^*$ with respect to the compact open topology of the class $\mathcal{A}$. This follows from the fact that $f \in \mathcal{S}^*$ if and only if $f/z \in K(0,2)$ [11, p. 46] and from the fact that $K(0,2)$ is the closure of $\mathcal{D}/z$ [11, p. 32]. It will therefore suffice to prove Theorem 2 for functions in $\mathcal{D}$. Further, since $\operatorname{Im} I_f \in h^\infty$ for $f \in \mathcal{S}^*$ (as seen in Lemma 4), we have

$$\operatorname{Im} I_f(z) = \frac{1}{2\pi} \int_0^{2\pi} P(e^{it}, z) \operatorname{Im} I_f(e^{it})\, dt, \qquad z \in \mathbb{D},$$

where $P(e^{it}, z) = \operatorname{Re}(e^{it}+z)/(e^{it}-z)$ is the Poisson kernel (cf. [1, Ch. 6]). It will thus be enough to show the following statement in order to prove Theorem 2.

Lemma 8. *For $f \in \mathcal{D}$ as in* (7) *we have $|\operatorname{Im} I_f(e^{it})| < \pi/2$ for all $t \in (\theta_1, \theta_{n+1})$, $t \neq \theta_j$.*

First we will show that this is true in the limit case where t tends to θ_j.

To that end, for $\theta \in (\theta_j, \theta_{j+1})$ set $\gamma_\theta(t) = f^{-1}(tf(e^{i\theta}))$, $0 \le t \le 1$, where the branch of f^{-1} is chosen such that $\gamma_\theta(1) = e^{i\theta}$. Then $\gamma_\theta(0) = 0$ and

$$\gamma_\theta'(t) f'(\gamma_\theta(t)) = f(e^{i\theta}) \qquad \text{for } 0 \le t \le 1. \tag{9}$$

For $\theta, \theta^* \in \mathbb{R}$ with $\theta, \theta^* \in (\theta_j, \phi_j)$ or $\theta, \theta^* \in (\phi_j, \theta_{j+1})$ set

$$\gamma_{\theta,\theta^*}(t) = f^{-1}\left(f(e^{i\theta}) + t(f(e^{i\theta^*}) - f(e^{i\theta}))\right), \qquad 0 \le t \le 1,$$

where the branch of f^{-1} is chosen such that $\gamma_{\theta,\theta^*}(0) = e^{i\theta}$. Then $\gamma_{\theta,\theta^*}(1) = e^{i\theta^*}$ and

$$\gamma_{\theta,\theta^*}'(t) f'(\gamma_{\theta,\theta^*}(t)) = f(e^{i\theta^*}) - f(e^{i\theta}) \qquad \text{for } 0 \le t \le 1. \tag{10}$$

Lemma 9. *For $f \in \mathcal{D}_n$ as in* (7) *and $j = 1, \ldots, n$ we have*

$$\lim_{\theta \to \theta_j^+} \operatorname{Im} I_f(e^{i\theta}) = \frac{\pi}{2} \qquad \textit{and} \qquad \lim_{\theta \to \theta_{j+1}^-} \operatorname{Im} I_f(e^{i\theta}) = -\frac{\pi}{2}.$$

In particular, the bound $\pi/2$ in Theorem 2 is sharp.

Proof. Let $\theta_j < \theta < \theta^* < \phi_j$. Applying the relations (9) and (10) and Cauchy's Integral Theorem and using also the mapping properties of functions in $\mathcal{D}_n$ as described at the beginning of this section, we get

$$\begin{aligned}\operatorname{Im} I_f(e^{i\theta}) &= \operatorname{Im} \int_0^{e^{i\theta^*}} \frac{f'(\zeta)}{f(e^{i\theta})} \log \frac{e^{i\theta} f'(\zeta)}{f(e^{i\theta})}\, d\zeta + \operatorname{Im} \int_{e^{i\theta^*}}^{e^{i\theta}} \frac{f'(\zeta)}{f(e^{i\theta})} \log \frac{e^{i\theta} f'(\zeta)}{f(e^{i\theta})}\, d\zeta \\ &= \frac{f(e^{i\theta^*})}{f(e^{i\theta})} \int_0^1 \arg \frac{e^{i\theta}}{\gamma'_{\theta^*}(t)}\, dt + \left(1 - \frac{f(e^{i\theta^*})}{f(e^{i\theta})}\right) \int_0^1 \arg \frac{e^{i\theta}}{\gamma'_{\theta^*,\theta}(t)}\, dt \\ &= \frac{f(e^{i\theta^*})}{f(e^{i\theta})} \int_0^1 \arg \frac{\gamma'_{\theta^*,\theta}(t)}{\gamma'_{\theta^*}(t)}\, dt + \int_0^1 \arg \frac{e^{i\theta}}{\gamma'_{\theta^*,\theta}(t)}\, dt.\end{aligned}$$

Letting $\theta \to \theta_j$ on both sides of this equation, we obtain

$$\lim_{\theta \to \theta_j^+} \operatorname{Im} I_f(e^{i\theta}) = \lim_{\theta \to \theta_j^+} \int_0^1 \arg \frac{e^{i\theta}}{\gamma'_{\theta^*,\theta}(t)}\, dt \tag{11}$$

for all $\theta_j < \theta^* < \phi_j$. For $\theta_j < \theta < \theta^* < \phi_j$ the curve $\gamma_{\theta^*,\theta}$ is one-to-one and maps into the unit circle. Also, $\gamma_{\theta^*,\theta}(0) = e^{i\theta^*}$, $\gamma_{\theta^*,\theta}(1) = e^{i\theta}$ and $\arg \gamma_{\theta^*,\theta}$ is decreasing. Hence, $\theta - \pi/2 \le \arg \gamma'_{\theta^*,\theta}(t) \le \theta^* - \pi/2$ for $0 \le t \le 1$, and thus, because of (11),

$$\theta_j - \theta^* + \frac{\pi}{2} \le \lim_{\theta \to \theta_j^+} \operatorname{Im} I_f(e^{i\theta}) \le \frac{\pi}{2}.$$

Since this holds for all $\theta_j < \theta^* < \phi_j$, we obtain the desired result in the case that $\theta \to \theta_j^+$.

The second asserted relation can be proved in a similar way. ■

We are now ready to give the proof of Lemma 8.

Proof of Lemma 8. Suppose Lemma 8 is wrong. Then, because of Lemma 9, there is a $w = e^{is} \in \mathbb{T}$, $s \ne \theta_j$, such that, putting $a + ib := I_f(w)$, $a, b \in \mathbb{R}$, we have

$$|b| = |\operatorname{Im} I_f(w)| = \max_{z \in \overline{\mathbb{D}}} |\operatorname{Im} I_f(z)| \ge \frac{\pi}{2} \tag{12}$$

and consequently also

$$\operatorname{Im} iw I'_f(w) = \frac{d}{dt} \operatorname{Im} I_f(e^{it}) \Big|_{t=s} = 0. \tag{13}$$

By Lemma 6 we have $f'(w) \ne 0$ and therefore I_f has an analytic extension to a neighborhood of w.

Set $r := iwf'(w)/f(w)$ and $\varphi := \arg(-ir)$. Then, due to (8), $r \in \mathbb{R}$ and $|\varphi| = \pi/2$. Further, it follows from Lemma 6 and (12) that $\varphi b < 0$ and thus

$|b-\varphi| = \pi/2+|b|$. Taking the imaginary part of (5), we get $0 = 1+r(\varphi-b)$ or $r = 1/(b-\varphi)$ and this, together with (5) and (13), yields

$$\begin{aligned} \operatorname{sgn}(b) iwI_f'(w) &= \operatorname{sgn}(b) \operatorname{Re} iwI_f'(w) \\ &= \operatorname{sgn}(b) r \left(\log\left|\frac{wf'(w)}{f(w)}\right| - 1 - \operatorname{Re} I_f(w) \right) \\ &= -\frac{1}{|b|+\frac{\pi}{2}} \left(\log\left(|b|+\frac{\pi}{2}\right) + 1 + a \right). \end{aligned} \tag{14}$$

Observe that $z \mapsto \int_0^z f'(\zeta)\log f'(\zeta)\,d\zeta$ has at least a double zero at the origin and that $h(z) := f(z)/z$ is analytic at 0 with $h(0) = f'(0) = 1$; hence, we see from the representation (4) that $I_f(0) = 0$. Because of this and (12) it follows from the mapping properties of the tangent (cf. [8, p. 277]) that $F(z) := \tan(i\pi I_f(z)/(4b))$, $z \in \mathbb{D}$, is an analytic function in $\mathbb{D}$ with $F(0) = 0$ that has an analytic extension to a neighborhood N of w and satisfies $|F(z)| \le |F(w)| = 1$ for all $z \in \mathbb{D} \cup (\mathbb{T} \cap N)$. The Clunie-Jack Lemma yields that

$$1 \le \frac{wF'(w)}{F(w)} = \frac{i\pi wI_f'(w)}{2b \sin\frac{i\pi I_f(w)}{2b}}. \tag{15}$$

Since

$$\sin\frac{i\pi I_f(w)}{2b} = \sin\left(\frac{i\pi a}{2b} - \frac{\pi}{2}\right) = -\cosh\frac{\pi a}{2b},$$

this and (14) give

$$1 \le -\frac{\operatorname{sgn}(b) i\pi wI_f'(w)}{2|b|\cosh\frac{\pi a}{2|b|}} = \frac{\pi(\log(|b|+\frac{\pi}{2})+1+a)}{2|b|(|b|+\frac{\pi}{2})\cosh\frac{\pi a}{2|b|}}.$$

But this is impossible for $a \in \mathbb{R}$ and $|b| \ge \pi/2$ by Lemma 3.

This completes the proof of Lemma 8 and hence Theorem 2. ■

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