Ranking of Alternatives with Ordered Weighted Averaging Operators

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Multiattribute decision making is an important part of the decision process for both individual and group problems. We incorporate the fuzzy set theory and the basic nature of subjectivity due to ambiguity to achieve a flexible decision approach suitable for uncertain and fuzzy environments. Let us consider the analytic hierarchy process (AHP) in which the labels are structured as fuzzy numbers. To obtain the scoring that corresponds to the best alternative or the ranking of the alternatives, we need to use a total order for the fuzzy numbers involved in the problem. In this article, we consider a definition of such a total order, which is based on two subjective aspects: the degree of optimism/pessimism reflected with the ordered weighted averaging (OWA) operators. A numerical example is given to illustrate the approach. © 2004 Wiley Periodicals, Inc.

1. INTRODUCTION

The aim of this article is the solution of a hierarchy multicriteria decision making (MCDM), where the preferences are stated by fuzzy number.

Since its foundation in 1965 by Zadeh,[1] the theory of fuzzy sets has experienced great growth and obtained great importance. This fact is because of, fundamentally, the ability of the fuzzy sets to express ambiguity and vagueness that are inherent in the human language. Such characteristics are present in the processes of decision making, which appear in multiple real and practical situations. All this has given rise to the extensive development of the decision theory in a fuzzy environment.

A very important contribution to the use of fuzzy sets in decision problems was made by Bellman and Zadeh,[2] who considered how the fuzzy sets can be used to describe the constraints on the alternatives as well as the objectives in certain decision problems. The corresponding solution then is given by the fuzzy set obtained as the "confluence" of constraints and objectives. These ideas have been used as a starting point in many other researches.

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In many situations, we use measures or quantities that are not exact but approximate. In those cases, the concept of fuzzy number is more adequate than that of real number. Based on the usual operations with real numbers, the Zadeh's extension principle gives rise to an arithmetic with fuzzy numbers that is commonly used.

The analytic hierarchy process (AHP) proposed by Saaty[3,4] is a very popular approach of MCDM that involves qualitative data and has been applied during the last 20 years in many situations of decision making. The AHP has received wide applications in a number of different fields. The method uses a reciprocal matrix of decision obtained by the method of pair comparisons (MPC) such that the information is given in a linguistic form. The eigenvector is used in order to obtain the importance of the criteria/alternatives and the additive weighting method is used to calculate the utility for each alternative across all criteria.

It also was suggested that with the aid of the ordered weighted averaging (OWA) operators we can provide a unification of types of decision making attitudes used in solving these problems.

In this article, we consider the tools that are needed to solve the AHP in which the preferences are expressed by means of linguistic labels, in which their values are stated as triangular Zadeh's numbers.

2. THE AHP: THE MODEL OF SAATY

In the MPC, criteria and alternatives are presented in pairs to one or more experts or decision makers. It is necessary to evaluate individual alternatives, derive weights for the criteria, construct the overall rating on the alternatives, and identify the best alternative.

Let us denote the alternatives by $A_1, A_2, \ldots, A_n$ (n is the number of compared alternatives), their actual weights by $w_1, w_2, \ldots, w_n$, and the matrix of the ratios of all weights by $\mathbf{W} = [w_i/w_j]$. The matrix of pairwise comparisons $\mathbf{A} = [a_{ij}]$ represents the intensities of the decision maker's preference between individual pairs of alternatives (A_i versus A_j, for all $i, j = 1, 2, n$ chosen usually from a given scale represented in Table I).

$$\mathbf{A} = \begin{pmatrix} 1 & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ 1/a_{12} & 1 & \cdots & a_{2j} & \cdots & a_{2j} \\ \vdots & & & & & \\ 1/a_{1n} & 1/a_{2n} & \cdots & 1/a_{jn} & \cdots & 1 \end{pmatrix}$$

Table I. Table of labels associated with Saaty's model.

Definition	Saaty's scale	Triangular number
A_i and A_j are equally important	1	[1, 1, 1]
A_i is moderately more important than A_j	3	[2, 3, 4]
A_i is strongly more important than A_j	5	[2, 3, 4]
A_i is very strongly more important than A_j	7	[6, 7, 8]
A_i is extremely more important than A_j	9	[8, 9, 9]
The scales 2, 4, 6, and 8 also are used and represent compromises among the tabulated scale	2, 4, 6, and 8	in the same way

The element a_{ij} is considered to be the estimate of the ratios w_i/w_j, where $\mathbf{w}$ is the vector of actual weights of the alternatives, which is what we want to find. All of the ratios are positive and satisfy the reciprocity property: $a_{ij} = 1/a_{ji}$ $(i, j = 1, 2, \ldots, n)$.

The next step consists of the computation of the vector of priorities obtained from the given matrix.[3] In mathematical terms, the principal eigenvector $\mathbf{Aw} = \lambda\mathbf{w}$ is computed and when normalized becomes the vector of priorities; but the eigenvector is possible to calculate whenever we dispose of a computer. In the absence of a computer, to solve the problem, it is necessary to apply some estimates as.

1. *The crudest.* Sum the elements in each row and normalize by dividing each sum by the total of the all of the sums; thus, the results now add up to unity. The first entry of the resulting vector is the priority of the priority of the first activity, the second of the second activity, etc.
2. *The better.* Take the sum of the elements in each column and form the reciprocals of these sums. To normalize so that these numbers add to unity, divide each reciprocal by the sum of the reciprocals.
3. *Good.* Divide the elements of each column by the sum of that column (normalize the column) and then add the elements in each resulting row and divide this sum by the number of elements in the row. This is a process of averaging over the normalized columns.
4. *Good.* Multiply the n elements in each row and take the nth root. Normalize the resulting numbers

$$w_i = \left(\prod_{i=1}^{n} a_{ij}\right)^{1/n}, \quad i = 1, 2, \ldots, n$$

If the matrix is consistent, the eigenvector and the solution of all of these four vectors would be the same. Therefore, without generalized loss, we will use the last estimator of the weighted vector.

At this moment, we suppose that the decision maker is consistent and therefore we don't consider the consistency index and related terms.

Based on these comparisons, the AHP computes the importance of criteria, the weights of criteria levels, and, finally, the preferences for alternatives.

3. TRIANGULAR FUZZY NUMBERS

DEFINITION 1.1. *We give the term triangular fuzzy number as determined by two linear functions a, b: $[0, 1] \to R$, in which its graphs describe a triangle (Figure 1). More precisely, for any three real numbers $A \leqslant M \leqslant B$, we have the triangular number $\tau_{(A,M,B)} = [a(\alpha), b(\alpha)]$ determined by the following functions:*

$$a(\alpha) = A + \alpha(M - A); \quad b(\alpha) = B - \alpha(B - M)$$

where

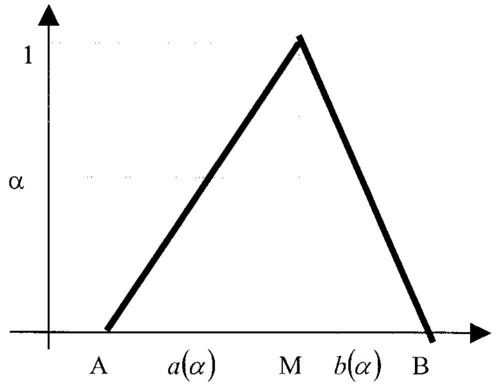

Figure 1. Representation of a triangular fuzzy number.

$a(\alpha)$ is continuous and strictly increasing on [A, M]
$b(\alpha)$ is continuous and strictly decreasing on [M, B]

Let us consider the label that corresponds with the expression "A_i is moderately more important than A_j" with an associated value of 3. *The corresponding triangular graded number, if the label is symmetric with amplitude* 1, *is the following*:

$$a(\alpha) = 2 + \alpha, \quad b(\alpha) = 4 - \alpha$$

In particular, we have

$$\begin{cases} \alpha = 0 \ \rightarrow [2, 4] \\ \alpha = 1 \ \rightarrow [3, 3] \\ \tau_{(A,M,B)} = [2, 3, 4] \end{cases}$$

3.1. Operations

For any positive graded numbers $\psi(\alpha) = [a(\alpha), b(\alpha)]$, $\psi_1(\alpha) = [a_1(\alpha), b_1(\alpha)]$, and $\psi_2(\alpha) = [a_2(\alpha), b_2(\alpha)]$ and any real number P (in our case $P > 0$), we are interested in the operations related to AHP problems, which are defined as follows:

a. *Product by a scalar*

$$(P\psi)(\alpha) := \{Px : x \in \psi(\alpha)\} = \begin{cases} [Pa(\alpha), Pb(\alpha)], & \text{if } P \geqslant 0 \\ [Pb(\alpha), Pa(\alpha)], & \text{if } P \leqslant 0 \end{cases}$$

b. *Inverse*

$$(\psi)^{-1}(\alpha) = \{1/x : x \in \psi(\alpha)\} = [1/a_2(\alpha), 1/a_1(\alpha)]$$

c. *Addition*

$$(\psi_1 + \psi_2)(\alpha) = \{x + y : x \in \psi_1(\alpha),\ y \in \psi_2(\alpha)\}$$
$$= [a_1(\alpha) + a_2(\alpha),\ b_1(\alpha) + b_2(\alpha)]$$

d. *Product*

$$(\psi_1 \circ \psi_2)(\alpha) = \{x \circ y : x \in \psi_1(\alpha),\ y \in \psi_2(\alpha)\} = [a_1(\alpha) \circ a_2(\alpha),\ b_1(\alpha) \circ b_2(\alpha)]$$

e. *Division*

$$(\psi_1/\psi_2)(\alpha) = \{x/y : x \in \psi_1(\alpha),\ y \in \psi_2(\alpha)\} = [a_1(\alpha)/b_2(\alpha),\ b_1(\alpha)/a_2(\alpha)]$$

f. *Root*

$$(\psi)^{1/n}(\alpha) = \{x^{1/n} : x \in \psi(\alpha)\} = [a_1(\alpha)^{1/n},\ a_2(\alpha)^{1/n}]$$

4. THE OWA OPERATOR

In a fuzzy environment, ranking fuzzy numbers is a very important decision-making procedure. Some of these ranking methods have been compared and reviewed by Bortolan and Degani,[5] and, more recently, by Chen and Hwang.[6] But we are interested in:

DEFINITION 1.[7,8] *An OWA operator of dimension n is a function*

$$F : R^n \rightarrow R$$

that has an associated n vector **W**

$$\mathbf{W} = [w_1, w_2, \ldots, w_n]^T$$

such that

1. $w_i \in [0, 1]$
2. $\Sigma_i w_i = 1$

Furthermore,

$$F(a_1, a_2, \ldots, a_n) = \sum_j w_j b_j \tag{1}$$

where b_j is the jth *largest of the a_i.*

A fundamental aspect of these operators is the reordering process, which associates the arguments with the weights. If we represent the ordered arguments b_j by a vector **B**, called ordered argument vector, we can express Equation 1 as

$$F_{\mathbf{W}}(a_1, a_2, \ldots, a_n) = \mathbf{W} \cdot \mathbf{B}$$

In particular, a weight $\mathbf{w}_i$ is not associated with a specific argument but with an ordered position of the aggregate. This ordering operation essentially provides a non-linear aspect to this aggregation operation.

A number of properties can be associated with this operator. It is first noted that the OWA aggregation is *commutative,* i.e., the aggregation is indifferent to the initial indexing of the argument. A second characteristic associated with these operators is *monotonic.* Thus, if $\hat{a}_i \geqslant a_i$ for all i, then

$$F(\hat{a}_1, \hat{a}_2, \ldots, \hat{a}_n) \geqslant f(a_1, a_2, \ldots, a_n)$$

Another characteristic associated with these operators is that of *idempotency.* In particular, if $a_i = a$ for all i, then

$$F(a_1, a_2, \ldots, a_n) = a$$

The satisfaction of these three conditions, as noted by Dubois and Prade,[9] assures these operators of being in the class of operators called mean operators. It also can be shown, from the fact that an OWA operator is a mean operator, that the OWA aggregation is bounded by the Min and Max of the arguments. For any OWA aggregation F

$$\mathrm{Min}_i[a_i] \leqslant F(a_1, a_2, \ldots, a_n) \leqslant \mathrm{Max}_i[a_i]$$

If we take into account the decision maker's attitude, we obtain the well-known weighted vectors.[10]

Pessimistic attitude. If we select $\mathbf{W}$, where $\mathbf{W} = \mathbf{W}_* = [0, 0, \ldots, 1]^T$, this is the aggregation rule used in the pessimistic strategy.
Optimistic attitude. If we select $\mathbf{W}$, where $\mathbf{W} = \mathbf{W}^* = [1, 0, \ldots, 0]^T$, this is used in the optimistic strategy.
Normative approach. If we select $\mathbf{W}$, where $\mathbf{W} = \mathbf{W}_{\mathrm{AVG}} = [1/n, 1/n, \ldots, 1/n]^T$, essentially, this function is the normative strategy.

A second measure introduced by Yager[10,11] is called the dispersion (or entropy) of $\mathbf{W}$ and is defined as

$$H(\mathbf{W}) = -\sum w_i \ln w_i$$

It was shown that this helps measure the degree to which $\mathbf{W}$ takes into account all of the information in the aggregation. O'Hagan[12] suggested a methodology for obtaining the OWA weighting vector based on the use of these characterizing measures. This approach, which only requires the specification of just the α-value, generates a class of OWA weights that are called Maximum Entropy Operator Weighted Averaging (ME-OWA) weights.

The determination of these weights $w_l, \ldots, w_n$, from a degree of optimism α given by the decision maker requires the solution of the following mathematical programming problem:

$$\text{maximize:} \quad H(\mathbf{W}) = -\sum w_i \ln w_i$$

subject to

1. $\alpha = [1/(n - 1)] \sum_{i=1}^{n} ((n - i)w_i)$
2. $\sum_i w_i = 1$
3. $w_i \in [0, 1]$

In this mathematical programming formulation, Restriction 1 is the imposition of the condition that the desired α-value is attained. Constraints 2 and 3 just assure us that the weights satisfy the basic requirements of the OWA weights. The objective function used in this approach is one of trying to maximize the dispersion or entropy, which calculate the weights to be the ones that use as much information as possible in the aggregation.

In addition, we assume the decision-making attitude is captured by a degree of optimism of $\alpha = 0.75$,

$$\mathbf{w}_1 = [1]$$

$$\mathbf{w}_3 = [0.62, 0.27, 0.11]$$

$$\mathbf{w}_5 = [0.46, 0.26, 0.15, 0.08, 0.05]$$

A second measure introduced by Yager,[10] which we call the orness measure is defined as

$$\text{orness}(\mathbf{W}) = \frac{1}{n-1} \sum_{i=1}^{n} ((n-1)w_i)$$

It can be shown easily that

1. orness $(\mathbf{W}^*) = 1$
2. orness $(\mathbf{W}_*) = 0$
3. orness $(\mathbf{W}_{\text{AVG}}) = 0.5$

In this case, for the same value of orness it is possible to obtain several associated vectors with components

$$\mathbf{W}_\alpha = \left[\frac{1-\alpha}{2}, \alpha, \frac{1+\alpha}{2}\right]$$

$$\text{orness}(\mathbf{W}) = \frac{1}{2}\left[2 \cdot \frac{1-\alpha}{2} + \alpha\right] = 0.5$$

A measure of *andness* can be defined as andness $(\mathbf{W}) = 1 -$ orness $(\mathbf{W})$.

5. CASE STUDY

School Selection Example (Saaty[3])

Three high schools A, B, and C, were analyzed from the standpoint of the author's son according to their desirability (Figure 2). Six independent characteristics

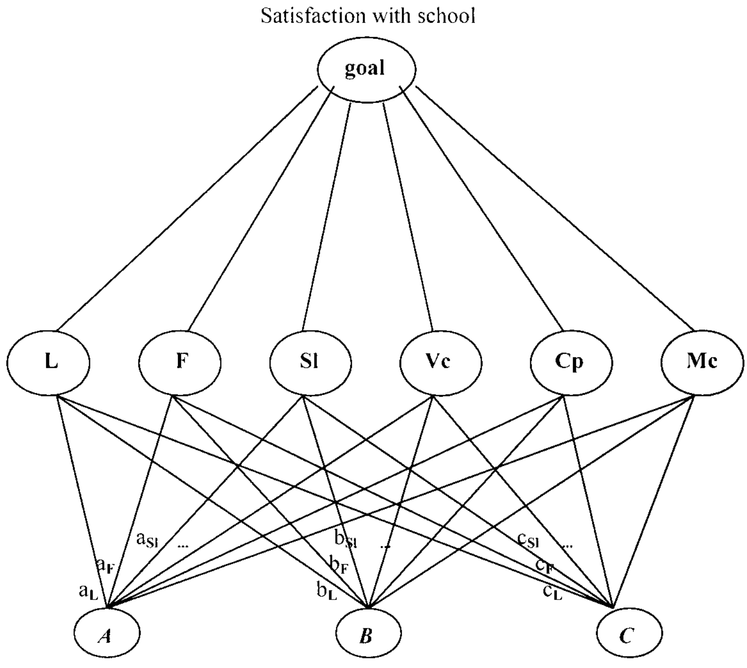

Figure 2. School satisfaction hierarchy.

were selected for comparison learning, friends, school life, vocational training, college preparation, and music classes. The pairwise judgment matrices are shown in Tables II and III.

The priority vector of this matrix is given in the last column.

To obtain the overall ranking of the schools, we multiply the transpose of the last column of the Table III by the last column of the Table II, taking into account the operations given in Section 3.1, and the final result is

$$A = [0.1808, 0.3701, 0.7287]$$

$$B = [0.1621, 0.3760, 0.8379]$$

$$C = [0.1238, 0.2443, 0.5196]$$

Table II. Comparison of characteristic with respect to overall satisfaction with school.

	L	F	Sl	Vt	Cp	Mc	Solution
L	1, 1, 1	3, 4, 5	2, 3, 4	1, 1, 1	2, 3, 4	3, 4, 5	[0.2111, 0.3160, 0.4526]
F	1/5,1/4,1/3	1, 1, 1	6, 7, 8	2, 3, 4	1/6,1/5,1/4	1, 1, 1	[0.0997, 0.1392, 0.1963]
Sl	1/4,1/3,1/2	1/8,1/7,1/6	1, 1, 1	1/6,1/5,1/4	1/6,1/5,1/4	1/7,1/6,1/5	[0.0259, 0.0361, 0.0531]
Vt	1, 1, 1	1/4,1/3,1/2	4, 5, 6	1, 1, 1	1, 1, 1		[0.0922, 0.1251, 0.1784]
Cp	1/4,1/3,1/2	4, 5, 6	4, 5, 6	1, 1, 1	1, 1, 1	2, 3, 4	[0.1642, 0.2360, 0.3401]
Mc	1/5,1/4,1/3	1, 1, 1	5, 6, 7	2, 3, 4	1/4,1/3,1/2	1, 1, 1	[0.1034, 0.1477, 0.2155]

L, learning; F, friends; Sl, school life; Vt, vocational training; Cp, college preparation; Mc, music classes.

Table III. Comparison of schools with respect to the six characteristics.

	A	B	C	Solution
L				
A	1, 1, 1	1/4,1/3,1/2	1/3,1/2, 1	[0.0980, 0.1571, 0.2990]
B	2, 3, 4	1, 1, 1	2, 3, 4	[0.3561, 0.5936, 0.9494]
C	1, 2, 3	1/4,1/3,1/2	1, 1, 1	[0.1413, 0.2493, 0.4313]
F				
A	1, 1, 1	1, 1, 1	1, 1, 1	[0.3333, 0.3333, 0.3333]
B	1, 1, 1	1, 1, 1	1, 1, 1	[0.3333, 0.3333, 0.3333]
C	1, 1, 1	1, 1, 1	1, 1, 1	[0.3333, 0.3333, 0.3333]
Sl				
A	1, 1, 1	4, 5, 6	1, 1, 1	[0.3938, 0.4545, 0.5225]
B	1/6,1/5,1/4	1, 1, 1	1/6,1/5,1/4	[0.0751, 0.0910, 0.1141]
C	1, 1, 1	4, 5, 6	1, 1, 1	[0.1450, 0.1734, 0.2131]
Vt				
A	1, 1, 1	8, 9, 9	6, 7, 8	[0.6638, 0.7720, 0.8866]
B	1/9,1/9,1/8	1, 1, 1	1/6,1/5,1/4	[0.0483, 0.0545, 0.0671]
C	1/8,1/7,1/6	4, 5, 6	1, 1, 1	[0.1450, 0.1734, 0.2131]
Cp				
A	1, 1, 1	1/3,1/2, 1	1, 1, 1	[0.1699, 0.2500, 0.4190]
B	1, 2, 3	1, 1, 1	1, 2, 3	[0.2451, 0.5000, 0.8715]
C	1, 1, 1	1/3,1/2, 1	1, 1, 1	[0.1699, 0.2500, 0.4190]
Mc				
A	1, 1, 1	5, 6, 7	3, 4, 5	[0.5100, 0.6909, 0.9261]
B	1/7,1/6,1/5	1, 1, 1	1/4,1/3,1/2	[0.0681, 0.0914, 0.1314]
C	1/5,1/4,1/3	2, 3, 4	1, 1, 1	[0.1524, 0.2176, 0.3116]

It is possible to see that the alternative C is dominated for the other two alternatives, but there is no relation of dominance between A and B.

This is an possible explanation for the solution of Saaty. If we use the OWA operator,

- $\mathbf{W}^* = [1, 0, 0]$ and then A = 0.7287 is not preferred to B = 0.8379
- $\mathbf{W}_* = [0, 0, 1]$ and then A = 0.1808 is preferred to B = 0.1621
- $\mathbf{W}_{avg} = [0, 1, 0]$ and then A = 0.3701 is not preferred to B = 0.3760 and it coincides with the Saaty value
- $\mathbf{W}_{0.75} = [0.62, 0.27, 0.11]$ and then A = 0.5716 is not preferred to B = 0.6388
- $\mathbf{W}_{0.25} = [0.11, 0.27, 0.62]$ and then A = 0.2922 is not preferred, but is very close to B = 0.2942; in this case, it is possible to point out that the $\alpha = 0.25$ is the point of compromise for choosing A or B.

Other operators with other degrees of optimism are possible but this is not subject of this study.

6. CONCLUSIONS

We have incorporated the fuzzy set theory and the basic nature of subjectivity due to ambiguity to achieve a flexible decision approach suitable for uncertain and linguistic labels. We extend the basic concept of the AHP to the case in which the linguistic information is structured as a fuzzy number, and we prove that the central values of this resulting triangular fuzzy number coincide with that obtained in the classical AHP of Saaty's method.

This obtained final value is possible to achieve with the grade of optimism/pessimism that the decision maker brings into decision making. This is performed by means of the OWA operator, i.e., pointing to the three values of the triangular fuzzy number with the respective weights of the OWA operator.

Acknowledgments

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