Applied Mathematics and Computation 218 (2011) 912–918

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

A subclass of parabolic starlike and uniformly convex functions

Oh Sang Kwon

Department of Mathematics, Kyungsung University, Busan 608-736, Republic of Korea

ARTICLE INFO

Dedicated to Professor H. M. Srivastava on the Occasion of his Seventieth Birth Anniversary

Keywords:
Uniformly convex functions
Parabolic starlike functions
Salagean operator
Subordination
Fekete-Szegö problems

ABSTRACT

Let A be the class of analytic functions in the open unit disk U. A function f in A satisfying the normalization is said to be in the class SP_n if D^nf is a parabolic starlike function, where D^n is a notation of the Salagean operator. In this paper, several basic properties and characteristics of the class SP_n are investigated. These include subordination, convolution properties, class-preserving integral operators, and Fekete-Szegö problems.

1. Introduction

Let A be the class of functions analytic in the open unit disk

$$U := \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$$

and let A_0 be the family of functions f in A satisfying the normalization condition:

$$f(0) = f'(0) - 1 = 0.$$

A function f in A_0 is said to be uniformly convex in U if f is a univalent convex function along with the property that, for every circular arc γ contained in U, with center ζ also in U, the image curve $f(\gamma)$ is a convex arc. The class of uniformly convex functions is denoted by UCV. It is well known from [6,10] that

$$f \in UCV \Longleftrightarrow \left|\frac{zf''(z)}{f'(z)}\right| < Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\}, \quad (z \in U). \tag{1.1}$$

Condition (1.1) implies that $1 + \frac{zf''(z)}{f'(z)}$ lies in the interior of the parabolic region

$$R := \{w : w = u + iv \text{ and } v^2 < 2u - 1\}$$

for every value of $z \in U$.

A function $f \in A_0$ is said to be in the class of parabolic starlike functions, denoted by SP (cf. [10]), if

$$\frac{zf'(z)}{f(z)} \in R, \quad (z \in U).$$

And let P be the class of functions with positive real parts. Let

E-mail address: oskwon@ks.ac.kr

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doi:10.1016/j.amc.2011.02.009

$$\phi(a,c,z) := \sum_{k=0}^{\infty} \frac{(a)_k}{(c)_k} z^{k+1}, \quad (z \in U; c \neq 0, -1, -2, \ldots),$$

where $(\lambda)_n$ is the Pochhammer symbol defined, in terms of Gamma functions, by

$$(\lambda)_n := \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)} = \begin{cases} 1, & (n=0), \\ \lambda(\lambda+1)\cdots(\lambda+n-1), & (n \in \mathbb{N} := \{1,2,3,\ldots\}). \end{cases}$$

Further, let (cf. [2])

$$\mathcal{L}(a,c)f(z) = \phi(a,c;z) * f(z), \quad (f \in A),$$

in terms of the Hadamard product (or convolution). Note that $\mathcal{L}(a,a)$ is the identity operator and

$$\mathcal{L}(a,c) = \mathcal{L}(a,b)\mathcal{L}(b,c), \quad (b,c \neq 0,-1,-2,\ldots).$$

It is well known that, if $c > a > 0$, then $\mathcal{L}$ maps A into itself. Salagean introduced the following operator which is popularly known as the Salagean derivative operator (cf. [10]):

$$D^0 f(z) = f(z),$$
$$D^1 f(z) = Df(z) = zf'(z)$$

and, in general,

$$D^n f(z) = D(D^{n-1} f(z)) \quad (n \in \mathbb{N} = \{1,2,3,\ldots\}).$$

We can easily find from that

$$D^n f(z) = z + \sum_{k=2}^{\infty} k^n a_k z^k \quad (f \in A; n \in \mathbb{N}_0 := \{0,1,2,\ldots\}).$$

Suppose that the functions f and F are in A. We say that f is subordinate to F in U, written as $f \prec F$, if F is univalent in U,

$$f(0) = F(0) \quad \text{and} \quad f(U) \subset F(U).$$

Definition 1.1. Let SP_n $(n \in \mathbb{N}_0)$ be the class of functions $f \in A_0$ satisfying the inequality:

$$\left| \frac{z(D^n f(z))'}{D^n f(z)} - 1 \right| < \mathrm{Re}\left\{ \frac{z(D^n f(z))'}{D^n f(z)} \right\}, \quad (z \in U). \tag{1.2}$$

It follows that

$$SP_1 = UCV \quad \text{and} \quad SP_0 = SP.$$

Definition 1.2. Let $S_n(\frac{1}{2})(n \in \mathbb{N}_0)$ be the class of functions $f \in A_0$ satisfies as following;

$$\mathrm{Re}\left\{ \frac{z(D^n f(z))'}{D^n f(z)} \right\} > \frac{1}{2}.$$

2. Basic properties of the class SP_n

We need the following results in our investigation of the class SP_n.

Lemma 2.1 (*see [8]*). *Let F and G be univalent convex functions in U. Then the Hadamard product F∗G is also univalent convex in U.*

Lemma 2.2 (*see [9]*). *Let the functions F and G be univalent convex in U. Also let $f \prec F$ and $g \prec G$. Then $f*g \prec F*G$.*

Lemma 2.3 (*see [8]*). *Let each of the functions f and g be univalent starlike of order 1/2. Then, for every function $F \in A$,*

$$\frac{f(z) * g(z)F(z)}{f(z) * g(z)} \in \overline{\mathcal{CH}}\{F(U)\}, \quad (z \in U),$$

where $\overline{\mathcal{CH}}$ denotes the closed convex hull.

It can be verified that the Riemann map q of U onto the region R, satisfying $q(0) = 1$ and $q'(0) > 0$, is given by

$$q(z) = 1 + \frac{2}{\pi^2}\left(\log\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)^2 = 1 + \frac{8}{\pi^2}\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=0}^{n-1}\frac{1}{2k+1}\right)z^n = \sum_{n=1}^{\infty}B_n z^n = 1 + \frac{8}{\pi^2}\left(z + \frac{2}{3}z^2 + \frac{23}{45}z^3 + \frac{44}{105}z^4 + \cdots\right), \quad (z \in U). \tag{2.1}$$

We define the function G by

$$G(z) = \frac{1}{z}\left[h(z) * z\, exp\left(\int_o^z \frac{q(s)-1}{s}ds\right)\right], \tag{2.2}$$

where $h(z) = z + \sum_{n=2}^{\infty}\frac{1}{k^n}z^k$.

Theorem 2.4. *Let $n \in \mathbb{N}_0$ and let $G(z)$ be defined by (2.2). Then $G(z)$ is a convex univalent function. Furthermore, if $f \in SP_n$, then*

$$\frac{f(z)}{z} \prec G(z).$$

Proof. We first note that

$$G(z) = \frac{h(z)}{z} * exp\left(\int_o^z \frac{q(s)-1}{s}ds\right), \quad (z \in U), \tag{2.3}$$

where each member of the Hadamard product in (2.3) is known to be convex univalent function (cf. [5]). Therefore, by Lemma 2.1, $G(z)$ is a univalent convex function.

Next, if $f \in SP_n$, then

$$\frac{z(D^n f(z))'}{D^n f(z)} \prec q(z).$$

Thus there exists a function ψ satisfying the Schwarz Lemma such that

$$\frac{D^n f(z)}{z} = exp\left(\int_o^z \frac{q(\psi(s))-1}{s}ds\right), \quad (z \in U).$$

Since $q(z) - 1$ is a univalent convex function, a result of [4] yields

$$\frac{D^n f(z)}{z} \prec exp\left(\int_o^z \frac{q(s)-1}{s}ds\right), \quad (z \in U).$$

We can deduce the following result easily,

$$\frac{f(z)}{z} \prec G(z).$$

The proof Theorem 2.4 is evidently completed. □

Theorem 2.5. *Let $n \in \mathbb{N}_0$. If $f \in SP_n$, then*

$$G(-r) \leqslant \left|\frac{f(z)}{z}\right| \leqslant G(r), \quad (|z| = r) \tag{2.4}$$

and

$$\left|arg\left(\frac{f(z)}{z}\right)\right| \leqslant max_{\theta\in[0,2\pi]}\{arg(G(re^{i\theta}))\}, \quad (z = re^{i\theta}), \tag{2.5}$$

where $G(z)$ is defined by (2.2). Equality holds true in (2.4) and (2.5) for some $z \neq 0$ if and only if f is a rotation of $zG(z)$.

Proof. Let $f \in SP_n$. Then, by Theorem 2.4 and Lindelöf's principle of subordination, we get

$$inf_{|z|\leqslant r}Re\{G(z)\} \leqslant inf_{|z|\leqslant r}Re\left\{\frac{f(z)}{z}\right\} \leqslant sup_{|z|\leqslant r}Re\left\{\frac{f(z)}{z}\right\} \leqslant sup_{|z|\leqslant r}\left|\frac{f(z)}{z}\right| \leqslant sup_{|z|\leqslant r}Re\{G(z)\}. \tag{2.6}$$

Since $G(z)$ is a univalent convex function and has real coefficients, $G(U)$ is a convex region symmetric with respect to real axis. Hence,

$$inf_{|z|\leqslant r}Re\{G(z)\} = inf_{-r\leqslant x\leqslant r}G(z) = G(-r) \tag{2.7}$$

and

$$sup_{|z|\leqslant r} Re\{G(z)\} = sup_{-r\leqslant x\leqslant r} G(z) = G(r). \tag{2.8}$$

Thus, (2.6) gives the assertion (2.4) of Theorem 2.5. Also we readily have the assertion (2.5) of Theorem 2.5. And the sharpness in inequalities (2.4) and (2.5) is also a consequence of the principle of subordination. This completes the proof of Theorem 2.5. □

Theorem 2.6. *Let m and n be nonnegative integers with* $m \geqslant n$. *If* $f \in S_n(\frac{1}{2})$ *and* $g \in SP_m$, *then*

$$D^n f * D^m g \in SP_m.$$

In particular, if $f \in S_n(\frac{1}{2})$ *and* $g \in SP_n$, *then*

$$D^n f * D^n g \in SP_n.$$

Proof. Let $f \in S_n(\frac{1}{2})$ and $g \in SP_m$. Then

$$D^n f \in S^*\left(\frac{1}{2}\right) \quad \text{and} \quad D^m g \in SP \subset S^*\left(\frac{1}{2}\right).$$

The commutative and associative properties of the Hadamard product yield

$$z(D^n f * D^m g)' = D^n f * \{z(D^m g)'\}.$$

Therefore, using Lemma 2.3, we get

$$\frac{z(D^n f * D^m g)'}{D^n f * D^m g} = \frac{D^n f * \frac{z(D^m g)'}{D^m g} D^m g}{D^n f * D^m g} \subset R. \tag{2.9}$$

This completes the proof of Theorem 2.6. □

Corollary 2.1. (see [7]) *If* $f \in S^*(\frac{1}{2})$ *and* $g \in SP$, *then* $f*g \in SP$. *In particular, if* $f \in SP$ *and* $g \in SP$, *then* $f*g \in SP$.

Theorem 2.7. *Let* $f \in SP_n$. *Then the function F(z) defined by the integral transformation*

$$F(z) = \frac{c+1}{z^c}\int_0^z t^{c-1} f(t)dt \quad (z \in U, c > -1).$$

is also in the class SP_n.

Proof. We begin by noting that

$$F(z) = \mathcal{L}(c+1, c+2) f(z).$$

Let $h(z) = z + \sum_{k=2}^{\infty} \frac{1}{k^n} z^k$. Then

$$D^n F(z) = h(z) * F(z) = h(z) * \mathcal{L}(c+1, c+2) f(z)$$

and

$$z(D^n F(z))' = z(h(z) * \mathcal{L}(c+1, c+2) f(z))' = \mathcal{L}(c+1, c+2)(z(h(z) * f(z))') = \phi(c+1, c+2; z) * z(D^n f)'.$$

Using a result of Bernardi [1], it can be verified that

$$\phi(c+1, c+2; z) \in S^*\left(\frac{1}{2}\right).$$

Also, by hypothesis, $D^n f(z) \in SP \subset S^*(\frac{1}{2})$. Thus, using Lemma 2.3, we get

$$\frac{z(D^n F(z))'}{D^n F(z)} = \frac{\phi(c+1, c+2; z) * \frac{z(D^n f(z))'}{D^n f(z)} D^n f(z)}{\phi(c+1, c+2; z) * D^n f(z)} \in R,$$

which completes the proof of Theorem 2.7. □

3. The Fekete-Szegö problem for the Class SP_n

Let the function *f*, given by

$$f(z) = z + a_2z^2 + a_3z^3 + \cdots, \quad (z \in U), \tag{3.1}$$

be in the class SP_n. Then there exists a function $w \in A$, satisfyting

$$w(0) = 0 \quad \text{and} \quad |w(z)| < 1, \quad (z \in U),$$

such that

$$\frac{z(D^nf(z))'}{D^nf(z)} = q(w(z)), \quad (z \in U). \tag{3.2}$$

Let the function $p_1 \in P$ be defined by

$$p_1(z) = \frac{1 + w(z)}{1 - w(z)} = 1 + c_1z + c_2z^2 + \cdots. \tag{3.3}$$

Then, by using (2.1), (4.2) and (4.3) in the form

$$w(z) = \frac{p_1(z) - 1}{p_1(z) + 1},$$

we find that

$$a_2 = \frac{2^{2-n}}{\pi^2}c_1, \tag{3.4}$$

$$a_3 = \frac{2}{3^n\pi^2}\left(c_2 - \frac{1}{6}\left(1 - \frac{24}{\pi^2}\right)c_1^2\right), \tag{3.5}$$

$$a_4 = \frac{4^{1-n}}{3\pi^2}\left[c_3 - \frac{1}{3}\left(1 - \frac{18}{\pi^2}\right)c_1c_2 + \frac{2}{45}\left(1 - \frac{45}{2\pi^2} + \frac{180}{\pi^4}\right)c_1^3\right]. \tag{3.6}$$

These expressions shall be used throughout the rest of the paper.

Define the function $k(z, \rho, v)$ in SP_n by

$$k(z, \rho, v) = h(z) * z\ exp\left(\int_0^z \left[q\left(\frac{e^{i\rho}\zeta(\zeta + v)}{1 + v\zeta}\right) - 1\right]\frac{d\zeta}{\zeta}\right), \quad (0 \leqslant \rho \leqslant 2\pi, 0 \leqslant v \leqslant 1), \tag{3.7}$$

where $h(z)$ is defined by (2.2) ([11]).

Note that $k(z,0,1) = zG(z)$ defined by (2.2) and that $k(z,\rho,0)$ is an odd function. We also need the following Lemma in our investigation: (see [3])

Lemma 3.1. *Let $g \in P$, where*

$$g(z) = 1 + c_1z + c_2z^2 + \cdots = 1 + D(z). \tag{3.8}$$

Then

$$|c_n| \leqslant 2, \quad (n \in \mathbb{N}) \tag{3.9}$$

and

$$\left|c_2 - \frac{1}{2}\mu c_1^2\right| \leqslant 2 + \frac{1}{2}(|\mu - 1| - 1)|c_1|^2. \tag{3.10}$$

Furthermore, if we define the sequence $\{A_n\}_{n=1}^{\infty}$ by

$$\sum_{n=1}^{\infty}(-1)^{n-1}\gamma_{n-1}\{D(z)\}^n = \sum_{n=1}^{\infty}A_nz^n, \tag{3.11}$$

where

$$\gamma_0 = 1 \quad \text{and} \quad \gamma_n = \frac{1}{2^n}\left[1 + \frac{1}{2}\sum_{j=1}^{n} {}_nC_jB_j\right], \tag{3.12}$$

and the sequence $\{B_n\}_{n=1}^{\infty}$ is given by

$$h(z) = 1 + \sum_{n=1}^{\infty}B_nz^n, \tag{3.13}$$

then

$$|A_n| \leqslant 2, \quad (n \in \mathbb{N}). \tag{3.14}$$

Theorem 3.2. *Let the function f, given by (3.1), be in the class SP_n. Then,*

$$|\mu a_2^2 - a_3| \leqslant \begin{cases} \frac{4}{3^{n+1}\pi^2}\left(\frac{3^{n+1}2^{4-2n}}{\pi^2}\mu - 2 - \frac{24}{\pi^2}\right), & (\mu \geqslant \sigma_1), \\ \frac{4}{3^n\pi^2}, & (\sigma_2 \leqslant \mu \leqslant \sigma_1), \\ \frac{4}{3^{n+1}\pi^2}\left(2 + \frac{24}{\pi^2} - \frac{3^{n+1}2^{4-2n}}{\pi^2}\mu\right), & (\mu \leqslant \sigma_2), \end{cases} \tag{3.15}$$

where

$$\sigma_1 = \frac{2^{2n-4}\pi^2}{3^{n+1}}\left(5 + \frac{24}{\pi^2}\right) \quad \text{and} \quad \sigma_2 = \frac{2^{2n-4}\pi^2}{3^{n+1}}\left(\frac{24}{\pi^2} - 1\right).$$

Each of the estimates in (3.15) is sharp.

Proof. Using (3.4) and (3.5), we write

$$|\mu a_2^2 - a_3| = \frac{1}{3^{n+1}\pi^2}\left|\left(\frac{2^{4-2n}3^{n+1}}{\pi^2}\mu + 1 - \frac{24}{\pi^2}\right)c_1^2 - 6c_2\right|, \tag{3.16}$$

$$\leqslant \frac{1}{3^{n+1}\pi^2}\left(\left|\frac{2^{4-2n}3^{n+1}}{\pi^2}\mu - 5 - \frac{24}{\pi^2}\right||c_1|^2 + 6|c_1^2 - c_2|\right). \tag{3.17}$$

If $\mu \geqslant \sigma_1$, then the expression inside the first modulus on the right-hand side of (3.17) is nonnegative. Thus, by applying Lemma 3.1, we get

$$|\mu a_2^2 - a_3| \leqslant \frac{4}{3^{n+1}\pi^2}\left(\frac{3^{n+1}2^{4-2n}}{\pi^2}\mu - 2 - \frac{24}{\pi^2}\right), \tag{3.18}$$

which is the first part of assertion (3.15). Equality in (3.18) holds true if and only if $|c_1| = 2$. Thus the function f is $k(z,0,1)$ or one of its rotations for $\mu > \sigma_1$. Next, if $\mu \leqslant \sigma_2$, we rewrite (3.16) as

$$|\mu a_2^2 - a_3| = \frac{1}{3^{n+1}\pi^2}\left|6c_2 + \left(\frac{24}{\pi^2} - 1 - \frac{2^{4-2n}3^{n+1}}{\pi^2}\mu\right)c_1^2\right|, \tag{3.19}$$

$$\leqslant \frac{1}{3^{n+1}\pi^2}\left\{6|c_2| + \left(\frac{24}{\pi^2} - 1 - \frac{2^{4-2n}3^{n+1}}{\pi^2}\mu\right)|c_1|^2\right\}. \tag{3.20}$$

The estimates $|c_2| \leqslant 2$ and $|c_1| \leqslant 2$, after simplication, yield the second part of the assertion (3.15), in which equality holds true if and only if f is a rotation of $k(z,0,1)$ for $\mu < \sigma_2$.

If $\mu = \sigma_1$, then

$$\frac{2^{4-2n}3^{n+1}}{\pi^2}\mu - 5 - \frac{24}{\pi^2} = 0.$$

Therefore, equality holds true if and only if $|c_1^2 - c_2| = 2$. This happens if and only if

$$\frac{1}{p_1(z)} = \frac{1+v}{2}\left(\frac{1+z}{1-z}\right) + \frac{1-v}{2}\left(\frac{1-z}{1+z}\right), \quad (0 < v < 1; z \in U).$$

Thus the function f is $k(z,\pi,v)$ or one of its rotations.

If $\mu = \sigma_2$, then equality holds true if and only if $|c_2| = 2$. Equivalently, we have

$$p_1(z) = \frac{1+v}{2}\left(\frac{1+z}{1-z}\right) + \frac{1-v}{2}\left(\frac{1-z}{1+z}\right), \quad (0 < v < 1; z \in U).$$

Thus, the function f is $k(z,0,v)$ or one of its rotations.

Finally, we see that

$$|\mu a_2^2 - a_3| = \frac{1}{3^{n+1}\pi^2}\left|\left(\frac{2^{4-2n}3^{n+1}}{\pi^2}\mu - 2 - \frac{24}{\pi^2}\right)c_1^2 + 3(c_1^2 - 2c_2)\right|$$

and

$$max\left|\frac{2^{4-2n}3^{n+1}}{\pi^2}\mu-2-\frac{24}{\pi^2}\right|\leqslant 3,\ (\sigma_2\leqslant\mu\leqslant\sigma_1).$$

Therefore, using Lemma 4.1, we get

$$|\mu a_2^2-a_3|\leqslant\frac{1}{3^{n+1}\pi^2}\left\{\left|\frac{2^{4-2n}3^{n+1}}{\pi^2}\mu-2-\frac{24}{\pi^2}\right||c_1^2|+3|c_1^2-2c_2|\right\}\leqslant 4\cdot 3^{-n}\pi^2.$$

If $\sigma_2 < \mu < \sigma_1$, then equality holds true if and only if $|c_1| = 0$ and $|c_2| = 2$. Equivalently, we have

$$p_1(z)=\frac{1+z^2}{1-z^2},\quad (0\leqslant v\leqslant 1; z\in U).$$

Thus the function f is $k(z,0,0)$ or one of its rotations. The proof of Theorem 4.2 is evidently completed. □

Acknowledgements

The research was supported Kyungsung University Research Grants in 2011.

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