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On the order of starlikeness of the shifted Gauss hypergeometric function

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Abstract

We determine for several ranges of real parameters the order of starlikeness of the shifted Gauss hypergeometric function and we give some consequences of our results, in particular some mapping properties of the Carlson–Shaffer convolution operator.

Keywords: Gauss hypergeometric function; Univalence; Order of starlikeness; Order of convexity; Zerofreeness in the unit disk; Carlson–Shaffer convolution operator

1. Introduction and statement of results

Let $\mathbb{D} := \{z \in \mathbb{C}\colon\ |z| < 1\}$ be the open unit disk in the complex plane $\mathbb{C}$ and let $\mathcal{H}$ denote the set of all functions which are analytic in $\mathbb{D}$. For a function $f \in \mathcal{H}$ with $f(0) = 0 \neq f'(0)$ its order of starlikeness (with respect to zero) is defined by

$$\sigma(f) := \inf_{z\in\mathbb{D}} \operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) \in [-\infty, 1],$$

and if at least $f'(0) \neq 0$ then the order of convexity of f is defined by

$$\kappa(f) := \sigma(zf') = 1 + \inf_{z\in\mathbb{D}} \operatorname{Re}\left(\frac{zf''(z)}{f'(z)}\right) \in [-\infty, 1].$$

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As is well known, f is starlike, i.e. $\sigma(f) \geqslant 0$, if and only if f is univalent in $\mathbb{D}$ (i.e. one-to-one) with $f(\mathbb{D})$ being starlike with respect to zero; and f is convex, i.e. $\kappa(f) \geqslant 0$, if and only if f is univalent in $\mathbb{D}$ with $f(\mathbb{D})$ being convex. It is further known that if $\kappa(f) \geqslant -1/2$ then f is univalent in $\mathbb{D}$ with $f(\mathbb{D})$ being convex in (at least) one direction, see [12,21] and [16, Theorem 2.24, pp. 71, 73].

If $\sigma(f) > -\infty$ then $z \mapsto zf'(z)/f(z)$ can have no pole in $\mathbb{D}$, i.e. $z \mapsto f(z)/z$ has no zero in $\mathbb{D}$. For this reason we make the convention that $\sigma(f) := -\infty$ only if $z \mapsto f(z)/z$ has no zero in $\mathbb{D}$ and $\operatorname{Re}(zf'(z)/f(z))$ is not bounded from below in $\mathbb{D}$, whereas $\sigma(f)$ is considered to be not defined if $z \mapsto f(z)/z$ has a zero in $\mathbb{D}$. And, of course, we shall also make the corresponding convention for $\kappa(f)$, i.e. $\kappa(f) := -\infty$ only if f' has no zero in $\mathbb{D}$ and $\operatorname{Re}(zf''(z)/f'(z))$ is not bounded from below in $\mathbb{D}$, whereas $\kappa(f)$ is considered to be not defined if f' has a zero in $\mathbb{D}$.

In this paper we determine for several ranges of real parameters a, b, c the order of starlikeness of the shifted Gauss hypergeometric function $z \mapsto zF(a,b,c,z)$. Here the Gauss hypergeometric function $z \mapsto F(a,b,c,z)$ depends on the three parameters $a, b, c \in \mathbb{C}$, $-c \notin \mathbb{N} := \{0, 1, 2, \ldots\}$, and is defined for $z \in \mathbb{D}$ by

$$F(a,b,c,z) := \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} z^n, \tag{1}$$

where $(a)_n$ is the Pochhammer symbol, i.e. $(a)_0 := 1$ and $(a)_{n+1} := (a)_n(a+n)$ for all $n \in \mathbb{N}$, so that in particular $(1)_n = n!$ and $F(a,b,c,z) = 1$ if $abz = 0$.

Our results will also yield the zerofreeness in $\mathbb{D}$ as well as the order of convexity of the hypergeometric function, and if $b = 1$ or if $c = 2$ then they further yield the order of convexity of the shifted hypergeometric function.

As another consequence of our results on the order of starlikeness of the shifted hypergeometric function, we shall also obtain three mapping properties of the Carlson–Shaffer convolution operator on classes of starlike functions of order α and two results on the convolution of starlike or convex functions of order α.

1.1. Results on the order of starlikeness of the shifted hypergeometric function

In what follows we summarize known as well as new results on the order of starlikeness $\sigma(zF(a,b,c,z))$ of the shifted hypergeometric function $z \mapsto zF(a,b,c,z)$.

Theorem 1.

(a) [8, Theorem 1.1, Remark 2.3] *If* $0 < a \leqslant b \leqslant c$ *then*

$$1 - \frac{ab}{c+b} \leqslant \sigma\bigl(zF(a,b,c,z)\bigr) = 1 - \frac{F'(a,b,c,-1)}{F(a,b,c,-1)} \leqslant 1 - \frac{ab}{2c}.$$

(b) [8, Remark 1.2] *If* $-1 \leqslant a < 0 < b \leqslant c \leqslant a+b+1$ *then*

$$\sigma\bigl(zF(a,b,c,z)\bigr) = 1 + \frac{F'(a,b,c,1)}{F(a,b,c,1)} = -\infty.$$

(c) [19, Theorem 2], [8, Remark 1.2] *If* $-1 \leqslant a < 0 < b \leqslant a+b+1 < c$ *then*

$$\sigma\bigl(zF(a,b,c,z)\bigr) = 1 + \frac{F'(a,b,c,1)}{F(a,b,c,1)} = 1 + \frac{ab}{c-a-b-1}.$$

(d) *If* $a < 0 < b$ *and if* $c \geqslant b - a + 1$ *then*

$$\sigma\big(zF(a,b,c,z)\big) = 1 + \frac{F'(a,b,c,1)}{F(a,b,c,1)} = 1 + \frac{ab}{c-a-b-1} \geqslant 1 - \frac{b}{2}.$$

(e) *If* $0 < a < c < b \leqslant c - a + 1$ *then* $\sigma(zF(a,b,c,z)) = -\infty$.

(f) *If* $0 < a \leqslant c \leqslant b \leqslant c + 1 < a + b$ *then*

$$\sigma\big(zF(a,b,c,z)\big) = 1 + \frac{(c-b)(c-a)}{a+b-c-1} + \frac{c-a-b}{2} < \frac{1}{2}.$$

(g) *If* $1 < a \leqslant c \leqslant b \leqslant c + a - 1$ *then*

$$1 - \frac{b}{2} \leqslant \sigma\big(zF(a,b,c,z)\big) = 1 + \frac{(c-b)(c-a)}{a+b-c-1} + \frac{c-a-b}{2} \leqslant 1 - \frac{a}{2} < \frac{1}{2}.$$

In particular, in all cases (a)–(g) *we have* $F(a,b,c,z) \neq 0$ *for all* $z \in \mathbb{D}$, *and in the cases* (b), (e)–(g) *the function* $z \mapsto zF(a,b,c,z)$ *is not convex.*

Part (c) has first been proved by Silverman [19, Theorem 2]. Parts (a)–(c) are special or limiting cases of the following more general result which has been obtained from the continued fraction of Gauss using a characterization, due to Wall, of Hausdorff moment sequences by means of (continued) g-fractions.

Theorem 2. *(See [8, Theorem 1.1, Remark 2.3].) If* $a, b, c \in \mathbb{R}$ *such that* $0 < a \leqslant b < c$ *or* $-1 \leqslant a < 0 < b < c$, *and if* $r \in (0,1]$, *then the function* $z \mapsto zF(a,b,c,rz)$ *has the order of starlikeness*

$$\sigma\big(zF(a,b,c,rz)\big) = 1 + \frac{\rho F'(a,b,c,\rho)}{F(a,b,c,\rho)} = 1 + a\rho\frac{\int_0^1 (1-\rho t)^{-a-1} t^b (1-t)^{c-b-1}\,\mathrm{d}t}{\int_0^1 (1-\rho t)^{-a} t^{b-1} (1-t)^{c-b-1}\,\mathrm{d}t},$$

where $\rho := -r$ *if* $a > 0$ *respectively* $\rho := r$ *if* $a < 0$. *In particular it holds that*

$$\sigma\big(zF(a,c,c,rz)\big) = 1 + \frac{a\rho}{1-\rho} \leqslant \sigma\big(zF(a,b,c,rz)\big) \leqslant 1 + \frac{ab\rho}{2c},$$

and in the case $0 < a \leqslant b < c$ *we have the better estimates*

$$1 - \frac{abr}{c+br} \leqslant \sigma\big(zF(a,b,c,rz)\big) \leqslant 1 - \frac{abr}{c+cr}.$$

In Theorem 2 the case $a < 0$ and $r = 1$ is to be considered as a limiting one, in which $1 + ar/(1-r) = -\infty$, and it yields the two parts (b) and (c) of Theorem 1 if one uses in addition the known asymptotic behaviour for $z \to 1$ of the hypergeometric function.

Theorem 2 generalizes results of Wilken and Feng [22] as well as of Ruscheweyh and Singh [17, Theorem 1, Corollary 1], but indeed it does not contain completely the following theorem of Ruscheweyh and Singh.

Theorem 3. *(See Ruscheweyh and Singh [17, Theorem 1].) If* $a, c, r \in \mathbb{R}$, $a \neq 0 < r \leqslant 1 < c$ *and if* $-1/r \leqslant a \leqslant c - 1 + 1/r$ *then*

$$\sigma\big(zF(a, c-1, c, rz)\big) = 1 + \frac{\rho F'(a, c-1, c, \rho)}{F(a, c-1, c, \rho)} = 2 - c + \left(\int_0^1 \left(\frac{1-\rho}{1-t\rho}\right)^a t^{c-2}\,\mathrm{d}t\right)^{-1},$$

where $\rho := -r$ *if* $a > 0$ *respectively* $\rho := r$ *if* $a < 0$.

This theorem is remarkable inasmuch as a limit case of it yields a result on the order of starlikeness of the shifted confluent (Kummer) hypergeometric function, see [17, Theorem 8]. Unfortunately, this is not the case for Theorem 2.

Ruscheweyh has remarked that a proof given by Lewis [9, Lemma 1], which uses Jack's lemma and the hypergeometric differential equation, also remains valid under more general assumptions, and so he obtained the following estimate.

Theorem 4. *(See Ruscheweyh [16, Theorem 2.12, p. 60].) If* $a \in \mathbb{C}$, $b \in (0, +\infty)$ *such that* $2\,\mathrm{Re}(a) \leqslant b + 1$ *then* $\sigma(zF(a, b, b - \bar{a} + 1, z)) \geqslant 1 - b/2$.

Note that here a is allowed to be complex and compare to Theorem 1(d), (g). In Section 3 we will obtain Theorem 1(d) as a consequence of more general results which will also be proved by means of Jack's lemma and the hypergeometric differential equation.

Parts (e)–(g) of Theorem 1 follow respectively from its parts (b)–(d) by an application of the following theorem which will be proved in Section 2 by means of the well-known Euler identity

$$F(a, b, c, z) = (1 - z)^{c-a-b} F(c - b, c - a, c, z) \tag{2}$$

and the known asymptotic behaviour for $z \to 1$ of the hypergeometric function.

Theorem 5. *Let* $a, b, c \in \mathbb{R}$, $c \notin -\mathbb{N}$ *such that* $F(a, b, c, z) \neq 0$ *for all* $z \in \mathbb{D}$. *If either*

(a) $a, b \notin -\mathbb{N}$, $\quad c - a - b < 0$, *or*

(b) $c - b, c - a \notin -\mathbb{N}$, $\quad c - a - b > 0$

then

$$\sigma\big(zF(a, b, c, z)\big) = \sigma\big(zF(c - b, c - a, c, z)\big) + \frac{1}{2}(c - a - b).$$

1.2. Results on the order of convexity of the hypergeometric function

From (1) it follows that

$$F'(a, b, c, z) := \frac{\mathrm{d}}{\mathrm{d}z} F(a, b, c, z) = \frac{ab}{c} F(a + 1, b + 1, c + 1, z) \tag{3}$$

and consequently

$$\kappa\big(F(a, b, c, z)\big) = \sigma\big(zF(a + 1, b + 1, c + 1, z)\big) \quad \text{if } abc \neq 0 \tag{4}$$

and if both sides are defined, i.e. if the function in (3) has no zeros in $\mathbb{D}$. Hence we get from Theorems 1 and 2 at once the following two corollaries on the order of convexity $\kappa(F(a, b, c, z))$ of the hypergeometric function $z \mapsto F(a, b, c, z)$.

Corollary 6.

(a) *If* $-1 < a \leqslant b \leqslant c$ *and if* $abc \neq 0$ *then*

$$1 - \frac{(a+1)(b+1)}{c+b+2} \leqslant \kappa\big(F(a,b,c,z)\big) = 1 - \frac{F''(a,b,c,-1)}{F'(a,b,c,-1)} \leqslant 1 - \frac{(a+1)(b+1)}{2(c+1)}.$$

(b) *If* $-2 \leqslant a < -1 < b \leqslant c \leqslant a+b+2$ *and if* $bc \neq 0$ *then*

$$\kappa\big(F(a,b,c,z)\big) = 1 + \frac{F''(a,b,c,1)}{F'(a,b,c,1)} = -\infty.$$

(c) *If* $-2 \leqslant a < -1 < b \leqslant a+b+2 < c$ *and if* $bc \neq 0$ *then*

$$\kappa\big(F(a,b,c,z)\big) = 1 + \frac{F''(a,b,c,1)}{F'(a,b,c,1)} = 1 + \frac{(a+1)(b+1)}{c-a-b-2}.$$

(d) *If* $a < -1 < b \neq 0$ *and if* $c \geqslant b - a$ *then*

$$\kappa\big(F(a,b,c,z)\big) = 1 + \frac{F''(a,b,c,1)}{F'(a,b,c,1)} = 1 + \frac{(a+1)(b+1)}{c-a-b-2} \geqslant \frac{1-b}{2}.$$

(e) *If* $-1 < a < c < b \leqslant c - a$ *and if* $bc \neq 0$ *then* $\kappa(F(a,b,c,z)) = -\infty$.

(f) *If* $-1 < a \leqslant c \leqslant b \leqslant c+1 < a+b+1$ *and if* $ac \neq 0$ *then*

$$\kappa\big(F(a,b,c,z)\big) = 1 + \frac{(c-b)(c-a)}{a+b-c} + \frac{c-a-b-1}{2} < \frac{1}{2}.$$

(g) *If* $0 < a \leqslant c \leqslant b \leqslant c + a$ *then*

$$\frac{1-b}{2} \leqslant \kappa\big(F(a,b,c,z)\big) = 1 + \frac{(c-b)(c-a)}{a+b-c} + \frac{c-a-b-1}{2} \leqslant \frac{1-a}{2} < \frac{1}{2}.$$

In particular, in all cases (a)–(g) *we have* $F'(a,b,c,z) \neq 0$ *for all* $z \in \mathbb{D}$.

Corollary 7. *(See [8, Corollary 1.4].) If* $a, b, c \in \mathbb{R} \setminus \{0\}$ *such that* $-1 < a \leqslant b < c$ *or* $-2 \leqslant a < -1 < b < c$*, and if* $r \in (0,1]$*, then the function* $z \mapsto F(a,b,c,rz)$ *has the order of convexity*

$$\begin{aligned}\kappa\big(F(a,b,c,rz)\big) &= 1 + \frac{\rho F''(a,b,c,\rho)}{F'(a,b,c,\rho)} \\ &= 1 + (a+1)\rho \frac{\int_0^1 (1-\rho t)^{-a-2} t^{b+1} (1-t)^{c-b-1}\,\mathrm{d}t}{\int_0^1 (1-\rho t)^{-a-1} t^{b} (1-t)^{c-b-1}\,\mathrm{d}t}\end{aligned}$$

where $\rho := -r$ *if* $a > -1$ *respectively* $\rho := r$ *if* $a < -1$. *In particular it holds that*

$$\kappa\big(F(a,c,c,rz)\big) = 1 + \frac{(a+1)\rho}{1-\rho} \leqslant \kappa\big(F(a,b,c,rz)\big) \leqslant 1 + \frac{(a+1)(b+1)\rho}{2(c+1)},$$

and in the case $-1 < a \leqslant b < c$ *we have the better estimates*

$$1 - \frac{(a+1)(b+1)r}{(c+1)+(b+1)r} \leqslant \kappa\big(F(a,b,c,rz)\big) \leqslant 1 - \frac{(a+1)(b+1)r}{(c+1)(1+r)}.$$

Here again the case $a < -1$ and $r = 1$ is to be considered as a limiting one.

1.3. Results on the order of convexity of the shifted hypergeometric function

As $(2)_n = (n+1)(1)_n$ for all $n \in \mathbb{N}$, and hence $z(zF(a,b,2,z))' = zF(a,b,1,z)$, we obtain from the five parts (a)–(c), (e) and (f) of Theorem 1 the following corollary on the order of convexity $\kappa(zF(a,b,2,z)) = \sigma(zF(a,b,1,z))$ of the function $z \mapsto zF(a,b,2,z)$.

Corollary 8.

(a) *If* $0 < a \leqslant b \leqslant 1$ *then*

$$1 - \frac{ab}{b+1} \leqslant \kappa\big(zF(a,b,2,z)\big) = 1 - \frac{F'(a,b,1,-1)}{F(a,b,1,-1)} \leqslant 1 - \frac{ab}{2}.$$

(b) *If* $0 < -a \leqslant b \leqslant 1$ *then*

$$\kappa\big(zF(a,b,2,z)\big) = 1 + \frac{F'(a,b,1,1)}{F(a,b,1,1)} = -\infty.$$

(c) *If* $0 < b < -a \leqslant 1$ *then*

$$\kappa\big(zF(a,b,2,z)\big) = 1 + \frac{F'(a,b,1,1)}{F(a,b,1,1)} = 1 - \frac{ab}{b+a}.$$

(d) *If* $0 < a < 1 < b \leqslant 2 - a$ *then* $\kappa(zF(a,b,2,z)) = -\infty$.

(e) *If* $0 < a \leqslant 1 \leqslant b \leqslant 2 < a + b$ *then*

$$\kappa\big(zF(a,b,2,z)\big) = 1 + \frac{(1-b)(1-a)}{a+b-2} + \frac{1-a-b}{2} < \frac{1}{2}.$$

In particular, in all cases (a)–(e) *we have* $F(a,b,1,z) \neq 0$ *for all* $z \in \mathbb{D}$.

As $(n+1)(1)_n = (2)_n$ for all $n \in \mathbb{N}$, we also have $z(zF(a,1,c,z))' = zF(a,2,c,z)$, and we obtain from Theorem 1 also the following corollary on the order of convexity $\kappa(zF(a,1,c,z)) = \sigma(zF(a,2,c,z))$ of the function $z \mapsto zF(a,1,c,z)$ if we use in addition that the hypergeometric function $F(a,b,c,z)$ is symmetric in a and b.

Corollary 9.

(a) *If* $0 < a \leqslant c$ *and* $c \geqslant 2$ *then*

$$0 \leqslant 1 - \frac{2a}{c + \max(a,2)} \leqslant \kappa\big(zF(a,1,c,z)\big) = 1 - \frac{F'(a,2,c,-1)}{F(a,2,c,-1)} \leqslant 1 - \frac{a}{c}.$$

(b) *If* $-1 \leqslant a < 0$ *and* $2 \leqslant c \leqslant a + 3$ *then*

$$\kappa\big(zF(a,1,c,z)\big) = 1 + \frac{F'(a,2,c,1)}{F(a,2,c,1)} = -\infty.$$

(c) *If* $-1 \leqslant a < 0$ *and* $c > a + 3$ *then*

$$\kappa\big(zF(a,1,c,z)\big) = 1 + \frac{F'(a,2,c,1)}{F(a,2,c,1)} = 1 + \frac{2a}{c-a-3} = \frac{c+a-3}{c-a-3}.$$

(d) *If* $a < 0$ *and* $c \geqslant 3 - a$ *then*

$$\kappa\big(zF(a,1,c,z)\big) = 1 + \frac{F'(a,2,c,1)}{F(a,2,c,1)} = 1 + \frac{2a}{c-a-3} = \frac{c+a-3}{c-a-3} \geqslant 0.$$

(e) *If* $0 < a < 1 < a+1 \leqslant c < 2$ *then* $\kappa(zF(a,1,c,z)) = -\infty$.

(f) *If* $0 < a \leqslant 1 \leqslant c < a+1$ *or* $1 < a \leqslant c \leqslant 2$ *or* $2 \leqslant c \leqslant a \leqslant c+1$ *then*

$$\kappa\big(zF(a,1,c,z)\big) = \frac{(c-2)(c-a)}{a-c+1} + \frac{c-a}{2} = \frac{c-a}{2}\frac{a+c-3}{a-c+1} \leqslant \frac{c-a}{2} < \frac{1}{2}.$$

In particular, in all cases (a)–(f) *we have* $F(a,2,c,z) \neq 0$ *for all* $z \in \mathbb{D}$.

Proof. The parts (a)–(d) are clear, for the lower bound in part (a) we also use the symmetry of $F(a,b,c,z)$ in a and b. The hypothesis of Theorem 1(e) implies $a < 1$ and therefore, for $b = 2$, it is equivalent to the hypothesis of (e). For $b = 2$ the hypothesis of Theorem 1(f), i.e. $0 < a \leqslant c \leqslant 2 \leqslant c+1 < a+2$, i.e. $0 < a \leqslant c \leqslant 2$ and $1 \leqslant c < a+1$, is equivalent to either the first or the second hypothesis of (f). If we use the symmetry of $F(a,b,c,z)$ in a and b and if in the hypothesis of Theorem 1(f) we replace a by 2 and b by a, then this hypothesis becomes $0 < 2 \leqslant c \leqslant a \leqslant c+1 < 2+a$, i.e. the third hypothesis of (f). For $b = 2$ the hypothesis of Theorem 1(g), i.e. $1 < a \leqslant c \leqslant 2 \leqslant c+a-1$, is already contained in the second hypothesis of (f). Finally, if in the hypothesis of Theorem 1(g) we replace a by 2 and b by a, then this hypothesis becomes $1 < 2 \leqslant c \leqslant a \leqslant c+1$, i.e. again the third hypothesis of (f). □

It seems to be much more difficult to determine for arbitrary real parameters a, b, c for the shifted hypergeometric function its order of convexity than its order of starlikeness, there exists only one result due to Silverman [19, Theorem 4] which can be stated in the following form.

Theorem 10. *(See Silverman [19, Theorem 4].) If* $-1 \leqslant a < 0 < b < c$ *and* $c - a - b - 1 > \max(1, -ab)$ *then*

$$\kappa\big(zF(a,b,c,z)\big) = 1 + \frac{ab}{c-a-b-1+ab}\left(2 + \frac{(a+1)(b+1)}{c-a-b-2}\right).$$

Note that Corollary 8(c) and Corollary 9(c) are special cases of Theorem 10. (One should note that [19, Theorem 4] and its proof contain three misprints; there should stand three times $(c-a-b-2)_2$ instead of $(c-a-b-1)_2$, namely one time on p. 578 in Eq. (11) and two times on p. 579 at the end of the proof.)

2. Proof of Theorem 5

If we replace in hypothesis (a) of Theorem 5 simultaneously a by $c-b$ and b by $c-a$ then we obtain hypothesis (b), whereas this replacement leaves the conclusion of Theorem 5 unchanged. Therefore, in what follows, it is enough to prove the theorem under its hypothesis (b), i.e. we assume that $a, b, c \in \mathbb{R}$, $c, c-a, c-b \notin -\mathbb{N}$, that $c-a-b > 0$ and that $F(a,b,c,z) \neq 0$ for all $z \in \mathbb{D}$. Then the Euler identity (2) implies that also $F(c-b, c-a, c, z) \neq 0$ for all $z \in \mathbb{D}$ and a logarithmic differentiation of (2) yields that

$$1 + \frac{zF'(c-b,c-a,c,z)}{F(c-b,c-a,c,z)} = 1 + \frac{zF'(a,b,c,z)}{F(a,b,c,z)} + (c-a-b)\frac{z}{1-z}. \tag{5}$$

As $c-a-b>0$ and as $z \mapsto z/(1-z)$ maps $\mathbb{D}$ onto the half-plane $\operatorname{Re}(w) > -\frac{1}{2}$, we obtain from (5), by taking the infimum of the real part of (5) over $\mathbb{D}$, that

$$\sigma\big(zF(c-b,c-a,c,z)\big) \geqslant \sigma\big(zF(a,b,c,z)\big) - \frac{1}{2}(c-a-b). \tag{6}$$

By the minimum principle there exists a sequence $z_n \in \mathbb{D}$ such that $|z_n| \to 1$ and

$$\operatorname{Re}\left(1 + \frac{z_n F'(a,b,c,z_n)}{F(a,b,c,z_n)}\right) \to \sigma\big(zF(a,b,c,z)\big) \quad \text{for } n \to \infty.$$

By taking a convergent subsequence of this bounded sequence z_n, we may even assume that there exists a point $\zeta \in \mathbb{C}$, $|\zeta| = 1$, such that $z_n \to \zeta$ for $n \to \infty$. If we can choose this sequence z_n in such a way that $\zeta \neq 1$ then it follows that

$$\operatorname{Re}\left(\frac{z_n}{1-z_n}\right) \to \operatorname{Re}\left(\frac{\zeta}{1-\zeta}\right) = -\frac{1}{2} \quad \text{for } n \to \infty$$

and therefore, using (5), that we even have equality in (6), i.e. the conclusion of Theorem 5. But if $z_n \to 1$ then we cannot conclude so easily that we have equality in (6), as $z \mapsto z/(1-z)$ has a pole at $z = 1$. So we have to examine whether this case $\zeta = 1$ can happen at all and, when it can happen, whether we then have nevertheless equality in (6). For that we shall use the special sequence $\zeta_n := 1/n + (1-1/n)\mathrm{e}^{\mathrm{i}\pi/n} \in \mathbb{D}$, $n \geqslant 2$, which satisfies $\zeta_n \to 1$ and

$$\operatorname{Re}\left(\frac{\zeta_n}{1-\zeta_n}\right) = \operatorname{Re}\left(\frac{1}{1-\zeta_n} - 1\right) = \operatorname{Re}\left(\frac{\frac{n}{n-1}}{1-\mathrm{e}^{\mathrm{i}\pi/n}}\right) - 1 = \frac{1}{2}\frac{n}{n-1} - 1 \to -\frac{1}{2} \tag{7}$$

for $n \to \infty$.

According to (3) we have

$$\frac{zF'(a,b,c,z)}{F(a,b,c,z)} = \frac{ab}{c} z \frac{F(a+1,b+1,c+1,z)}{F(a,b,c,z)}. \tag{8}$$

If $a \in -\mathbb{N}$ or if $b \in -\mathbb{N}$ then the denominator $F(a,b,c,z)$ is a polynomial. If $a, b \notin -\mathbb{N}$ then, as we assume that $c-a-b>0$, it is well known that the hypergeometric series (1) converges absolutely and uniformly for all z in the closed unit disk $\overline{\mathbb{D}}$. Hence, in both cases $z \mapsto F(a,b,c,z)$ is continuous on $\overline{\mathbb{D}}$ and, according to the Chu–Vandermonde identity respectively Gauss's (hypergeometric) summation formula, we have in both cases that

$$F(a,b,c,1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \neq 0, \infty, \tag{9}$$

as we assume that $c, c-a, c-b \notin -\mathbb{N}$ and $c-a-b>0$, thus also $c-a-b \notin -\mathbb{N}$.

Remark. If $c-a-b>0$, $c \notin -\mathbb{N}$, but $c-a \in -\mathbb{N}$ or $c-b \in -\mathbb{N}$, then we have $F(a,b,c,1) = 0$ and, as $z \mapsto F(a,b,c,z)$ is continuous on $\overline{\mathbb{D}}$ and as we assume that $F(a,b,c,z) \neq 0$ for all $z \in \mathbb{D}$, it then follows from [6, Theorem 2] that $\sigma(zF(a,b,c,z)) = -\infty$.

This remark will be stated as part (c) in Corollary 11 which will be given below, after the end of this proof.

If $ab = 0$ then, as $c-a-b>0$, it follows from (5) and (8) that we have equality in (6). Hence, in what follows, we assume in addition that $ab \neq 0$ and we consider now for the numerator $F(a+1,b+1,c+1,z)$ in (8) three cases.

First case. In this case we assume that $a \in -\mathbb{N}$ or $b \in -\mathbb{N}$ or $c-a-b-1>0$.

If $a \in -\mathbb{N}$ or if $b \in -\mathbb{N}$ then, as $c, c-a, c-b \notin -\mathbb{N}$ and $ab \neq 0$, it follows that $a+1 \in -\mathbb{N}$ or $b+1 \in -\mathbb{N}$ and consequently that $c-a-b-1 \notin -\mathbb{N}$. Therefore, in this first case, the before-mentioned properties of $F(a,b,c,z)$ also hold correspondingly for $F(a+1,b+1,c+1,z)$, i.e. $z \mapsto F(a+1,b+1,c+1,z)$ is continuous on $\overline{\mathbb{D}}$ and we have

$$F(a+1,b+1,c+1,1) = \frac{\Gamma(c+1)\Gamma(c-a-b-1)}{\Gamma(c-a)\Gamma(c-b)} \neq 0, \infty.$$

In this case it follows that the function on the right-hand side of (8) is continuous on $\mathbb{D} \cup \{1\}$ and that it has at $z=1$ the value

$$\frac{ab}{c}\frac{F(a+1,b+1,c+1,1)}{F(a,b,c,1)} = \frac{ab}{c}\frac{\Gamma(c+1)\Gamma(c-a-b-1)}{\Gamma(c)\Gamma(c-a-b)} = \frac{ab}{c-a-b-1} \neq 0, \infty.$$

Accordingly the case $\zeta = 1$ cannot happen if $ab/(c-a-b-1) > 0$. On the other hand, if $ab/(c-a-b-1) < 0$ and the case $\zeta = 1$ happens, then it follows that

$$\sigma\big(zF(a,b,c,z)\big) = 1 + \frac{ab}{c}\frac{F(a+1,b+1,c+1,1)}{F(a,b,c,1)} = 1 + \frac{ab}{c-a-b-1}$$

and, as $\zeta_n \to 1$ for $n \to \infty$, with (5), (7) and (8) also that

$$\begin{aligned}\operatorname{Re}\left(1 + \frac{\zeta_n F'(c-b,c-a,c,\zeta_n)}{F'(c-b,c-a,c,\zeta_n)}\right) &= \operatorname{Re}\left(1 + \frac{\zeta_n F'(a,b,c,\zeta_n)}{F'(a,b,c,\zeta_n)} + (c-a-b)\frac{\zeta_n}{1-\zeta_n}\right) \\ &\to \sigma\big(zF(a,b,c,z)\big) - \frac{1}{2}(c-a-b) \quad \text{for } n \to \infty,\end{aligned}$$

so that we have equality in (6).

Second case. In this case we assume that $a, b \notin -\mathbb{N}$ and $c-a-b-1<0$.

According to the Euler identity (2) we have

$$F(a+1,b+1,c+1,z) = (1-z)^{c-a-b-1}F(c-b,c-a,c+1,z).$$

As $c+1-(c-a)-(c-b) = a+b+1-c > 0$, we know that $z \mapsto F(c-b,c-a,c+1,z)$ is continuous on $\overline{\mathbb{D}}$, and with Gauss's summation formula (9) applied to $F(c-b,c-a,c+1,z)$ we obtain the asymptotic equality

$$F(a+1,b+1,c+1,z) \sim \frac{\Gamma(c+1)\Gamma(a+b+1-c)}{\Gamma(a+1)\Gamma(b+1)}(1-z)^{c-a-b-1} \quad \text{for } z \to 1,\ z \in \mathbb{D}.$$

Since $z \mapsto F(a,b,c,z)$ is continuous on $\overline{\mathbb{D}}$, it then follows with (9) and since

$$\frac{ab}{c}\frac{\Gamma(c+1)\Gamma(a+b+1-c)\Gamma(c-b)\Gamma(c-a)}{\Gamma(a+1)\Gamma(b+1)\Gamma(c)\Gamma(c-a-b)} = \frac{\Gamma(a+b+1-c)\Gamma(c-b)\Gamma(c-a)}{\Gamma(c-a-b)\Gamma(a)\Gamma(b)} \neq 0, \infty$$

that we also have the asymptotic equality

$$\frac{zF'(a,b,c,z)}{F(a,b,c,z)} \sim \frac{\Gamma(a+b+1-c)\Gamma(c-b)\Gamma(c-a)}{\Gamma(c-a-b)\Gamma(a)\Gamma(b)}(1-z)^{c-a-b-1} \quad \text{for } z \to 1,\ z \in \mathbb{D}.$$

Here, as $c-a-b > 0 > c-a-b-1$, it holds that $\Gamma(a+b+1-c)/\Gamma(c-a-b) > 0$,

$$\left|(1-z)^{c-a-b-1}\right| \to +\infty \quad \text{for } z \to 1,\ z \in \mathbb{D}, \quad \text{and}$$
$$\left|\operatorname{Im}\bigl((1-z)^{c-a-b-1}\bigr)\right| < \tan\left((a+b+1-c)\frac{\pi}{2}\right)\operatorname{Re}\bigl((1-z)^{c-a-b-1}\bigr) \quad \text{for } z \in \mathbb{D},$$

where $\tan((a+b+1-c)\pi/2) > 0$ since $0 < a+b+1-c < 1$, and therefore it holds that

$$\operatorname{Re}\bigl((1-z)^{c-a-b-1}\bigr) \to +\infty \quad \text{for } z \to 1, z \in \mathbb{D}.$$

Hence, if $\Gamma(c-b)\Gamma(c-a)/(\Gamma(a)\Gamma(b)) > 0$ then it follows for every sequence $z_n \in \mathbb{D}$ with $z_n \to 1$ for $n \to \infty$ that $\operatorname{Re}(z_n F'(a,b,c,z_n)/F(a,b,c,z_n)) \to +\infty$, and therefore, in this case, the case $\zeta = 1$ cannot happen. On the other hand, if $\Gamma(c-b)\Gamma(c-a)/(\Gamma(a)\Gamma(b)) < 0$ then it follows for all $z_n \in \mathbb{D}$ with $z_n \to 1$ for $n \to \infty$ that $\operatorname{Re}(z_n F'(a,b,c,z_n)/F(a,b,c,z_n)) \to -\infty$, and so $\sigma(zF(a,b,c,z)) = -\infty$. But then, using in particular the sequence ζ_n and (7) as well as (5), it also follows that $\sigma(zF(c-b,c-a,c,z)) = -\infty$, and so we have equality in (6), both sides being equal to $-\infty$.

Third case. In this case we assume that $a, b \notin -\mathbb{N}$ and $c-a-b-1 = 0$.

In this case we have, see [1, 15.3.10, p. 559], [2, p. 110], the asymptotic equality

$$F(a+1,b+1,c+1,z) \sim -\frac{\Gamma(c+1)}{\Gamma(a+1)\Gamma(b+1)}\log(1-z) \quad \text{for } z \to 1,\ z \in \mathbb{D}.$$

Since $z \mapsto F(a,b,c,z)$ is continuous on $\overline{\mathbb{D}}$ and since $c-a-b-1 = 0$ and so

$$\frac{\Gamma(c-b)\Gamma(c-a)}{\Gamma(a)\Gamma(b)} = \frac{ab}{c}\,\frac{\Gamma(c+1)\Gamma(c-b)\Gamma(c-a)}{\Gamma(a+1)\Gamma(b+1)\Gamma(c)\Gamma(c-a-b)} = ab \neq 0, \infty,$$

it then follows with (9) that we also have the asymptotic equality

$$\frac{zF'(a,b,c,z)}{F(a,b,c,z)} \sim -ab\log(1-z) \quad \text{for } z \to 1,\ z \in \mathbb{D}.$$

Here it holds that $\operatorname{Re}(\log(1-z)) = \log(|1-z|) \to -\infty$ for $z \to 1$, $z \in \mathbb{D}$. So, as in the second case, it follows that the case $\zeta = 1$ cannot happen if $ab > 0$ and, using the sequence ζ_n, that both sides of (6) are equal to $-\infty$ if $ab < 0$.

The proof of Theorem 5 is now finished.

An inspection of the preceding proof yields the following corollary.

Corollary 11. *Suppose that $a, b, c \in \mathbb{R}$, $c \notin -\mathbb{N}$, $c-a-b > 0$ and $F(a,b,c,z) \neq 0$ for $z \in \mathbb{D}$.*

(a) *If $c-b, c-a \notin -\mathbb{N}$, and if $a \in -\mathbb{N}$ or $b \in -\mathbb{N}$ or $c-a-b-1 > 0$, then*

$$-\infty \leqslant \sigma\bigl(zF(a,b,c,z)\bigr) \leqslant \lim_{\substack{z\to 1\\ z\in\mathbb{D}}} \operatorname{Re}\left(1 + \frac{zF'(a,b,c,z)}{F(a,b,c,z)}\right) = 1 + \frac{ab}{c-a-b-1}$$
$$(= 1 \text{ if } c-a-b-1 = 0).$$

(b) *If $c-b, c-a, a, b \notin -\mathbb{N}$, $c-a-b-1 \leqslant 0$ and $\Gamma(c-b)\Gamma(c-a)/(\Gamma(a)\Gamma(b)) < 0$ then*

$$\sigma\bigl(zF(a,b,c,z)\bigr) = \lim_{\substack{z\to 1\\ z\in\mathbb{D}}} \operatorname{Re}\left(1 + \frac{zF'(a,b,c,z)}{F(a,b,c,z)}\right) = -\infty.$$

(c) *If $c-b \in -\mathbb{N}$ or if $c-a \in -\mathbb{N}$ then $\sigma(zF(a,b,c,z)) = -\infty$.*

(d) *It holds that*

$$\sigma\bigl(zF(c-b,c-a,c,z)\bigr) \geqslant \sigma\bigl(zF(a,b,c,z)\bigr) - \frac{1}{2}(c-a-b).$$

(e) *If* $c-a-b<0$, *instead of* $c-a-b>0$, *but if all the other hypotheses are satisfied, then*

$$\sigma\bigl(zF(c-b,c-a,c,z)\bigr) \leqslant \sigma\bigl(zF(a,b,c,z)\bigr) - \frac{1}{2}(c-a-b).$$

Corollary 11(a) will be used in the next section in the proof of Theorem 1(d).

Remark. If $c-a-b>0$, but if $c-b \in -\mathbb{N}$ or if $c-a \in -\mathbb{N}$, so that the hypothesis (b) in Theorem 5 is not satisfied, then also the conclusion of Theorem 5 does not necessarily hold. We consider two examples:

(1) If $a<0<b=c$ then $c-a-b>0=c-b\in-\mathbb{N}$, $F(c-b,c-a,c,z)=1$, $F(a,b,c,z)=(1-z)^{-a}$, and so $\sigma(zF(c-b,c-a,c,z))=1$, but $\sigma(zF(a,b,c,z))=-\infty$ according to Corollary 11(c).
(2) If $2c<a<-1<c<0<b=c+1$ then $c-a-b>0$, $c-b=-1\in-\mathbb{N}$, $-1<(c-a)/c<0$, $F(c-b,c-a,c,z)=1-((c-a)/c)z$, and so $\sigma(zF(c-b,c-a,c,z))=1-(c-a)/(a-2c)$, but $\sigma(zF(a,b,c,z))=-\infty$ according to Corollary 11(c).

3. Proof of Theorem 1(d) and more general results

As already mentioned, in this section Theorem 1(d) and more general results will be proved. The tools to be used are on the one hand the well-known Jack's lemma and on the other hand the hypergeometric differential equation

$$(1-z)zF'' + \bigl(c-(a+b+1)z\bigr)F' - abF = 0, \quad z \in \mathbb{D}, \tag{10}$$

which is satisfied by the hypergeometric function $z \mapsto F(a,b,c,z)$.

Jack's lemma appeared without the statement concerning the case of equality in [6, Lemma 1] and for $k=1$ already in [13, Problem III.291, p. 141], where at the beginning of the solution on p. 326 the result is ascribed to Löwner.

Lemma 12. *(See Jack's lemma [6, Lemma 1].) Let* $z_0 \in \mathbb{C}\setminus\{0\}$ *and suppose that* $z \mapsto w(z)$ *is holomorphic for* $|z|<|z_0|$ *and also at* $z=z_0$, *that* $w(z)$ *has at* $z=0$ *a zero of order* $k \geqslant 1$, *but that* $w(z_0)\neq 0$. *If* $|w(z)| \leqslant |w(z_0)|$ *whenever* $|z|<|z_0|$ *then it follows that* $z_0 w'(z_0)/w(z_0) \geqslant k$, *and moreover* $z_0w'(z_0)/w(z_0)=k$ *if and only if there exists* $\eta \in \mathbb{C}\setminus\{0\}$ *such that* $w(z)=\eta z^k$ *for all* $|z|<|z_0|$.

With respect to Jack's lemma and in particular the case of equality see also [4]. By an application of the maximum principle to $f(z)=w(z)/z^k$, as in the proof of the Schwarz Lemma, Lemma 12 can be deduced from the following lemma.

Lemma 13. *Suppose that* $z_0 \in \mathbb{C}\setminus\{0\}$ *and* $z \mapsto f(z)$ *is holomorphic at* $z=z_0$ *with* $f(z_0)\neq 0$. *If in a neighbourhood of* z_0 *it holds that* $|f(z)| \leqslant |f(z_0)|$ *whenever* $|z|<|z_0|$ *then it follows that* $z_0f'(z_0)/f(z_0) \geqslant 0$, *with* $f'(z_0)=0$ *if and only if* f *is constant.*

This lemma also appeared in [13, Problem III.144, p. 112], but there it is even assumed that $z \mapsto f(z)$ is holomorphic for $|z| \leqslant |z_0|$ with $|f(z)| \leqslant |f(z_0)|$. But indeed the solution in [13, pp. 282, 283] does only use the assumptions of Lemma 13. For from these assumptions if follows, if f is not constant, that $z \mapsto f(z) - f(z_0)$ cannot have a multiple zero at $z = z_0$, i.e. $f'(z_0) \neq 0$, and that for the two curves $t \mapsto \gamma(t) := f(z_0 e^{it})$ and $t \mapsto \delta(t) := f(z_0)e^{it}$ at $t = 0$ their derivatives $\gamma'(0) = iz_0 f'(z_0)$ and $\delta'(0) = i f(z_0)$ must have the same argument, i.e. $z_0 f'(z_0)/f(z_0) > 0$.

Using Jack's lemma and (10) we can now prove the following general result.

Theorem 14. *Suppose that $a, b, c \in \mathbb{C} \setminus \{0\}$, $c \notin -\mathbb{N}$, $r \in (0, 1]$, $\alpha \in (-\infty, 1)$, $d := 2(1-\alpha)$. Then $\sigma(zF(a, b, c, rz)) \geqslant \alpha$ provided that for all $s \in (1, +\infty)$ and all $t \in \mathbb{R}$ it holds that*

$$\begin{aligned} &\left|d\left(1 - s - (d + s)t^2\right) + (2a - d)(2b - d) - d + i2d(a + b - d)t\right|r \\ &\quad \leqslant \left|d\left(1 - s - (d + s)t^2\right) - d(2c - 1 - d) + i2d(c - 1 - d)t\right| \neq 0. \end{aligned} \tag{11}$$

Proof. We define for $z \in \mathbb{D}$ the two functions

$$G(z) := \frac{zF'(a, b, c, z)}{F(a, b, c, z)} \quad \text{and} \quad w(z) := \frac{G(z)}{G(z) + d}, \quad \text{so that} \quad G(z) = \frac{dw(z)}{1 - w(z)}. \tag{12}$$

Since $d = 2(1-\alpha) > 0$ and since $F(a, b, c, z)$ is holomorphic in $\mathbb{D}$ with $F(a, b, c, 0) = 1$ and $F'(a, b, c, 0) = ab/c \neq 0$, both functions G and w are meromorphic in $\mathbb{D}$ with $w(0) = G(0) = 0$, $G'(0) = ab/c \neq 0$ and $w'(0) = G'(0)/d \neq 0$. We have $\sigma(zF(a, b, c, rz)) \geqslant \alpha$, i.e. $\operatorname{Re}(1 + G(z)) > \alpha$ whenever $|z| < r$, if and only if $|w(z)| < 1$ whenever $|z| < r$. Now we assume that $\sigma(zF(a, b, c, rz)) \geqslant \alpha$ does not hold and we will show that then (11) cannot hold for all $s \in (1, +\infty)$ and all $t \in \mathbb{R}$. As, by assumption, $\sigma(zF(a, b, c, rz)) \geqslant \alpha$ does not hold, there exists $z_0 \in \mathbb{C}$ with $0 < |z_0| < r$ such that $|w(z)| \leqslant |w(z_0)| = 1$ whenever $|z| < |z_0|$. Then, according to Jack's lemma, we have $z_0 w'(z_0)/w(z_0) \geqslant 1$, with equality if and only if there exists $\eta \in \mathbb{C} \setminus \{0\}$ such that $w(z) = \eta z$ for all $|z| < |z_0|$. If the case of equality holds then we get from (12) that $F(a, b, c, z) = (1 - \eta z)^{-d}$ and a comparison of the coefficients in the power series expansions at $z = 0$ of both sides yields that $\eta = 1$ and so $zF(a, b, c, z) = z(1 - z)^{-d} = z(1 - z)^{-2(1-\alpha)}$ for $z \in \mathbb{D}$, so that $\sigma(zF(a, b, c, z)) = \alpha$ and consequently $\sigma(zF(a, b, c, rz)) \geqslant \alpha$. But this last conclusion is just the negation of our previously made assumption, hence the case of equality cannot happen, i.e. we even have $z_0 w'(z_0)/w(z_0) > 1$.

As already mentioned, the hypergeometric function $z \mapsto F(a, b, c, z)$ satisfies the hypergeometric differential equation (10) and therefore also the equation

$$(1 - z)z^2 F''/F + \left(c - (a + b + 1)z\right)zF'/F - abz = 0, \quad z \in \mathbb{D}. \tag{13}$$

From $z(zF'/F)' = z(F'/F + zF''/F - z(F'/F)^2) = zF'/F + z^2F''/F - (zF'/F)^2$ and from $G = zF'/F$ it follows that $z^2F''/F = zG' + G^2 - G$, and accordingly we get from (13) that the function G defined by (12) satisfies the (Riccati) differential equation

$$(1 - z)\left(zG' + G^2\right) + \left(c - 1 - (a + b)z\right)G - abz = 0, \quad z \in \mathbb{D}. \tag{14}$$

From $G = dw/(1 - w) = d/(1 - w) - d$ it follows that $zG' = dzw'/(1 - w)^2$. Substituting these relations into (14) and multiplying the result by $(1 - w)^2/w$ we obtain that the function w defined by (12) satisfies the differential equation

$$d(1-z)\left(\frac{zw'}{w}+dw\right)+d\big(c-1-(a+b)z\big)(1-w)-ab\frac{z}{w}(1-w)^2=0, \quad z\in\mathbb{D}. \tag{15}$$

From this equation it follows for $z=z_0$, since $z_0w'(z_0)/w(z_0)>1\geqslant r>|z_0|$ and $d>0$, in particular that $w(z_0)\neq 1$, i.e. $G(z_0)\neq\infty$, i.e. $F(a,b,c,z_0)\neq 0$. Now we still define for $z\in\mathbb{D}$ the function

$$p(z):=\frac{1+w(z)}{1-w(z)}=1+\frac{2}{d}G(z), \quad \text{so that} \quad w(z)=\frac{p(z)-1}{p(z)+1}. \tag{16}$$

Substituting into (15) the three relations $w=(p-1)/(p+1)$ and $1-w=2/(p+1)$ and $(1-w)^2/w=4/((p+1)(p-1))$, and multiplying the result by $p^2-1=(p+1)(p-1)$, we finally obtain the equation

$$\begin{aligned} &d(1-z)\left(\big(p^2-1\big)\frac{zw'}{w}+dp^2+1\right)+2d\big(c-1-d-(a+b-d)z\big)p \\ &\quad -d(2c-1-d)-\big((2a-d)(2b-d)-d\big)z=0, \quad z\in\mathbb{D}. \end{aligned} \tag{17}$$

As $|w(z_0)|=1\neq w(z_0)$ we get from (16) that $p(z_0)\neq\infty$ and $\mathrm{Re}(p(z_0))=0$, i.e. $p(z_0)=\mathrm{i}t_0$ with $t_0:=\mathrm{Im}(p(z_0))\in\mathbb{R}$. Setting $s_0:=z_0w'(z_0)/w(z_0)\in(1,+\infty)$ we can then rewrite Eq. (17) for $z=z_0$ in the form

$$\begin{aligned} &\Big(d\big(1-s_0-(d+s_0)t_0^2\big)+(2a-d)(2b-d)-d+\mathrm{i}2d(a+b-d)t_0\Big)z_0 \\ &\quad =d\big(1-s_0-(d+s_0)t_0^2\big)-d(2c-1-d)+\mathrm{i}2d(c-1-d)t_0. \end{aligned} \tag{18}$$

From (18) it follows, as $|z_0|<r$, that (11) does not hold for $s=s_0$ and $t=t_0$. The proof of Theorem 14 is now finished. □

The next thing that we should do now is to study under which conditions on the numbers a,b,c,r,d it is true that (11) does hold for all $s\in(1,+\infty)$ and all $t\in\mathbb{R}$. But for the sake of simplicity we consider only the case of real parameters a,b,c and $r=1$. In this case we obtain the following result.

Theorem 15. *Suppose that $a,b\in\mathbb{R}\setminus\{0\}$, $c\in(\frac{1}{2},+\infty)$, $\alpha\in(-\infty,1)$ and $d:=2(1-\alpha)>0$. If either in the case $ab(a-d)(b-d)\geqslant 0$ it holds that*

$$\begin{aligned} &d(2c-1-d)\geqslant\big|(2a-d)(2b-d)-d\big|, \\ &\quad \textit{i.e.} \quad (c-a-b-1)d+2ab\geqslant 0 \quad \textit{and} \quad (c+a+b-d)d-2ab\geqslant 0, \\ &\quad \textit{i.e.} \quad d(c-1)\geqslant d(a+b)-2ab\geqslant d(d-c), \end{aligned} \tag{19}$$

or if in the case $ab(a-d)(b-d)<0$ it holds that

$$\begin{aligned} &d(2c-1-d)\geqslant\sqrt{\big((2a-d)(2b-d)-d\big)^2-16ab(a-d)(b-d)}, \\ &\quad \textit{i.e. if} \quad (c-a-b-1)(c+a+b-d)d+2ab(d+1)\geqslant 0 \\ &\quad \textit{and if} \quad 2c-1-d>0 \quad \textit{or} \quad (c-a-b-1)d+2ab>0 \\ &\qquad\qquad \textit{or} \quad (c+a+b-d)d-2ab>0, \end{aligned} \tag{20}$$

then $\sigma(zF(a,b,c,z))\geqslant\alpha\geqslant\frac{3}{2}-c$. In particular this conclusion also holds if

$$d(2c-1-d)\geqslant\big|(2a-d)(2b-d)-d\big| \quad \textit{and} \quad |c-1-d|\geqslant|a+b-d|. \tag{21}$$

Proof. As $a, b, c, d, s, t \in \mathbb{R}$ we have

$$\begin{aligned}
&\bigl|d\bigl(1-s-(d+s)t^2\bigr)-d(2c-1-d)+\mathrm{i}2d(c-1-d)t\bigr|^2 \\
&\quad -\bigl|d\bigl(1-s-(d+s)t^2\bigr)+(2a-d)(2b-d)-d+\mathrm{i}2d(a+b-d)t\bigr|^2 \\
&= \bigl(d\bigl(1-s-(d+s)t^2\bigr)-d(2c-1-d)\bigr)^2+4d^2(c-1-d)^2t^2 \\
&\quad -\bigl(d\bigl(1-s-(d+s)t^2\bigr)+(2a-d)(2b-d)-d\bigr)^2-4d^2(a+b-d)^2t^2 \\
&= d^2(2c-1-d)^2-\bigl((2a-d)(2b-d)-d\bigr)^2 \\
&\quad -2d\bigl(1-s-(d+s)t^2\bigr)\bigl(d(2c-1-d)+(2a-d)(2b-d)-d\bigr) \\
&\quad +4d^2\bigl((c-1-d)^2-(a+b-d)^2\bigr)t^2 \\
&= d^2(2c-1-d)^2-\bigl((2a-d)(2b-d)-d\bigr)^2 \\
&\quad +2d\bigl(d(2c-1-d)+(2a-d)(2b-d)-d\bigr)(s-1) \\
&\quad +\Bigl(2d\bigl(d(2c-1-d)+(2a-d)(2b-d)-d\bigr)(d+s) \\
&\quad +4d^2\bigl((c-1-d)^2-(a+b-d)^2\bigr)\Bigr)t^2.
\end{aligned}$$

From this it follows, since $d > 0$, that for $r = 1$ the condition (11) does hold for all $s \in (1, +\infty)$ and all $t \in \mathbb{R}$ if and only if the following four conditions are satisfied:

$$\begin{aligned}
&d(2c-1-d) \geqslant 0, \\
&d^2(2c-1-d)^2-\bigl((2a-d)(2b-d)-d\bigr)^2 \geqslant 0, \\
&d(2c-1-d)+(2a-d)(2b-d)-d \geqslant 0, \\
&2d\bigl(d(2c-1-d)+(2a-d)(2b-d)-d\bigr)(d+1) \\
&\quad +4d^2\bigl((c-1-d)^2-(a+b-d)^2\bigr) \geqslant 0.
\end{aligned}$$

Here the first condition $d(2c-1-d) \geqslant 0$ is needed to guarantee that always

$$d\bigl(1-s-(d+s)t^2\bigr)-d(2c-1-d) \leqslant d\bigl(1-s-(d+s)t^2\bigr) \leqslant d(1-s) < 0,$$

so that in (11) the lower modulus can never vanish. From this first condition it also follows, as $d = 2(1-\alpha) > 0$, that $c \geqslant (d+1)/2 > 1/2$ and $\alpha \geqslant 3/2 - c$.

Remark. Here it pays that in the proof of Theorem 14 it was possible to show that $s_0 = 1$ cannot occur, otherwise we would now need that $d(2c-1-d) > 0$.

Above the first three conditions taken together are equivalent to the condition

$$d(2c-1-d) \geqslant \bigl|(2a-d)(2b-d)-d\bigr|.$$

From the two equations

$$\begin{aligned}
&d(2c-1-d)+(2a-d)(2b-d)-d = 2\bigl((c-a-b-1)d+2ab\bigr), \\
&d(2c-1-d)-(2a-d)(2b-d)+d = 2\bigl((c+a+b-d)d-2ab\bigr),
\end{aligned}$$

we obtain the relations

$$
\begin{aligned}
&2d\big(d(2c-1-d)+(2a-d)(2b-d)-d\big)(d+1)\\
&\quad+4d^2\big((c-1-d)^2-(a+b-d)^2\big)\\
&=4d\big((c-a-b-1)d+2ab\big)(d+1)+4d^2\big((c-1-d)^2-(a+b-d)^2\big)\\
&=4d\big(\big((c-a-b-1)d+2ab\big)(d+1)+d(c-a-b-1)(c+a+b-2d-1)\big)\\
&=4d\big((c-a-b-1)(c+a+b-d)d+2ab(d+1)\big)\\
&=4\big((c-a-b-1)d(c+a+b-d)d\\
&\quad+2abd(c+a+b-d-c+a+b+1-2a-2b+2d)\big)\\
&=4\Big(\big((c-a-b-1)d+2ab\big)\big((c+a+b-d)d-2ab\big)+4ab(a-d)(b-d)\Big)\\
&=d^2(2c-1-d)^2-\big((2a-d)(2b-d)-d\big)^2+16ab(a-d)(b-d).
\end{aligned}
$$

From the preceding relations we can then conclude that taken together the four conditions from above are in the case $ab(a-d)(b-d)\geqslant 0$ equivalent to (19) respectively that in the case $ab(a-d)(b-d)<0$ they are equivalent to (20), and moreover that in particular these four conditions are a consequence of (21). The proof of Theorem 15 is now finished. □

In the particular case $\alpha=0$, i.e. $d=2$, Theorem 15 takes the following form.

Corollary 16. *Let $a,b,c\in\mathbb{R}\setminus\{0\}$. If either in the case $ab(a-2)(b-2)\geqslant 0$ it holds that*

$$c\geqslant\frac{3}{2}+\left|(a-1)(b-1)-\frac{1}{2}\right|,\quad \textit{i.e.}\quad c-1\geqslant(a-1)(b-1)\geqslant 2-c,$$

or if in the case $ab(a-2)(b-2)<0$ it holds that

$$c\geqslant\frac{3}{2}+\sqrt{\left((a-1)(b-1)-\frac{1}{2}\right)^2-ab(a-2)(b-2)},$$

$$\textit{i.e. if}\quad c>\frac{3}{2}\quad\textit{and if}\quad (c-a-b-1)(c+a+b-2)+3ab\geqslant 0,$$

then $\sigma(zF(a,b,c,z))\geqslant 0$, i.e. the function $z\mapsto zF(a,b,c,z)$ is starlike. In particular this conclusion also holds if

$$c\geqslant\frac{3}{2}+\left|(a-1)(b-1)-\frac{1}{2}\right|\quad\textit{and}\quad |c-3|\geqslant|a+b-2|.$$

It is also worth to note, since $(2a-d)(2b-d)=4ab-2(a+b)d+d^2$, that in the proof of Theorem 15 we have not really used that $a,b\in\mathbb{R}\setminus\{0\}$ but indeed only that $ab\in\mathbb{R}\setminus\{0\}$ and $a+b\in\mathbb{R}$. These two conditions are also satisfied if $a\in\mathbb{C}\setminus\{0\}$ and $b=\bar{a}$, in which case we have $ab(a-d)(b-d)=|a|^2|a-d|^2\geqslant 0$. Therefore we also obtain from the proof of Theorem 15 the following corollary.

Corollary 17. *Suppose that $a\in\mathbb{C}\setminus\{0\}$, $c\in(\frac{1}{2},+\infty)$, $\alpha\in(-\infty,1)$ and $d:=2(1-\alpha)>0$. Then $\sigma(zF(a,\bar{a},c,z))\geqslant\alpha\geqslant\frac{3}{2}-c$ provided that*

$$d(2c-1-d)\geqslant\big||2a-d|^2-d\big|,\quad\textit{i.e. if}\quad d(c-1)\geqslant 2\big(d\,\mathrm{Re}(a)-|a|^2\big)\geqslant d(d-c).$$

In particular $\sigma(zF(a,\bar{a},c,z)) \geqslant 0$, *i.e. the function* $z \mapsto zF(a,\bar{a},c,z)$ *is starlike, provided that*

$$c \geqslant \frac{3}{2} + \left| |a-1|^2 - \frac{1}{2} \right|, \quad \textit{i.e. if} \quad c-1 \geqslant |a-1|^2 \geqslant 2-c.$$

Since we are interested in the order of starlikeness $\sigma(zF(a,b,c,z))$, the next thing we should do is to determine for $a,b,c \in \mathbb{R} \setminus \{0\}$, $c > 1/2$, the smallest possible $d = d(a,b,c) > 0$ for which either $ab(a-d)(b-d) \geqslant 0$ and (19) holds or for which $ab(a-d)(b-d) < 0$ and (20) holds. For then $\alpha = 1 - d/2$ would be the largest lower bound obtainable from Theorem 15 for $\sigma(zF(a,b,c,z))$. But this determination of $d(a,b,c)$ would be rather lengthy, not very illuminating and would need the consideration of many different cases. Therefore we pick out the only case in which we obtain a sharp result, namely Theorem 1(d).

Proof of Theorem 1(d). We suppose that $a < 0 < b$ and $c \geqslant b - a + 1$. Then $c - a - b - 1 \geqslant -2a > 0$ and for $\alpha := 1 + ab/(c-a-b-1) < 1$ it follows that $0 < d := 2(1-\alpha) = -2ab/(c-a-b-1) \leqslant b$, that $ab(a-d)(b-d) \geqslant 0$, that $(c-a-b-1)d + 2ab = 0$, that $(c+a+b-d)d - 2ab = cd + (d-a)(b-d) - ab > 0$, and also that $c-1-d > a+b-d$ as well as $c-1-d \geqslant c-1-b \geqslant -a \geqslant d-a-b$. Accordingly in Theorem 15 the condition (19), which is required in the case $ab(a-d)(b-d) \geqslant 0$, and even the stronger condition (21) are both satisfied, and so we obtain from Theorem 15 that $\sigma(zF(a,b,c,z)) \geqslant \alpha = 1 - d/2 \geqslant 1 - b/2$. Therefore we have in particular $F(a,b,c,z) \neq 0$ for all $z \in \mathbb{D}$ and, since also $c-a > c-a-b > c-a-b-1 \geqslant -2a > 0$ as well as $c > c-b \geqslant -a+1 > 1$, we obtain then from Corollary 11(a) that also $\sigma(zF(a,b,c,z)) \leqslant \alpha$. Taken together we have $\sigma(zF(a,b,c,z)) = \alpha$, so that now Theorem 1(d) is proved. □

Since we have $zF'(a,b,c,z) = (ab/c)zF(a+1,b+1,c+1,z)$ and $z(zF(a,b,2,z))' = zF(a,b,1,z)$ and $z(zF(a,1,c,z))' = zF(a,2,c,z)$, we get from Theorem 15 and Corollaries 16 and 17 the following seven corollaries on orders of convexity.

Corollary 18. *Let be* $a,b,c \in \mathbb{R} \setminus \{0,-1\}$, $c \in (-\frac{1}{2}, +\infty)$, $\alpha \in (-\infty, 1)$ *and* $d := 2(1-\alpha) > 0$. *If either in the case* $(a+1)(b+1)(a+1-d)(b+1-d) \geqslant 0$ *it holds that*

$$\begin{aligned} &d(2c+1-d) \geqslant \left|(2a+2-d)(2b+2-d) - d\right|, \\ &\quad \textit{i.e.} \quad (c-a-b-2)d + 2(a+1)(b+1) \geqslant 0 \quad \textit{and} \\ &\quad\quad (c+a+b+3-d)d - 2(a+1)(b+1) \geqslant 0, \\ &\quad \textit{i.e.} \quad dc \geqslant d(a+b+2) - 2(a+1)(b+1) \geqslant d(d-c-1), \end{aligned}$$

or if in the case $(a+1)(b+1)(a+1-d)(b+1-d) < 0$ *it holds that*

$$\begin{aligned} &d(2c+1-d) \\ &\quad \geqslant \sqrt{\big((2a+2-d)(2b+2-d) - d\big)^2 - 16(a+1)(b+1)(a+1-d)(b+1-d)}, \\ &\quad \textit{i.e. if} \quad (c-a-b-2)(c+a+b+3-d)d + 2(a+1)(b+1)(d+1) \geqslant 0 \\ &\quad \textit{and if} \quad 2c+1-d > 0 \quad \textit{or} \quad (c-a-b-2)d + 2(a+1)(b+1) > 0 \\ &\quad\quad \textit{or} \quad (c+a+b+3-d)d - 2(a+1)(b+1) > 0, \end{aligned}$$

then $\kappa(F(a,b,c,z)) \geqslant \alpha \geqslant \frac{1}{2} - c$. *In particular this conclusion also holds if*

$$d(2c+1-d) \geqslant \left|(2a+2-d)(2b+2-d)-d\right| \quad \text{and} \quad |c-d| \geqslant |a+b+2-d|.$$

Corollary 19. *Let* $a,b,c \in \mathbb{R} \setminus \{0,-1\}$. *If either in the case* $(a^2-1)(b^2-1) \geqslant 0$ *it holds that*

$$c \geqslant \frac{1}{2} + \left|ab - \frac{1}{2}\right|, \quad \text{i.e.} \quad c \geqslant ab \geqslant 1-c,$$

or if in the case $(a^2-1)(b^2-1) < 0$ *it holds that*

$$c \geqslant \frac{1}{2} + \sqrt{\left(ab - \frac{1}{2}\right)^2 - (a^2-1)(b^2-1)},$$
$$\text{i.e. if} \quad c > \frac{1}{2} \quad \text{and if} \quad (c-a-b-2)(c+a+b+1) + 3(a+1)(b+1) \geqslant 0,$$

then $\kappa(F(a,b,c,z)) \geqslant 0$, *i.e. the function* $z \mapsto F(a,b,c,z)$ *is convex. In particular this conclusion also holds if*

$$c \geqslant \frac{1}{2} + \left|ab - \frac{1}{2}\right| \quad \text{and} \quad |c-2| \geqslant |a+b|.$$

Corollary 20. *Let be* $a, c \in \mathbb{C} \setminus \{0,-1\}$, $c \in (-\frac{1}{2}, +\infty)$, $\alpha \in (-\infty, 1)$ *and* $d := 2(1-\alpha) > 0$. *Then* $\kappa(F(a,\bar{a},c,z)) \geqslant \alpha \geqslant \frac{1}{2} - c$ *provided that*

$$d(2c+1-d) \geqslant \left||2a+2-d|^2 - d\right|,$$
$$\text{i.e. if} \quad dc \geqslant 2\left(d\,\mathrm{Re}(a+1) - |a+1|^2\right) \geqslant d(d-c-1).$$

In particular $\kappa(F(a,\bar{a},c,z)) \geqslant 0$, *i.e. the function* $z \mapsto F(a,\bar{a},c,z)$ *is convex, provided that*

$$c \geqslant \frac{1}{2} + \left||a|^2 - \frac{1}{2}\right|, \quad \text{i.e. if} \quad c \geqslant |a|^2 \geqslant 1-c.$$

Corollary 21. *Suppose that* $a, b \in \mathbb{R} \setminus \{0\}$, $\alpha \in [\frac{1}{2}, 1)$ *and* $d := 2(1-\alpha) \in (0,1]$. *If either in the case* $ab(a-d)(b-d) \geqslant 0$ *it holds that*

$$d(1-d) \geqslant \left|(2a-d)(2b-d)-d\right|,$$
$$\text{i.e.} \quad -(a+b)d + 2ab \geqslant 0 \quad \text{and} \quad (a+b+1-d)d - 2ab \geqslant 0,$$
$$\text{i.e.} \quad d(1-d) \geqslant 2ab - d(a+b) \geqslant 0,$$

or if in the case $ab(a-d)(b-d) < 0$ *it holds that*

$$d(1-d) \geqslant \sqrt{\left((2a-d)(2b-d)-d\right)^2 - 16ab(a-d)(b-d)},$$
$$\text{i.e. if} \quad -(a+b)(a+b+1-d)d + 2ab(d+1) \geqslant 0 \quad \text{and if}$$
$$d < 1 \quad \text{or} \quad -(a+b)d + 2ab > 0 \quad \text{or} \quad (a+b+1-d)d - 2ab > 0,$$

then $\kappa(zF(a,b,2,z)) \geqslant \alpha$. *In particular this conclusion also holds if*

$$d(1-d) \geqslant \left|(2a-d)(2b-d)-d\right| \quad \text{and} \quad d \geqslant |a+b-d|.$$

Corollary 22. *Let $a \in \mathbb{C} \setminus \{0\}$, $\alpha \in [\frac{1}{2}, 1)$ and $d := 2(1-\alpha) \in (0,1]$. Then $\kappa(zF(a, \bar{a}, 2, z)) \geqslant \alpha$ provided that*

$$d(1-d) \geqslant \big||2a-d|^2 - d\big|, \quad \textit{i.e. if} \quad d(1-d) \geqslant 2\big(|a|^2 - d\,\mathrm{Re}(a)\big) \geqslant 0.$$

Corollary 23. *Suppose that $a \in \mathbb{R} \setminus \{0\}$, $c \in (\frac{1}{2}, +\infty)$, $\alpha \in (-\infty, 1)$ and $d := 2(1-\alpha) > 0$. If either in the case $a(a-d)(2-d) \geqslant 0$ it holds that*

$$\begin{aligned} &d(2c-1-d) \geqslant \big|(2a-d)(4-d)-d\big|, \\ &\quad \textit{i.e.} \quad (c-a-3)d + 4a \geqslant 0 \quad \textit{and} \quad (c+a+2-d)d - 4a \geqslant 0, \\ &\quad \textit{i.e.} \quad d(c-1) \geqslant d(a+2) - 4a \geqslant d(d-c), \end{aligned}$$

or if in the case $a(a-d)(2-d) < 0$ it holds that

$$\begin{aligned} &d(2c-1-d) \geqslant \sqrt{\big((2a-d)(4-d)-d\big)^2 - 32a(a-d)(2-d)}, \\ &\quad \textit{i.e. if} \quad (c-a-3)(c+a+2-d)d + 4a(d+1) \geqslant 0 \quad \textit{and if} \\ &\quad 2c-1-d > 0 \quad \textit{or} \quad (c-a-3)d + 4a > 0 \quad \textit{or} \quad (c+a+2-d)d - 4a > 0, \end{aligned}$$

then $\kappa(zF(a,1,c,z)) \geqslant \alpha \geqslant \frac{3}{2} - c$. In particular this conclusion also holds if

$$d(2c-1-d) \geqslant \big|(2a-d)(4-d)-d\big| \quad \textit{and} \quad |c-1-d| \geqslant |a+2-d|.$$

Corollary 24. *If $a, c \in \mathbb{R} \setminus \{0\}$ then $\kappa(zF(a,1,c,z)) \geqslant 0$, i.e. the function $z \mapsto zF(a,1,c,z)$ is convex, provided that*

$$c \geqslant \frac{3}{2} + \left|a - \frac{3}{2}\right|, \quad \textit{i.e. if} \quad c \geqslant a \geqslant 3 - c.$$

4. Applications to convolutions

In this section we consider for two functions $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$ in $\mathcal{H}$, i.e. which are analytic in $\mathbb{D}$, their convolution or Hadamard product

$$(f * g)(z) := \sum_{n=0}^{\infty} a_n b_n z^n,$$

which is also analytic in $\mathbb{D}$, i.e. $f * g \in \mathcal{H}$. For subsets $\mathcal{M}$ and $\mathcal{N}$ of $\mathcal{H}$ we define $\mathcal{M} * \mathcal{N} := \{f * g\colon\ f \in \mathcal{M},\ g \in \mathcal{N}\}$ and we also set $f * \mathcal{N} := \{f\} * \mathcal{N}$. For an introduction to convolutions in the present context see [15] or [16].

In particular we consider convolutions for functions in the classes $\mathcal{S}_\alpha$ and $\mathcal{K}_\alpha$ of normalized starlike respectively convex functions of order $\alpha \in (-\infty, 1]$. A function $f \in \mathcal{H}$ belongs to $\mathcal{S}_\alpha$ if and only if $f(0) = 0$, $f'(0) = 1$ and $\sigma(f) \geqslant \alpha$, and $f \in \mathcal{H}$ belongs to $\mathcal{K}_\alpha$ if and only if $f(0) = 0$, $f'(0) = 1$ and $\kappa(f) \geqslant \alpha$. So $f \in \mathcal{K}_\alpha$ if and only if $zf' \in \mathcal{S}_\alpha$. Special elements of $\mathcal{S}_\alpha$ and $\mathcal{K}_\alpha$ are

$$s_\alpha(z) := \frac{z}{(1-z)^{2(1-\alpha)}} \quad \text{respectively} \quad k_\alpha(z) := \int_0^1 \frac{z}{(1-tz)^{2(1-\alpha)}}\,\mathrm{d}t.$$

They satisfy $zk_\alpha' = s_\alpha$ as well as $\kappa(k_\alpha) = \sigma(s_\alpha) = \alpha$ and are extremal in various aspects in their respective classes. So, for instance, Ruscheweyh [16, Theorem 3.9, p. 119] has obtained the following theorem.

Theorem 25. *(See Ruscheweyh [16, Theorem 3.9, p. 119].) Suppose that* $f \in \mathcal{H}$ *with* $f(0) = 0$ *and* $f'(0) = 1$ *and that* $-\infty < \alpha \leqslant \beta \leqslant 1$. *Then the inclusion* $f * \mathcal{S}_\alpha \subseteq \mathcal{S}_\beta$ *holds if and only if* $f * s_\alpha \in \mathcal{S}_\beta$.

Using this theorem we can obtain from our results on the order of starlikeness of the shifted hypergeometric function three consequences with respect to the mapping properties of the Carlson–Shaffer convolution operator $L(a,c)$, see [3], which for $a, c \in \mathbb{C}$, $c \notin -\mathbb{N}$, is defined on $\mathcal{H}_0 := \{f \in \mathcal{H}\colon\ f(0) = 0\}$ by means of the multiplier function $z \mapsto zF(a,1,c,z)$, i.e.

$$L(a,c)\colon \mathcal{H}_0 \to \mathcal{H}_0, \quad f \mapsto L(a,c)f \quad \text{with } L(a,c)f(z) := zF(a,1,c,z) * f(z).$$

If $a \in -\mathbb{N}$ then $zF(a,1,c,z)$ is a polynomial of degree $1-a$ and so $L(a,c)f$ is a polynomial of degree at most $1-a$. But if $a \notin -\mathbb{N}$ then $L(a,c)$ is bijective with inverse $L(a,c)^{-1} = L(c,a)$. In particular $L(2,1)f(z) = zf'(z)$ and the well-known Euler integral representation for the hypergeometric function yields that

$$L(a,c)f(z) = \frac{z\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 \frac{f(tz)}{tz} t^{a-1}(1-t)^{c-a-1}\,\mathrm{d}t, \quad \operatorname{Re}(c) > \operatorname{Re}(a) > 0.$$

Using Theorem 25 we obtain, since with $b = 2 - 2\alpha$ we have $zF(a,b,c,z) = zF(a,1,c,z) * s_\alpha(z)$, from the three parts (a), (d), (g) of Theorem 1 the following corollary which generalizes results of Ruscheweyh and Singh, see [17, pp. 6–8].

Corollary 26.

(a) *Suppose that* $\alpha < 1$, *that* $0 < a < c$, *that* $0 < b := 2 - 2\alpha \leqslant c$ *and set*

$$\beta := \sigma\bigl(zF(a,b,c,z)\bigr) = 1 - \frac{F'(a,b,c,-1)}{F(a,b,c,-1)} = 1 - b\frac{\int_0^1 (1+t)^{-b-1} t^a (1-t)^{c-a-1}\,\mathrm{d}t}{\int_0^1 (1+t)^{-b} t^{a-1} (1-t)^{c-a-1}\,\mathrm{d}t}.$$

Then $\alpha < 1 - \frac{ab}{c+\max(a,b)} \leqslant \beta \leqslant 1 - \frac{ab}{2c} < 1$ *and* $L(a,c)\mathcal{S}_\alpha \subseteq \mathcal{S}_\beta$, *but* $L(a,c)\mathcal{S}_\alpha \nsubseteq \mathcal{S}_\gamma$ *for all* $\gamma \in (\beta, 1]$.

(b) *Suppose that* $\alpha < 1$, *that* $a < 0$, *with* $b := 2 - 2\alpha$ *that* $c \geqslant b - a + 1$, *and set*

$$\beta := \sigma\bigl(zF(a,b,c,z)\bigr) = 1 + \frac{F'(a,b,c,1)}{F(a,b,c,1)} = 1 + \frac{ab}{c-a-b-1}.$$

Then $\alpha \leqslant \beta < 1$ *and* $L(a,c)\mathcal{S}_\alpha \subseteq \mathcal{S}_\beta$, *but* $L(a,c)\mathcal{S}_\alpha \nsubseteq \mathcal{S}_\gamma$ *for all* $\gamma \in (\beta, 1]$.

(c) *Suppose that* $\alpha < \frac{1}{2}$, *that* $1 < a < c \leqslant b := 2 - 2\alpha \leqslant c - a + 1$ *and set*

$$\beta := \sigma\bigl(zF(a,b,c,z)\bigr) = 1 + \frac{(c-a)(c-b)}{a+b-c-1} + \frac{c-a-b}{2}.$$

Then $\alpha \leqslant \beta < \frac{1}{2}$ *and* $L(a,c)\mathcal{S}_\alpha \subseteq \mathcal{S}_\beta$, *but* $L(a,c)\mathcal{S}_\alpha \nsubseteq \mathcal{S}_\gamma$ *for all* $\gamma \in (\beta, 1]$.

Here, in (a) we have used in addition the symmetry of $F(a,b,c,z)$ and Theorem 2. Note also that the two parts (c) and (f) of Theorem 1 do not yield further results because we need that $\sigma(zF(a,b,c,z)) \geqslant 1-b/2$ in order to be able to apply Theorem 25. It may further be noted that in all three parts of Corollary 26 the multiplier function $z \mapsto zF(a,1,c,z)$ is prestarlike of order α, and consequently convex if $\alpha \leqslant 0$, and that in all three parts it can be shown that the inclusion $L(a,c)\mathcal{S}_\alpha \subseteq \mathcal{S}_\beta$ is indeed a strict inclusion.

Since $L(1,2)\mathcal{S}_\alpha = \mathcal{K}_\alpha$ Corollary 26(a) contains as the special case $a=1$ and $c=2$ the well-known result concerning the determination of the order of starlikeness of the convex functions of order α, see [11,22] and [23], in particular $\mathcal{K}_0 \subseteq \mathcal{S}_{1/2}$, which has been used for the last statement in Theorem 1. Since $L(1,2)$ is bijective and commutes with $L(a,c)$, Corollary 26 holds likewise with $\mathcal{K}_\alpha$, $\mathcal{K}_\beta$ and $\mathcal{K}_\gamma$ in place of $\mathcal{S}_\alpha$, $\mathcal{S}_\beta$ respectively $\mathcal{S}_\gamma$.

For $\beta < 1$ and $a = 2-2\beta$ we have the relations $zF(a,1,1,z) = s_\beta(z)$ and $zF(a,1,2,z) = zF(1,1,2,z) * zF(a,1,1,z) = L(1,2)s_\beta(z) = k_\beta(z)$, and consequently also $L(a,1)\mathcal{S}_\alpha = s_\beta * \mathcal{S}_\alpha$ as well as $L(a,2)\mathcal{S}_\alpha = k_\beta * \mathcal{S}_\alpha = s_\beta * \mathcal{K}_\alpha$. If moreover $\alpha < 1$ and $b = 2-2\alpha$ then we also have $zF(a,b,1,z) = (s_\alpha * s_\beta)(z)$ as well as $zF(a,b,2,z) = (k_\alpha * s_\beta)(z)$. Using these relations and Theorem 25, we can also deduce from Corollary 26(a) the following corollary which already has been obtained in [8, Theorems 1.8, 1.9] and which renders more precisely the well-known inclusions $\mathcal{S}_{1/2} * \mathcal{S}_{1/2} \subseteq \mathcal{S}_{1/2}$ and $\mathcal{K}_0 * \mathcal{K}_0 \subseteq \mathcal{K}_0$, which first have been proved by Ruscheweyh and Sheil-Small, see [10,14–16,18,20].

Corollary 27. *(See [8, Theorems 1.8, 1.9].)*

(a) *If* $\alpha \in [\frac{1}{2}, 1)$ *and* $\beta \in (\frac{1}{2}, 1)$ *then*

$$\gamma = \sigma(s_\alpha * s_\beta) = -\frac{(s_\alpha * s_\beta)'(-1)}{(s_\alpha * s_\beta)(-1)} = 1 - 2(1-\alpha)\frac{\int_0^1 (1+t)^{2\alpha-3} t^{2-2\beta}(1-t)^{2\beta-2}\,\mathrm{d}t}{\int_0^1 (1+t)^{2\alpha-2} t^{1-2\beta}(1-t)^{2\beta-2}\,\mathrm{d}t}$$

is the largest number $\gamma = \gamma(\alpha,\beta) \in [\frac{1}{2}, 1]$ *such that* $\mathcal{S}_\alpha * \mathcal{S}_\beta \subseteq \mathcal{S}_\gamma$, *and we have*

$$\gamma\left(\alpha, \frac{1}{2}\right) = \alpha \leqslant \max(\alpha,\beta) \leqslant \gamma(\alpha,\beta) \leqslant 1 - 2(1-\alpha)(1-\beta) \leqslant 1 = \gamma(\alpha, 1).$$

(b) *If* $\alpha \in [0, 1)$ *and* $\beta \in (0, 1)$ *then*

$$\delta = \sigma(k_\alpha * s_\beta) = -\frac{(k_\alpha * s_\beta)'(-1)}{(k_\alpha * s_\beta)(-1)} = 1 - 2(1-\alpha)\frac{\int_0^1 (1+t)^{2\alpha-3} t^{2-2\beta}(1-t)^{2\beta-1}\,\mathrm{d}t}{\int_0^1 (1+t)^{2\alpha-2} t^{1-2\beta}(1-t)^{2\beta-1}\,\mathrm{d}t}$$

is the largest number $\delta = \delta(\alpha,\beta) \in [0, 1]$ *such that* $\mathcal{K}_\alpha * \mathcal{S}_\beta \subseteq \mathcal{S}_\delta$, *and we have*

$$\delta(\alpha, 0) = \alpha \leqslant \max(\alpha,\beta) \leqslant \delta(\alpha,\beta) \leqslant 1 - (1-\alpha)(1-\beta) \leqslant 1 = \delta(\alpha, 1).$$

5. Concluding remarks

In Theorem 1 the order of starlikeness $\sigma(zF(a,b,c,z))$ of the shifted hypergeometric function $z \mapsto zF(a,b,c,z)$ has been determined for all parameters $a,b,c \in \mathbb{R}$ with $c > 0$ and (without loss of generality) with $a \leqslant b$ except for the following five ranges:

(a) $a < -1, \quad 0 < b < c < b-a+1,$ (b) $0 < a < c < b-a+1, \quad c+1 < b,$

(c) $a \leqslant b < 0 < c,$ (d) $0 < c < a \leqslant b,$ (e) $a < 0 < c < b.$

Then, naturally, one may ask what happens for parameters in these five ranges. Here, of course, the two ranges (a) and (b) are linked by the Euler identity (2) (respectively by (5)) and by Theorem 5, and likewise the two ranges (c) and (d) are linked.

For parameters in the range (e) we obtain the answer from the following result which was obtained by Klein [7] and by Hurwitz [5].

Theorem 28. *(See Klein [7], Hurwitz [5].) If* $a < 0 < c < b$ *then* $F(a, b, c, z)$ *has in the open interval* $(0, 1)$ *exactly* $-[a]$ *(simple) zeros if* $c \leqslant a + b$ *respectively* $-[c - b]$ *(simple) zeros if* $c \geqslant a + b$.

Here $[x]$ denotes the integer part of $x \in \mathbb{R}$, i.e. $[x] \in \mathbb{Z}$ with $[x] \leqslant x < [x] + 1$. Indeed Hurwitz did not consider the case where $a = -1, -2, -3, \ldots$, but in this case the result also follows from (2) and from the fact that for $\alpha, \beta > -1$ and $n \in \mathbb{N}$ the Jacobi polynomials

$$P_n^{(\alpha,\beta)}(x) = \frac{(\alpha + 1)_n}{n!} F\left(-n, n + \alpha + \beta + 1, \alpha + 1, \frac{1 - x}{2}\right)$$

are orthogonal polynomials on the interval $[-1, 1]$, with respect to the weight function $x \mapsto (1 - x)^\alpha (1 + x)^\beta$, and accordingly their zeros are all simple and included in $(-1, 1)$.

Thus, if $a < 0 < c < b$ then $F(a, b, c, z)$ has at least one zero in $(0, 1)$ and with Rolle's theorem it follows that also $(zF(a, b, c, z))'$ has a zero in $(0, 1)$, so that $\sigma(zF(a, b, c, z))$ and $\kappa(zF(a, b, c, z))$ are both not defined if $a < 0 < c < b$.

Looking at the two parts (c) and (d) of Theorem 1 one may wonder whether for $a < -1$ and $b > 0$ the condition $c \geqslant b - a + 1$ is sharp for the truth of the relation $\sigma(zF(a, b, c, z)) = 1 + F'(a, b, c, 1)/F(a, b, c, 1) = 1 + ab/(c - a - b - 1)$ or whether this relation also still remains true for some parameters $c < b - a + 1$, i.e. for some parameters in the range (a) from above. Indeed, one can show that for $a < -1$ and $b > 0$ this condition $c \geqslant b - a + 1$ is sharp for the truth of the before-mentioned relation, i.e. the bound $b - a + 1$ cannot be replaced by a smaller bound. For using (3) and (9) one can show by means of one page of calculations for $G(a, b, c, z) := zF'(a, b, c, z)/F(a, b, c, z)$ that

$$\lim_{\substack{z \to 1 \\ z \in \mathbb{D}}} G'(a, b, c, z) = \frac{ab(c - a - 1)(c - b - 1)}{(c - a - b - 1)^2 (c - a - b - 2)} \quad \text{if } \operatorname{Re}(c - a - b - 2) > 0,$$

and for $H(a, b, c, z) := 1 + zG''(a, b, c, z)/G'(a, b, c, z)$ that

$$\lim_{\substack{z \to 1 \\ z \in \mathbb{D}}} H(a, b, c, z) = \frac{(c - b + a - 1)(c - a + b - 1)}{(c - a - b - 1)(c - a - b - 3)} \quad \text{if } \operatorname{Re}(c - a - b - 3) > 0$$

and if $ab(c - a - 1)(c - b - 1) \neq 0$. But, if $a < -1$ and $b > 0$ then it follows that $b - a + 1 > b + 1 > a + 1 > a - b + 1$ and $b - a + 1 > a + b + 3$ and thus, if $b - a + 1 > c > \max(a + b + 3, b + 1)$, that then $\lim_{z \to 1, z \in \mathbb{D}} H(a, b, c, z) < 0$. This means that for these parameters the function $G(a, b, c, z)$ is near $z = 1$ not convex and, having real coefficients, then also not convex in the direction of the imaginary axis. Hence, for these parameters the relation $\sigma(zF(a, b, c, z)) = 1 + G(a, b, c, 1)$ cannot hold. Figures 1 and 2 illustrate this fact.

Here, for $a < -1$ and $b > 0$, one may conjecture that $G(a, b, c, z)$ is convex for $c \geqslant b - a + 1$, and one may ask for which parameters $c < b - a + 1$ it is true that $G(a, b, c, z)$ is starlike.

As in these two figures, it would seem that for all parameters in the range (a) from above the defining infimum of $\sigma(zF(a, b, c, z))$ is always attained on the boundary of $\mathbb{D}$ at two nonreal,

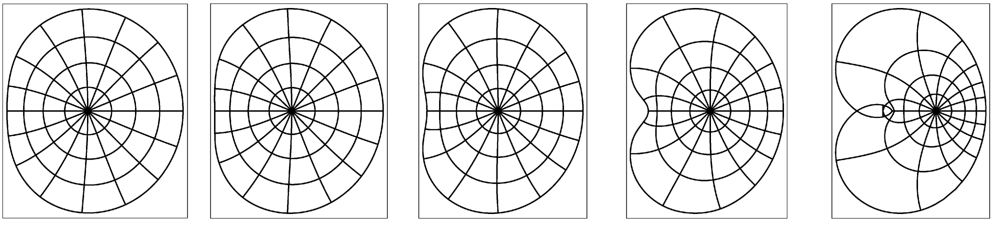

Fig. 1. $G(-3, 1, c, \mathbb{D})$ for $c = 6, 5, 4, 3, 2$, here $b - a + 1 = 5$.

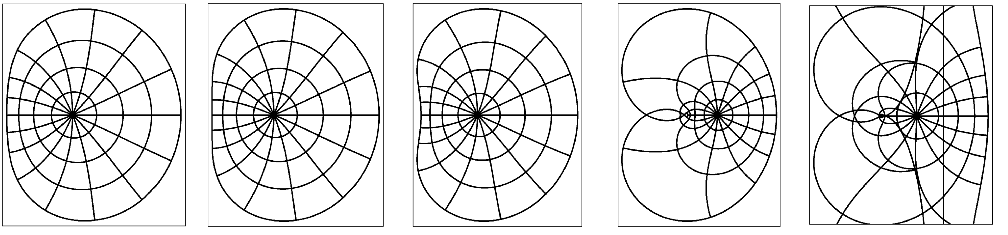

Fig. 2. $G(-5, 1, c, \mathbb{D})$ for $c = 8, 7, 6, 3, 2$, here $b - a + 1 = 7$.

complex conjugated points, as long as $F(a, b, c, z)$ is zerofree in $\mathbb{D}$. Because of the relation (5) we would likewise have the same behaviour for all parameters in the range (b) from above, as long as $F(a, b, c, z)$ is zerofree in $\mathbb{D}$. Thus, for parameters in the two ranges (a) and (b) it seems not to be possible, or at least really very difficult, to determine exactly the order of starlikeness $\sigma(zF(a, b, c, z))$.

For parameters in the two ranges (c) and (d) it seems that quite often, but not always, the defining infimum of $\sigma(zF(a, b, c, z))$ is attained at $z = -1$, as long as $F(a, b, c, z)$ is zerofree in $\mathbb{D}$. How could we prove this? I have no idea.

As already mentioned, it seems to be really much more difficult to determine for the shifted hypergeometric function $z \mapsto zF(a, b, c, z)$ its order of convexity $\kappa(zF(a, b, c, z))$ than its order of starlikeness $\sigma(zF(a, b, c, z))$. Therefore it is even interesting to know for which parameters $\kappa(zF(a, b, c, z))$ is not defined. Above we have noticed that $\kappa(zF(a, b, c, z))$ is not defined if $a < 0 < c < b$. Also $\kappa(zF(a, b, c, z))$ is not defined if the parameters a, b, c satisfy one of the four hypotheses in Theorem 1(a)–(d) and if $\sigma(zF(a, b, c, z)) < 0$. Because in Theorem 1(a) we have $\sigma(zF(a, b, c, z)) = 1 - F'(a, b, c, -1)/F(a, b, c, -1)$ and in Theorem 1(b)–(d) we have $\sigma(zF(a, b, c, z)) = 1 + F'(a, b, c, 1)/F(a, b, c, 1)$, and so, if $\sigma(zF(a, b, c, z)) < 0$, it then follows, since $1 + zF'/F = (zF)'/F$, that $(zF(a, b, c, z))'$ has a zero in $(-1, 1)$, so that $\kappa(zF(a, b, c, z))$ is not defined.

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