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Mapping Properties of Hypergeometric Functions and Convolutions of Starlike or Convex Functions of Order α

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Abstract. We determine the order of convexity of hypergeometric functions $z \mapsto F(a,b,c,z)$ as well as the order of starlikeness of shifted hypergeometric functions $z \mapsto zF(a,b,c,z)$, for certain ranges of the real parameters a, b and c. As a consequence we obtain the sharp lower bound for the order of convexity of the convolution $(f * g)(z) := \sum_{n=0}^{\infty} a_n b_n z^n$ when $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is convex of order $\alpha \in [0,1]$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$ is convex of order $\beta \in [0,1]$, and likewise we obtain the sharp lower bound for the order of starlikeness of $f * g$ when f, g are starlike of order $\alpha, \beta \in [1/2, 1]$, respectively. Further we obtain convexity in the direction of the imaginary axis for hypergeometric functions and for three ratios of hypergeometric functions as well as for the corresponding shifted expressions.

In the proofs we use the continued fraction of Gauss, a theorem of Wall which yields a characterization of Hausdorff moment sequences by means of (continued) g-fractions, and results of Merkes, Wirths and Pólya. Finally we state a subordination problem.

Keywords. Hadamard product, hypergeometric function, order of convexity, order of starlikeness, convexity in direction of the imaginary axis, continued fraction of Gauss, g-fraction, Hausdorff moment sequence.

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1. Introduction and statement of results

Let $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disk in the complex plane $\mathbb{C}$ and $\mathcal{H}$ the set of all functions analytic in $\mathbb{D}$. For a function $f \in \mathcal{H}$ with $f(0) = 0$ and $f'(0) \neq 0$ its *order of starlikeness* (with respect to zero) is defined by

$$\sigma(f) := \inf_{z \in \mathbb{D}} \operatorname{Re} \frac{zf'(z)}{f(z)} \in [-\infty, 1]$$

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and, when at least $f'(0) \neq 0$ then the *order of convexity* of f is defined by

$$\kappa(f) := \sigma(zf') = 1 + \inf_{z\in\mathbb{D}} \operatorname{Re} \frac{zf''(z)}{f'(z)} \in [-\infty, 1].$$

As is well-known, the function f is *starlike*, i.e. $\sigma(f) \geq 0$, if and only if f is univalent (i.e. one-to-one) in $\mathbb{D}$ with $f(\mathbb{D})$ being starlike with respect to zero; and f is *convex*, i.e. $\kappa(f) \geq 0$, if and only if f is univalent in $\mathbb{D}$ with $f(\mathbb{D})$ being convex. It is also known that if $\kappa(f) \geq -1/2$ then f is univalent in $\mathbb{D}$ with $f(\mathbb{D})$ being convex in (at least) one direction, see [26] and [19, p. 71, Thm. 2.24; p. 73].

1.1. Results on hypergeometric and related functions.

The *Gauss hypergeometric function* $z \mapsto F(a, b, c, z)$ depends on the three parameters $a, b, c \in \mathbb{C}$, $-c \notin \mathbb{N}_0 := \{0, 1, 2, \ldots\}$, and is defined for $z \in \mathbb{D}$ by the series expansion

$$F(a, b, c, z) := \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} z^n, \tag{1.1}$$

where $(a)_n$ is the *Pochhammer symbol*, i.e. $(a)_0 := 1$ and $(a)_{n+1} := (a)_n (a + n)$ for $n \in \mathbb{N}_0$, so that in particular $n! = (1)_n$ and $F(a, b, c, z) = 1$ when $abz = 0$.

In this paper we are dealing with starlikeness and convexity properties of hypergeometric and related functions. In particular we shall obtain for certain ranges of real parameters a, b, c the order of starlikeness of *shifted* hypergeometric functions $z \mapsto zF(a, b, c, z)$ and consequently also the order of convexity of hypergeometric functions. Results of this kind have previously been obtained by various authors, and in Section 2, beside giving the proofs, we shall also discuss and compare our results with the earlier ones.

Our main result is the following theorem.

Theorem 1.1. *If $a, b, c \in \mathbb{R}$ with $0 < a \leq b < c$ or $-1 \leq a < 0 < b < c$ and if $r \in (0, 1]$ then the function $z \mapsto zF(a, b, c, rz)$ has the order of starlikeness*

$$\begin{aligned}\sigma(zF(a, b, c, rz)) &= 1 + \frac{\rho F'(a, b, c, \rho)}{F(a, b, c, \rho)} \\ &= 1 + a\rho \frac{\int_0^1 (1 - \rho t)^{-a-1} t^b (1 - t)^{c-b-1} dt}{\int_0^1 (1 - \rho t)^{-a} t^{b-1} (1 - t)^{c-b-1} dt}\end{aligned}$$

where $\rho := -r$ if $a > 0$ and $\rho := r$ if $a < 0$. In particular we have

$$\sigma(zF(a, c, c, rz)) = 1 + \frac{a\rho}{1 - \rho} \leq \sigma(zF(a, b, c, rz)) \leq 1 + \frac{ab\rho}{2c}.$$

Remark 1.2. The case $a < 0$ and $r = 1$ in Theorem 1.1 is to be considered as a limiting one. In this case the lower bound $1 + ar/(1 - r)$ is equal to $-\infty$. Using known evaluations of the hypergeometric function at $z = 1$ one readily finds that

$$\sigma(zF(a,b,c,z)) = \begin{cases} 1 + \dfrac{ab}{c-a-b-1} & \text{if } a < 0 < b \le a+b+1 < c, \\ -\infty & \text{if } a < 0 < b \le c \le a+b+1. \end{cases}$$

The first of these two cases has already been obtained by Silverman [23, Thm. 2].

Remark 1.3. The integrals above arise from the Euler integral representation

$$F(a,b,c,z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 (1-tz)^{-a} t^{b-1} (1-t)^{c-b-1}\, dt, \tag{1.2}$$

(where Γ denotes the Gamma function) which holds for $\operatorname{Re} c > \operatorname{Re} b > 0$ and which yields the analytic continuation of the hypergeometric function from $\mathbb{D}$ to the cut-plane $\mathbb{C} \setminus [1, +\infty)$. They are perfectly suited for numerical purposes and may be used to obtain rough lower and upper bounds for $\sigma(zF(a,b,c,rz))$. The upper bound mentioned in the theorem, however, does not stem from such an estimate but is a simple consequence of the bound for the second coefficient of the functions in $\mathcal{S}_\alpha$, see below (1.4). We do not go any further into this matter.

From (1.1) we obtain

$$F'(a,b,c,z) = \frac{d}{dz}F(a,b,c,z) = \frac{ab}{c}F(a+1,b+1,c+1,z), \tag{1.3}$$

and consequently from Theorem 1.1 and Remark 1.2 the following corollary.

Corollary 1.4. *If $a,b,c \in \mathbb{R} \setminus \{0\}$ with $-1 < a \le b < c$ or $-2 \le a < -1 < b < c$ and if $r \in (0,1]$ then the function $z \mapsto F(a,b,c,rz)$ has the order of convexity*

$$\begin{aligned} \kappa(F(a,b,c,rz)) &= 1 + \frac{\rho F''(a,b,c,\rho)}{F'(a,b,c,\rho)} \\ &= 1 + (a+1)\rho \frac{\int_0^1 (1-\rho t)^{-a-2} t^{b+1} (1-t)^{c-b-1} dt}{\int_0^1 (1-\rho t)^{-a-1} t^{b} (1-t)^{c-b-1} dt} \end{aligned}$$

where $\rho := -r$ if $a > -1$ and $\rho := r$ if $a < -1$. In particular we have

$$\kappa(F(a,c,c,rz)) = 1 + \frac{(a+1)\rho}{1-\rho} \le \kappa(F(a,b,c,rz)) \le 1 + \frac{(a+1)(b+1)\rho}{2(c+1)}$$

and

$$\kappa(F(a,b,c,z)) = \begin{cases} 1 + \dfrac{(a+1)(b+1)}{c-a-b-2} & \textit{if } a < -1 < b \le a+b+2 < c,\ bc \ne 0, \\ -\infty & \textit{if } a < -1 < b \le c \le a+b+2,\ bc \ne 0. \end{cases}$$

Our method of proof combines the continued fraction of Gauss for a certain ratio of hypergeometric functions and a theorem of Wall which yields a characterization of Hausdorff moment sequences (i.e. totally monotone sequences) by means of (continued) g-fractions. In conjunction with results of Merkes and Wirths this method also yields mapping properties of such ratios and their shifts which seem to be of independent interest.

Theorem 1.5. *If* $a, b, c \in \mathbb{R}$ *with* $-1 \le a \le c$ *and* $0 < b \le c$ *then the functions*

$$z \mapsto \frac{zF(a+1,b,c,z)}{F(a,b,c,z)}, \qquad z \mapsto \frac{F(a+1,b,c,z)}{F(a,b,c,z)},$$

$$z \mapsto \frac{zF(a+1,b+1,c+1,z)}{F(a,b,c,z)}, \quad z \mapsto \frac{F(a+1,b+1,c+1,z)}{F(a,b,c,z)},$$

$$z \mapsto \frac{zF(a+1,b+1,c+1,z)}{F(a+1,b,c,z)}, \quad z \mapsto \frac{F(a+1,b+1,c+1,z)}{F(a+1,b,c,z)}$$

are analytic in $\mathbb{C} \setminus [1, +\infty)$ *and each function maps both the unit disk* $\mathbb{D}$ *and the half-plane* $\{z \in \mathbb{C} : \operatorname{Re} z < 1\}$ *univalently onto domains that are convex in the direction of the imaginary axis, except for the last function if* $a = -1$ *or* $b = c$ *(in which case it is constant* 1*).*

If, in addition, $a \neq 0$ *then the same is true for the two functions*

$$z \mapsto \frac{zF'(a,b,c,z)}{F(a,b,c,z)}, \qquad z \mapsto \frac{F'(a,b,c,z)}{F(a,b,c,z)}.$$

Moreover the coefficient sequences at $z = 0$ *of the first three functions on the right side are Hausdorff moment sequences (i.e. totally monotone sequences).*

Here a domain $\Omega \subseteq \mathbb{C}$ is called *convex in the direction of the imaginary axis* (cf. [14], [15]) if the intersection of Ω with any line parallel to the imaginary axis is either empty or a line segment. This theorem will be established in Section 3.

Indeed, with another restriction on a, Theorem 1.5 holds also for the hypergeometric function.

Theorem 1.6. *If* $a, b, c \in \mathbb{R}$ *with* $0 < a \le 1$ *and* $0 < b \le c$ *then the coefficient sequence at* $z = 0$ *of the hypergeometric function* $z \mapsto F(a,b,c,z)$ *is a Hausdorff moment sequence (i.e. a totally monotone sequence) and the function as well as the shifted function* $z \mapsto zF(a,b,c,z)$ *each maps both the unit disk* $\mathbb{D}$ *and the half-plane* $\{z \in \mathbb{C} : \operatorname{Re} z < 1\}$ *univalently onto domains that are convex in the direction of the imaginary axis.*

Remark 1.7. Theorem 1.6 holds, in a similar manner, for every $k \in \mathbb{N}_0$ for the generalized hypergeometric function

$${}_{k+1}F_k(a, b_1, b_2, \ldots, b_k; c_1, c_2, \ldots, c_k; z) := \sum_{n=0}^{\infty} \frac{(a)_n (b_1)_n (b_2)_n \cdots (b_k)_n}{(c_1)_n (c_2)_n \cdots (c_k)_n n!} z^n,$$

where $z \in \mathbb{D}$, provided that $0 < a \le 1$ and $0 < b_j \le c_j$ for $j = 1, 2, \ldots, k$.

Note that it has already been observed that under the above mentioned assumptions the coefficient sequences of F and ${}_{k+1}F_k$ are Hausdorff moment sequences, see Pólya [12].

1.2. Results on the convolution of starlike and convex functions. The *convolution* (or *Hadamard product*) $f * g$ of two functions $f, g \in \mathcal{H}$ with series expansions $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$ is defined by

$$(f * g)(z) := \sum_{n=0}^{\infty} a_n b_n z^n.$$

Obviously, we have $f * g \in \mathcal{H}$. We shall also use the notation

$$\mathcal{M} * \mathcal{N} := \{f * g : \ f \in \mathcal{M}, \, g \in \mathcal{N}\}, \qquad \mathcal{M}, \mathcal{N} \subseteq \mathcal{H}.$$

For an introduction to convolutions in the present context we refer to [19] or [18]. For $\alpha \in (-\infty, 1]$ let $\mathcal{S}_\alpha$ (resp. $\mathcal{K}_\alpha$) denote the set of all *normalized* functions in $\mathcal{H}$ which are *starlike* (resp. *convex*) *of order* α, i.e. which satisfy $f(0) = 0$, $f'(0) = 1$ and $\sigma(f) \geq \alpha$ (resp. $\kappa(f) \geq \alpha$). In particular, $\mathcal{S}_0$ (resp. $\mathcal{K}_0$) is the set of all normalized starlike (resp. convex) functions. Obviously,

$$\{\mathrm{id}\} = \mathcal{S}_1 \subsetneq \mathcal{S}_\beta \subsetneq \mathcal{S}_\alpha \quad \text{and} \quad \{\mathrm{id}\} = \mathcal{K}_1 \subsetneq \mathcal{K}_\beta \subsetneq \mathcal{K}_\alpha \quad \text{for } -\infty < \alpha < \beta < 1.$$

Special elements of $\mathcal{S}_\alpha$ and $\mathcal{K}_\alpha$ are the functions

$$s_\alpha(z) := \frac{z}{(1-z)^{2(1-\alpha)}} \in \mathcal{S}_\alpha \quad \text{and} \quad k_\alpha(z) := \int_0^1 \frac{z}{(1-tz)^{2(1-\alpha)}}\, dt \in \mathcal{K}_\alpha.$$

They satisfy $zk'_\alpha = s_\alpha$, $\kappa(k_\alpha) = \sigma(s_\alpha) = \alpha$ and are extremal in various aspects in their respective sets. For example, for the second coefficient $c_2(f) := f''(0)/2$ of the series expansion of a function $f \in \mathcal{H}$ at $z = 0$ we have

$$|c_2(f)| \leq c_2(s_\alpha) = 2(1-\alpha) \ \text{ for any } f \in \mathcal{S}_\alpha, \tag{1.4}$$

$$|c_2(f)| \leq c_2(k_\alpha) = 1-\alpha \qquad \text{for any } f \in \mathcal{K}_\alpha, \tag{1.5}$$

and equality occurs if and only if f is a rotation of s_α or k_α respectively, i.e. if there exists $\zeta \in \mathbb{C}$, $|\zeta| = 1$, such that $f(z) = s_\alpha(\zeta z)/\zeta$ or $f(z) = k_\alpha(\zeta z)/\zeta$. Note also that s_0 is the well-known Koebe function, while $k_0(z) = z/(1-z) = s_{1/2}(z)$. Much is known about the behavior of starlike and convex functions under convolutions, beginning with the convolution invariance of the classes $\mathcal{S}_{1/2}$ and $\mathcal{K}_0$ (the former Pólya-Schoenberg Conjecture), see Ruscheweyh and Sheil-Small [17, p. 125, Thm. 2.1; p. 129, Thm. 3.1], Suffridge [25, p. 178, Thm. 7; p. 180, Thm. 10], Lewis [6], Ruscheweyh [18, p. 497–499], [19, p. 49, Thm. 2.1], and Sheil-Small's recent monograph [22]. In particular, the following statements hold:

$$\mathcal{S}_\alpha * \mathcal{S}_\beta \subseteq \mathcal{S}_{\gamma(\alpha,\beta)}, \qquad \alpha, \beta \in [1/2, 1], \tag{1.6}$$

$$\mathcal{K}_\alpha * \mathcal{S}_\beta \subseteq \mathcal{S}_{\delta(\alpha,\beta)}, \qquad \alpha, \beta \in [0, 1], \tag{1.7}$$

where $\gamma(\alpha, \beta) \geq \max(\alpha, \beta)$ and $\delta(\alpha, \beta) \geq \max(\alpha, \beta)$. In what follows we will always assume that $\gamma(\alpha, \beta)$ and $\delta(\alpha, \beta)$ denote the largest numbers with the property stated in (1.6) and in (1.7), respectively. In fact, one then has

$$\gamma(\alpha, \beta) = \sigma(s_\alpha * s_\beta), \qquad \alpha, \beta \in [1/2, 1], \tag{1.8}$$

$$\delta(\alpha, \beta) = \sigma(k_\alpha * s_\beta), \qquad \alpha, \beta \in [0, 1]. \tag{1.9}$$

The relation (1.8) is in [19, p. 56, Thm. 2.7] and (1.9) can be proved in essentially the same way. We omit the details.

It may be worth stating explicitly that (1.7) and (1.9) are equivalent to

$$\mathcal{K}_\alpha * \mathcal{K}_\beta \subseteq \mathcal{K}_{\delta(\alpha,\beta)} \quad \text{and} \quad \delta(\alpha,\beta) = \kappa(k_\alpha * k_\beta), \qquad \alpha,\beta \in [0,1],$$

and $\delta(\alpha,\beta)$ is again best possible for this inclusion.

The problem which motivated the research presented in this paper was to explicitly evaluate the numbers $\gamma(\alpha,\beta)$ and $\delta(\alpha,\beta)$, and the connection with the hypergeometric function stems from the following representation formulas:

$$s_\alpha(z) = zF(2-2\alpha,1,1,z), \tag{1.10}$$

$$k_\alpha(z) = zF(2-2\alpha,1,2,z), \tag{1.11}$$

$$(s_\alpha * s_\beta)(z) = zF(2-2\alpha,2-2\beta,1,z), \tag{1.12}$$

$$(k_\alpha * s_\beta)(z) = zF(2-2\alpha,2-2\beta,2,z), \tag{1.13}$$

for $\alpha,\beta \in (-\infty,1]$ and for $z \in \mathbb{C} \setminus [1,+\infty)$. These are easily verified using the series expansions of the hypergeometric functions and the identity principle for analytic functions. Combining (1.8) with (1.12) and (1.9) with (1.13) we obtain the following results from Theorem 1.1 (which applies to the parameters in question since $F(a,b,c,z)$ is symmetric in a and b).

Theorem 1.8. *If $\alpha \in [1/2,1]$ and $\beta \in (1/2,1)$ then*

$$\begin{aligned}\gamma = \sigma(s_\alpha * s_\beta) &= -\frac{(s_\alpha * s_\beta)'(-1)}{(s_\alpha * s_\beta)(-1)} \\ &= 1 - 2(1-\alpha)\frac{\int_0^1 (1+t)^{2\alpha-3} t^{2-2\beta}(1-t)^{2\beta-2}\,dt}{\int_0^1 (1+t)^{2\alpha-2} t^{1-2\beta}(1-t)^{2\beta-2}\,dt}\end{aligned}$$

is the largest number $\gamma = \gamma(\alpha,\beta)$ such that $\mathcal{S}_\alpha * \mathcal{S}_\beta \subseteq \mathcal{S}_\gamma$. In particular we have

$$\gamma(\alpha,1/2) = \alpha \le \max(\alpha,\beta) \le \gamma(\alpha,\beta) \le 1 - 2(1-\alpha)(1-\beta) \le 1 = \gamma(\alpha,1).$$

Theorem 1.9. *If $\alpha \in [0,1]$ and $\beta \in (0,1)$ then*

$$\begin{aligned}\delta = \sigma(k_\alpha * s_\beta) &= -\frac{(k_\alpha * s_\beta)'(-1)}{(k_\alpha * s_\beta)(-1)} \\ &= 1 - 2(1-\alpha)\frac{\int_0^1 (1+t)^{2\alpha-3} t^{2-2\beta}(1-t)^{2\beta-1}\,dt}{\int_0^1 (1+t)^{2\alpha-2} t^{1-2\beta}(1-t)^{2\beta-1}\,dt}\end{aligned}$$

is the largest number $\delta = \delta(\alpha,\beta)$ such that $\mathcal{K}_\alpha * \mathcal{S}_\beta \subseteq \mathcal{S}_\delta$. In particular we have

$$\delta(\alpha,0) = \alpha \le \max(\alpha,\beta) \le \delta(\alpha,\beta) \le 1 - (1-\alpha)(1-\beta) \le 1 = \delta(\alpha,1).$$

Remark 1.10. Theorem 1.9 generalizes some well-known results due to Marx [8] and Strohhäcker [24], namely $\mathcal{K}_0 \subseteq \mathcal{K}_0 * \mathcal{S}_{1/2} \subseteq \mathcal{S}_{1/2}$, and due to MacGregor [7] and Wilken and Feng [29], namely $\mathcal{K}_\alpha \subseteq \mathcal{K}_\alpha * \mathcal{S}_{1/2} \subseteq \mathcal{S}_{\delta(\alpha,1/2)}$ for $\alpha \in [0,1]$ with

$$\delta(\alpha,1/2) = \sigma(k_\alpha * s_{1/2}) = \sigma(k_\alpha) = \frac{-k'_\alpha(-1)}{k_\alpha(-1)} = \begin{cases} \dfrac{\alpha - 1/2}{1-2^{1-2\alpha}} & \text{if } \alpha \neq 1/2, \\ \dfrac{1}{2\log 2} & \text{if } \alpha = 1/2. \end{cases}$$

In Section 4 we shall discuss a problem which is suggested by Theorem 1.5 and is related to the actual size of the complex sets $z(f*g)'/(f*g)(\mathbb{D})$ for f, g convex or starlike of various orders.

2. Proof of Theorem 1.1 and related comments

Proof of Theorem 1.1. From (1.3) we obtain

$$\text{(2.1)}\quad \frac{z(zF(a,b,c,z))'}{zF(a,b,c,z)} = 1 + \frac{zF'(a,b,c,z)}{F(a,b,c,z)} = 1 + \frac{ab}{c}\,\frac{zF(a+1,b+1,c+1,z)}{F(a,b,c,z)}.$$

Further, (1.1) implies

$$\text{(2.2)}\quad F(a+1,b,c,z) - F(a,b,c,z) = \frac{b}{c} zF(a+1,b+1,c+1,z)$$

(cf. [3, eq. 18]) which on the one hand combined with (2.1) yields

$$\text{(2.3)}\quad \frac{z(zF(a,b,c,z))'}{zF(a,b,c,z)} = 1 + \frac{zF'(a,b,c,z)}{F(a,b,c,z)} = 1 - a + a\frac{F(a+1,b,c,z)}{F(a,b,c,z)}$$

and on the other hand (cf. [3, eq. 27])

$$\text{(2.4)}\quad \frac{F(a+1,b,c,z)}{F(a,b,c,z)} = \frac{1}{1 - \dfrac{b}{c}\,\dfrac{zF(a+1,b+1,c+1,z)}{F(a+1,b,c,z)}}.$$

The ratio of hypergeometric functions in the denominator above at $z=0$ corresponds to the continued fraction of Gauss (cf. [3, p. 134] and [28, p. 337–339])

$$\text{(2.5)}\quad \frac{F(a+1,b+1,c+1,z)}{F(a+1,b,c,z)} \sim \frac{1}{1}\,-\,\frac{d_2 z}{1}\,-\,\frac{d_3 z}{1}\,-\,\frac{d_4 z}{1}\,-\,\cdots$$

with

$$\text{(2.6)}\quad d_n = d_n(a,b,c) := \begin{cases} \dfrac{(c-b+k-1)(a+k)}{(c+2k-2)(c+2k-1)} & \text{for } n = 2k \geq 2,\ k \geq 1, \\ \dfrac{(c-a+k-1)(b+k)}{(c+2k-1)(c+2k)} & \text{for } n = 2k+1 \geq 3,\ k \geq 1. \end{cases}$$

The correspondence $\sim$ in (2.5) means, by definition, that at $z=0$ the difference of the analytic function on the left-hand side and of the nth approximant of the continued fraction on the right-hand side has a zero of order n or, in the terminating case, that this difference is zero if d_{n+1} is the first vanishing coefficient.

From the correspondence (2.5) and (2.4) we obtain the correspondence

$$\frac{F(a+1,b,c,z)}{F(a,b,c,z)} \sim \frac{1}{1} - \frac{(1-g_0)g_1 z}{1} - \frac{(1-g_1)g_2 z}{1} - \frac{(1-g_2)g_3 z}{1} - \cdots \tag{2.7}$$

with

$$g_n = g_n(a,b,c) := \begin{cases} 0 & \text{for } n=0, \\ \dfrac{a+k}{c+2k-1} & \text{for } n=2k\geq 2,\ k\geq 1, \\ \dfrac{b+k-1}{c+2k-2} & \text{for } n=2k-1\geq 1,\ k\geq 1, \end{cases} \tag{2.8}$$

so that $d_n = (1-g_{n-1})g_n$ for all $n \geq 2$ and $b/c = (1-g_0)g_1$.

Now, if $-1 \leq a \leq c$ and $0 \leq b \leq c \neq 0$ then $0 \leq g_n \leq 1$ for all $n \geq 1$, and in this case the continued fraction (2.7) is called a g-fraction. But then, according to a theorem of Wall [28, p. 263, Thm. 69.2], the coefficients at $z = 0$ of the function $z \mapsto F(a+1,b,c,z)/F(a,b,c,z)$ on the left-hand side of (2.7) are the Hausdorff moments of a non-decreasing function on $[0,1]$ with a total increase of 1. Hence, if $-1 \leq a \leq c$ and $0 \leq b \leq c \neq 0$ then there exists a non-decreasing function $\mu_0\colon [0,1] \to [0,1]$ such that $\mu_0(1) - \mu_0(0) = 1$ and

$$\frac{F(a+1,b,c,z)}{F(a,b,c,z)} = \int_0^1 \frac{1}{1-tz}\, d\mu_0(t) \qquad \text{for} \quad z \in \mathbb{C} \setminus [1,+\infty), \tag{2.9}$$

by analytic continuation. Of course here the integral is of Riemann-Stieltjes type and the function μ_0 depends of the parameters a, b and c, i.e. $\mu_0(t) = \mu_0(a,b,c,t)$.

From (2.3) and the integral representation (2.9) it follows for $0 < a \leq b \leq c$ and $r \in (0,1]$ that the minimum of

$$\operatorname{Re} \frac{z(zF(a,b,c,z)'}{zF(a,b,c,z)} = 1 - a + a \operatorname{Re} \frac{F(a+1,b,c,z)}{F(a,b,c,z)} \tag{2.10}$$

for $|z| \leq r$ is attained at the point $z = -r$ and that this minimum is at least $1 - ar/(1+r) = \sigma(zF(a,c,c,rz))$, since we have $F(a,c,c,z) = (1-z)^{-a}$. This, combined with (2.1) and (1.2), proves Theorem 1.1 for positive values of a.

For $-1 \leq a < 0 < b \leq c$ and $r \in (0,1]$ the integral representation (2.9) implies that the minimum of (2.10) for $|z| \leq r$ is attained at the point $z = r$ and that this minimum is at least $1 + ar/(1-r) = \sigma(zF(a,c,c,rz))$ ($= -\infty$ if $r = 1$). The result follows as before. ■

Remark 2.1. The integral representation (2.9) combined with (2.3) yields that the logarithmic derivative $F'(a,b,c,z)/F(a,b,c,z)$ has no poles and consequently

$$F(a,b,c,z) \neq 0 \quad \text{for } z \in \mathbb{C} \setminus [1,+\infty) \text{ if } -1 \leq a \leq c,\ 0 \leq b \leq c \neq 0.$$

This result has already been proved by several authors, namely van Vleck [27], Hurwitz [5], Schafheitlin [21], Herglotz [4] and Runckel [16]. Using (1.3) we also obtain

$$F'(a,b,c,z) \neq 0 \quad \text{for } z \in \mathbb{C}\setminus[1,+\infty) \text{ if } -2 \leq a \leq c,\ -1 \leq b \leq c \neq -1,\ abc \neq 0.$$

We conclude this section with a discussion of the results on the order of starlikeness of shifted hypergeometric functions which we have obtained in Theorem 1.1.

Remark 2.2. Theorem 1.1 partially generalizes (but does not completely contain) two results of Ruscheweyh and Singh [20, p. 3, Thm. 1; p. 4, Cor. 1]. The lower bound $1 + a\rho/(1-\rho)$ in Theorem 1.1 is due to Merkes and Scott [10], who also used the continued fraction of Gauss, but in a different way. Ruscheweyh and Singh [20, p. 5, Thm. 2] have refined the procedure of Merkes and Scott in order to improve, in the case $a > 0$, the lower bound $1 - ar/(1+r)$ in Theorem 1.1, to obtain (note that there is a mistake in the displayed formula in [20]) instead the better lower bound

$$1 - \frac{ar}{1+r}\,\frac{b(c+1)+b(c-a)r}{c(c+1)+b(c-a)r} = 1 - \frac{ar}{1+r}\left(1 - \frac{c-b}{c}\,\frac{1}{1+\frac{b(c-a)}{c(c+1)}r}\right). \tag{2.11}$$

Remark 2.3. It is possible to obtain better bounds for $\sigma(zF(a,b,c,rz))$ in Theorem 1.1. For this one can use the approximants of the g-fraction (2.7) as well as the approximants of the continued fraction of Gauss which corresponds to the ratio of hypergeometric functions on the right-hand side of the relation

$$\frac{F(a+1,b,c,z)}{F(a,b,c,z)} = 1 + \frac{z}{1-z}\left(1 - \frac{c-b}{c}\,\frac{F(b,a+1,c+1,z)}{F(b,a,c,z)}\right) \tag{2.12}$$

which was used by Merkes and Scott [10]. The approximants simply have to be evaluated at $z = \rho$. For example, in the case $a > 0$ taking the second approximant in (2.7) and the first approximant in (2.12) we obtain via (2.3) that

$$1 - \frac{abr}{c+br} \leq \sigma(zF(a,b,c,rz)) \leq 1 - \frac{abr}{c+cr} \quad \text{for } 0 < a \leq b \leq c,\ r \in (0,1].$$

These bounds improve those given in Theorem 1.1 but the lower bound is not as good as (2.11), which is obtained by taking the second approximant in (2.12). Higher order evaluations are of course also possible.

Remark 2.4. Further results on the order of starlikeness of $z \mapsto zF(a,b,c,z)$ can be found in [1], [2], [11] and [13]. The basic tools used by these authors are coefficient conditions or the hypergeometric differential equation combined with first order differential subordination.

3. Proof of Theorem 1.5 and Theorem 1.6

Our proof uses again the integral representation for g-fractions and the following lemma which goes back to Merkes [9, Cor. 2.1, Thm. 3.1] and Wirths [30, p. 511].

Lemma 3.1. *Let* $\mu\colon [0,1] \to [0,1]$ *be non-decreasing with* $\mu(1) - \mu(0) = 1$. *Then the function*

$$z \mapsto \int_0^1 \frac{z}{1-tz}\, d\mu(t)$$

is analytic in the cut-plane $\mathbb{C} \setminus [1, +\infty)$ *and maps both the unit disk* $\mathbb{D}$ *and the half-plane* $\{z \in \mathbb{C} : \operatorname{Re} z < 1\}$ *univalently onto domains which are convex in the direction of the imaginary axis.*

Remark 3.2. When in Lemma 3.1 we have in addition that $m_1 := \int_0^1 t\, d\mu(t) > 0$ then its conclusion holds for the function $z \mapsto \int_0^1 (1-tz)^{-1} d\mu(t)$ as well.

For then $\mu'(t) := (1/m_1) \int_0^t s\, d\mu(s)$ is again a non-decreasing self-mapping of $[0,1]$ with $\mu'(1) - \mu'(0) = 1$ and $\int_0^1 (1-tz)^{-1} d\mu(t) = 1 + m_1 \int_0^1 z(1-tz)^{-1} d\mu'(t)$.

Proof of Theorem 1.5. Suppose that $-1 \le a \le c$ and $0 < b \le c$. The integral representation (2.9) yields

$$\frac{zF(a+1,b,c,z)}{F(a,b,c,z)} = \int_0^1 \frac{z}{1-tz}\, d\mu_0(t)$$

for $z \in \mathbb{C} \setminus [1, +\infty)$, where $\mu_0\colon [0,1] \to [0,1]$ is a non-decreasing function with $\mu_0(1) - \mu_0(0) = 1$. If we define

$$\mu_1(t) := \frac{1}{g_1} \int_0^t s\, d\mu_0(s),$$

where $g_1 = b/c > 0$ was defined by (2.8), then it follows from (2.2) and (2.9) that μ_1 is also a non-decreasing self-mapping of $[0,1]$ with $\mu_1(1) - \mu_1(0) = 1$,

$$\frac{F(a+1,b,c,z)}{F(a,b,c,z)} = 1 + g_1 \int_0^1 \frac{z}{1-tz}\, d\mu_1(t),$$

$$\frac{zF(a+1,b+1,c+1,z)}{F(a,b,c,z)} = \int_0^1 \frac{z}{1-tz}\, d\mu_1(t)$$

for $z \in \mathbb{C} \setminus [1, +\infty)$. Furthermore, using (1.3) and $d_2 = (c-b)(a+1)/(c(c+1))$ from (2.6) we obtain

$$\frac{d}{dz} \frac{F(a+1,b+1,c+1,z)}{F(a,b,c,z)}\bigg|_{z=0} = \frac{(a+1)(b+1)}{c+1} - \frac{ab}{c} = g_1 + d_2 \ge g_1 > 0$$

and consequently

$$\frac{F(a+1,b+1,c+1,z)}{F(a,b,c,z)} = 1 + (g_1 + d_2) \int_0^1 \frac{z}{1-tz}\, d\mu_2(t)$$

for $z \in \mathbb{C} \setminus [1, +\infty)$, where

$$\mu_2(t) := \frac{1}{g_1 + d_2} \int_0^t s\, d\mu_1(s)$$

is again a non-decreasing self-mapping of $[0,1]$ with $\mu_2(1) - \mu_2(0) = 1$.

Since, under the assumptions on a, b, c, the g_n defined by (2.8) satisfy $0 \le g_n \le 1$ for $n \ge 1$ and for the d_n defined by (2.6) we have $d_n = (1-g_{n-1})g_n$ for $n \ge 2$, the continued fraction of Gauss (2.5) is also a g-fraction and so the same theorem of Wall [28, p. 263, Thm. 69.2] which we have already used to prove (2.9) yields, when applied to (2.5), the existence of a non-decreasing self-mapping μ_3 of $[0,1]$ with $\mu_3(1) - \mu_3(0) = 1$ and

$$\frac{zF(a+1,b+1,c+1,z)}{F(a+1,b,c,z)} = \int_0^1 \frac{z}{1-tz}\, d\mu_3(t)$$

for $z \in \mathbb{C} \setminus [1,+\infty)$. Finally, in the same fashion, if in addition $a > -1$ and $b < c$, so that $d_2 > 0$, we obtain that

$$\frac{F(a+1,b+1,c+1,z)}{F(a+1,b,c,z)} = 1 + d_2 \int_0^1 \frac{z}{1-tz}\, d\mu_4(t)$$

for $z \in \mathbb{C} \setminus [1,+\infty)$, where

$$\mu_4(t) := \frac{1}{d_2} \int_0^t s\, d\mu_3(s)$$

is again a non-decreasing self-mapping of $[0,1]$ with $\mu_4(1) - \mu_4(0) = 1$.

Since the functions μ_k, $k = 0,1,2,3,4$, satisfy the assumptions of Lemma 3.1 we have verified the assertion about the first six functions in Theorem 1.5. Using in addition (1.3) and the assumption $a \ne 0$ this extends to the remaining two functions. ■

Proof of Theorem 1.6 and Remark 1.7. As already mentioned, Pólya [12] has observed that when the assumptions of Theorem 1.6 and Remark 1.7 are satisfied, the coefficient sequence at $z = 0$ of the hypergeometric function F as well as the generalized hypergeometric function ${}_{k+1}F_k$ is a Hausdorff moment sequence. In conjunction with Lemma 3.1 and Remark 3.2 this yields Theorem 1.6 and Remark 1.7, since the first moment, i.e. the first coefficient ab/c respectively $a(b_1 b_2 \cdots b_k)/(c_1 c_2 \cdots c_k)$, is positive by assumption. ■

4. A subordination problem

The relations (1.8) and (1.9) are actually based on a much stronger result, namely

$$\text{(4.1)} \quad \frac{z(f*g)'}{f*g}(\mathbb{D}) \subseteq \overline{\text{co}} \left(\frac{z(s_\alpha * s_\beta)'}{s_\alpha * s_\beta}(\mathbb{D}) \right) \qquad \text{for } f \in \mathcal{S}_\alpha,\ g \in \mathcal{S}_\beta,\ \alpha, \beta \in [1/2, 1],$$

$$\text{(4.2)} \quad \frac{z(f*g)'}{f*g}(\mathbb{D}) \subseteq \overline{\text{co}} \left(\frac{z(k_\alpha * s_\beta)'}{k_\alpha * s_\beta}(\mathbb{D}) \right) \qquad \text{for } f \in \mathcal{K}_\alpha,\ g \in \mathcal{S}_\beta,\ \alpha, \beta \in [0, 1],$$

where $\overline{\text{co}}$ denotes the closed convex hull. The inclusion (4.1) is contained in the proof of [19, p. 56, Thm. 2.7] and (4.2) can be proved in essentially the same way. Once again, we omit the details. On the other hand, from (1.12), (1.13) and Theorem 1.5 we see that the functions

$$(4.3)\quad z \mapsto \frac{z(s_\alpha * s_\beta)'(z)}{(s_\alpha * s_\beta)(z)} = 1 + \frac{zF'(2-2\alpha, 2-2\beta, 1, z)}{F(2-2\alpha, 2-2\beta, 1, z)}, \qquad \alpha, \beta \in [1/2, 1),$$

$$(4.4)\quad z \mapsto \frac{z(k_\alpha * s_\beta)'(z)}{(k_\alpha * s_\beta)(z)} = 1 + \frac{zF'(2-2\alpha, 2-2\beta, 2, z)}{F(2-2\alpha, 2-2\beta, 2, z)}, \qquad \alpha, \beta \in [0, 1),$$

map both the unit disk $\mathbb{D}$ and the half-plane $\{z \in \mathbb{C} : \operatorname{Re} z < 1\}$ univalently onto domains which are convex in the direction of the imaginary axis. Using the notion of subordination this observation may be formulated in a more attractive way.

A function $h \in \mathcal{H}$ is called *subordinate* to $H \in \mathcal{H}$, and one writes $h \prec H$, if there exists a function $w \in \mathcal{H}$ with $w(0) = 0$ and $w(\mathbb{D}) \subseteq \mathbb{D}$ such that $h(z) = H(w(z))$ for all $z \in \mathbb{D}$. In particular, if H is univalent then the subordination $h \prec H$ holds if and only if $h(0) = H(0)$ and $h(\mathbb{D}) \subseteq H(\mathbb{D})$.

Question. Do the subordinations

$$\frac{z(f*g)'}{f*g} \prec \frac{z(s_\alpha * s_\beta)'}{s_\alpha * s_\beta} \qquad \text{for } f \in \mathcal{S}_\alpha,\ g \in \mathcal{S}_\beta,\ \alpha, \beta \in [1/2, 1],$$

$$\frac{z(f*g)'}{f*g} \prec \frac{z(k_\alpha * s_\beta)'}{k_\alpha * s_\beta} \qquad \text{for } f \in \mathcal{K}_\alpha,\ g \in \mathcal{S}_\beta,\ \alpha, \beta \in [0, 1]$$

hold? In other words, do the inclusions (4.1) and (4.2) also hold without $\overline{\text{co}}$?

If the functions in (4.3) and (4.4) were convex univalent then we obviously would have an affirmative answer to this. But Theorem 1.5 provides only convexity in the direction of the imaginary axis, and there is numerical evidence that the functions (4.3) and (4.4) are not always convex. This, of course, does not rule out the possibility that the subordinations in question are actually valid.

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